LECTURE 1
Course Web Page
www3.cs.stonybrook.edu/~cse303

The webpage contains:
lectures notes slides;
very detailed solutions to majority of homework problems;
a lot of previous quizzes and tests;
all to be used for study
Course Text Book

ELEMENTS OF THE THEORY OF COMPUTATION

H.L. Lewis, Christos H. Papadimitriou

Prentice Hall, Second Edition

We will follow the book very closely
Course Objectives

The main objective of the course is to introduce abstract models of computation such as finite and push-down automata, and analyze their relationship and relative expressive power.

We will explore the connection between abstract machine models and formal languages, as specified by grammars.

The course will enhance students’ awareness of both the power and inherent limitations of algorithmic computation via the study of Turing machines and/or other abstract computational models.
Workload

There will be 4 quizzes, a midterm and a final examination. I will also give two PRACTICE TESTS for extra credit (30pts total).

None of the grades will be curved.

Quizzes and Tests problems will be mainly taken from problems solved or listed in the book and in our Lecture Notes, assigned homeworks, or past Tests and Quizzes published with solutions on the course Webpage.

Posted past Quizzes and Tests are designed to help you to learn what we covered in class and what you still may not understand.

You can take them for your own practice and then check your solutions with solutions posted by me.
Final grade computation

You can earn up to 300 points + 30 extra points = 330 points during the semester.

The grade will be determined in the following way:
# of earned points divided by 4 = % grade

The % grade is translated into a letter grade in a standard way as described in the course Syllabus
CHAPTER 1
SETS, RELATIONS, and LANGUAGES

1. Sets
2. Relations and Functions
3. Special types of binary relations
4. Finite and Infinite Sets
5. Fundamental Proof Techniques
6. Closures and Algorithms
7. Alphabets and languages
8. Finite Representation of Languages
Chapter 1
PART 1: Sets and Operations on Sets
Sets

Set  A set is a collection of objects

Elements The objects comprising a set are are called its elements or members

\[ a \in A \] denotes that \( a \) is an element of a set \( A \)

\[ a \notin A \] denotes that \( a \) is not an element of a set \( A \)

Empty Set is a set without elements and is denoted by \( \emptyset \)

Set can be defined by listing elements;

Example The set

\[ A = \{ a, \emptyset, \{ a, \emptyset \} \} \]

has 3 elements:

\[ a \in A, \emptyset \in A, \{ a, \emptyset \} \in A \]
Sets

Set can be **defined** by referring to other sets and to properties $P(x)$ that elements may or may not have. We write it as

$$B = \{x \in A : P(x)\}$$

**Example**

Let $N$ be a set in natural numbers

$$B = \{n \in N : n < 0\} = \emptyset$$
Operations on Sets

Set Inclusion \( A \subseteq B \) iff \( \forall a (a \in A \Rightarrow a \in B) \) is a true statement

Set Equality \( A = B \) iff \( A \subseteq B \) and \( B \subseteq A \)

Proper Subset \( A \subset B \) iff \( A \subseteq B \) and \( A \neq B \)

Subset Notations \( A \subseteq B \) for a subset (might be improper)

\( A \subset B \) for a proper subset

Power Set \( \mathcal{P}(A) = \{ B : B \subseteq A \} \)

Other Notation \( 2^A = \{ B : B \subseteq A \} \)
Operations on Sets

Union \[ A \cup B = \{ x : \quad x \in A \quad \text{or} \quad x \in B \} \]
We write: \[ x \in A \cup B \quad \text{iff} \quad x \in A \cup x \in B \]

Intersection \[ A \cap B = \{ x : \quad x \in A \quad \text{and} \quad x \in B \} \]
We write: \[ x \in A \cap B \quad \text{iff} \quad x \in A \cap x \in B \]

Relative Complement \[ A - B = \{ x : \quad x \in A \quad \cap \quad x \notin B \} \]
We write: \[ x \in (A - B) \quad \text{iff} \quad x \in A \cap x \notin B \]
Operations on Sets

Complement is defined only for $A \subseteq U$, where $U$ is called an universe

Complement $\neg A = U - A$

We write for $x \in U$, $x \in \neg A$ iff $x \notin A$

Algebra of sets consists of properties of sets that are TRUE for ALL sets involved

We use tautologies of propositional logic to prove BASIC properties of sets and we use the basic properties to prove more elaborated properties of sets
Operations on Sets

It is possible to form intersections and unions of more than two sets.

If \( \mathcal{F} \) is a collection of sets, we write \( \bigcup \mathcal{F} \) for the set whose elements are the elements of all of the sets in \( \mathcal{F} \).

**Example**  Let

\[
\mathcal{F} = \{\{a\}, \{\emptyset\}, \{a, \emptyset, b\}\}
\]

We get

\[
\bigcup \mathcal{F} = \{a, \emptyset, b\}
\]
Operations on Sets

We define formally

**Generalized Union** of any family \( \mathcal{F} \) of sets as

\[
\bigcup \mathcal{F} = \{ x : \exists S \in \mathcal{F} \ x \in S \}
\]

We write:

\[
x \in \bigcup \mathcal{F} \text{ iff } \exists S \in \mathcal{F} \ x \in S
\]

**Generalized Intersection** of any family \( \mathcal{F} \) of sets

\[
\bigcap \mathcal{F} = \{ x : \forall S \in \mathcal{F} \ x \in S \}
\]

We write:

\[
x \in \bigcap \mathcal{F} \text{ iff } \forall S \in \mathcal{F} \ x \in S
\]
Operations on Sets

**Ordered Pair** Given two sets $A$, $B$ we denote by $(a, b)$ an ordered pair, where $a \in A$ and $b \in B$

$a$ is a first coordinate, $b$ is the second coordinate

We define:

$$(a, b) = (c, d) \iff a = c \text{ and } b = d$$

**Cartesian Product** of two sets $A$ and $B$ is defined as

$$A \times B = \{(a, b) : a \in A \cap b \in B\}$$

We write

$$(a, b) \in A \times B \iff a \in A \cap b \in B$$
Binary Relations

Binary Relation $R$ defined in a set $A$ is any subset $R$ of a cartesian product of $A \times A$, i.e.

$$R \subseteq A \times A$$

Domain of $R$ Let $R \subseteq A \times A$, we define domain of $R$ as

$$D_R = \{a \in A \, (a, b) \in R\}$$

Range of $R$

Let $R \subseteq A \times A$, we define range of $R$ (set of values of $R$) as

$$V_R = \{b \in A \, (a, b) \in R\}$$
Chapter 1
PART 2 : Relations and Functions
n-ary Relations

Ordered tuple  Given sets $A_1, ... A_n$

An element $(a_1, a_2, ... a_n)$ such that $a_i \in A_i$ for $i = 1, 2, ... n$ is called an ordered tuple

Cartesian Product of sets $A_1, , A_n$

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ... a_n) : a_i \in A_i, i = 1, 2, ... n\}$$

n-ary Relation on sets $A_1, , A_n$ is any subset of $A_1 \times A_2 \times ... \times A_n$, i.e.

$$R \subseteq A_1 \times A_2 \times ... \times A_n$$
Function as Relation

Definition

\[ R \subseteq A \times B \text{ is a } \textbf{function} \text{ from } A \text{ to } B \text{ iff} \]

\[ \forall a \in A \ \exists! \ b \in B \ (a, b) \in R \]

Where \( \exists! \ b \in B \) means there is EXACTLY one \( b \in B \)

Because for all \( a \in A \) we have exactly one \( b \in B \), we write it as:

\[ a = R(b) \text{ for } (a,b) \in R \]

A is called a \textbf{domain} of a function \( R \) and we write:

\[ R : A \longrightarrow B \]

to denote that

\[ R \subseteq A \times B \]

is a \textbf{function} from A to B
FUNCTIONS

Function notation  We denote relations that are functions by letters f, g, h,... and write

\[ f : A \rightarrow B \]

to say that \( f \subseteq A \times B \) is a function from A to B, or that \( f \) maps A into B

Definition

Let \( f : A \rightarrow B \), \( A \) is called a domain of \( f \), and \( B \) is called a codomain of \( f \)

In our approach the graph of \( f \) and the function \( f \) are the same

\[ \text{Graph}f = f = \{(a, b) : b = f(a)\} \]
FUNCTIONS

onto function: \( f : A \overset{\text{onto}}{\rightarrow} B \) iff \( \forall b \in B \exists a \in A \ f(a) = b \)

1-1 function: \( f : A \rightarrow B \) is called a one- to -one function and denoted by
\( f : A \overset{1\text{-}1}{\rightarrow} B \) iff \( \forall x, y \in A (x \neq y \Rightarrow f(x) \neq f(y) ) \)

not 1-1 function: \( f : A \rightarrow B \) is not a one- to -one function iff \( \exists x, y \in A (x \neq y \cap f(x) = f(y) ) \)

1-1, onto function If \( f \) is a 1-1 and onto function we write it as \( f : A \overset{1\text{-}1,\text{onto}}{\rightarrow} B \)
FUNCTIONS

Composition  Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \), we define a new function \( h : A \rightarrow C \), called a composition of \( f \) and \( g \), as follows: for any \( x \in A \), \( h(x) = g(f(x)) \)

Composition notation  We denote a composition \( h \) of \( f \) and \( g \) as \( h = f \circ g \) as the function \( f \) goes FIRST and \( g \) SECOND

I.e. we define:
for all \( x \in A \), \((f \circ g)(x) = g(f(x))\)

Other Notation for composition of \( f \) and \( g \) is

\[(g \circ f)(x) = g(f(x)) \text{ for all } x \in A\]

Remember that both notations \( f \circ g \) or \( g \circ f \) are just NAMES for the same function \( h(x) = g(f(x)) \)! and BOTH mean:
we compose a function \( f : A \rightarrow B \) (first) with a function \( g : B \rightarrow C \) (second)
FUNCTIONS

Inverse function  Let $f : A \rightarrow B$ and $g : B \rightarrow A$
$g$ is called an **inverse** function to $f$  iff
$\forall a \in A (g \circ f)(a) = (f \circ g)(a) = g(f(a)) = a$ - we used here both names

Inverse function notation  If $g$ is an **inverse** function to $f$ we denote by $g = f^{-1}$

Identity function  $I : A \rightarrow A$ is called an **identity** on $A$
iff  $\forall a \in A I(a) = a$

Inverse and Identity  Let $f : A \rightarrow B$ and $f^{-1} : B \rightarrow A$ is an inverse to $f$
Then  $(f^{-1} \circ f)(a) = I(a) = a$, for all $a \in A$ and $(f \circ f^{-1})(b) = I(b) = b$ for all $b \in B$
FUNCTIONS: Image and Inverse Image

**Image** of a set $A \subseteq X$ under a function $f : X \rightarrow Y$ is denoted by $f[A]$ or $f(A)$ and defined as

$$f[A] = \{y \in Y : \exists x (x \in A \cap y = f(x))\}$$

We write

$$y \in f(A) \iff \exists x (x \in A \cap y = f(x))$$

**Inverse Image** of a set $B \subseteq Y$ under a function $f : X \rightarrow Y$ is denoted by $f^{-1}[B]$ or $f^{-1}(B)$ and defined as

$$f^{-1}[B] = \{x \in X : f(x) \in B\}$$

We write

$$x \in f^{-1}[B] \iff f(x) \in B$$
Sequences

Definition
A sequence of elements of a set $A$ is any function $f$ from the set of natural numbers $\mathbb{N}$ into $A$

$$f : \mathbb{N} \longrightarrow A$$

Any $f(n) = a_n$ is called the $n$-th term of the sequence $f$.

Notations
$$f = \{a_n\}_{n \in \mathbb{N}}, \quad \{a_n\}_{n \in \mathbb{N}}, \quad \{a_n\}$$
Sequences Example

Example
We define a sequence \( f \) of real numbers \( \mathbb{R} \) as follows

\[
f : \mathbb{N} \rightarrow \mathbb{R}
\]

\[
f(n) = n + \sqrt{n}
\]

We also use a shorthand notation for \( f \) and write

\[
a_n = n + \sqrt{n}
\]
We often write $f = \{a_n\}$ in an even shorter and informal form as

$$a_0 = 0, \quad a_1 = 1 + 1 = 2, \quad a_2 = 2 + \sqrt{2} \ldots \ldots$$

$$0, \quad 2, \quad 2 + \sqrt{2}, \quad 3 + \sqrt{3}, \quad \ldots \ldots n + \sqrt{n} \ldots \ldots$$
Observations

Observation 1: Sequence is ALAYS INFINITE (countably infinite) as by definition, the domain of the sequence function f is a set of N of natural numbers

Observation 2: We can enumerate elements of the sequence by any infinite subset of N; we usually take a set $N - \{0\}$

Definition: A set T is called COUNTABLY INFINITE iff there is a one to one (1-1) function f that maps N onto T, i.e.

$$f: N \rightarrow^{1-1,onto} T$$
Observations

**Observation 3**: We can choose as a SET of INDEXES of a sequence any COUNTABLY infinite set $T$, not only the set $N$ of natural numbers

We often choose $T = N - \{0\} = N^+$, i.e. we consider sequences that "start" with $n = 1$.

We usually write sequences as

$$a_1, \ a_2, \ a_3, \ .... \ a_n, \ .....$$

$$\{a_n\}_{n \in N^+}$$
Finite Sequences

Definition

A finite sequence of elements of a set $A$ is any function $f$ from a FINITE set $K$ into $A$.

Usually we take as $K$ a subset of natural numbers $N$ we write, for simplicity $K = \{1, 2, \ldots, n\}$, $n \in N$, and call $n$ the length of the sequence and write a finite sequence $f$ as

$$f : \{1, 2, \ldots, n\} \rightarrow A, \quad f(n) = a_n, \quad \text{or} \quad f = \{a_k\}_{k=1}^{n}$$

Case $n=0$

In this case the function $f$ is empty set we call it an empty sequence and denote by $e$. 
Example

Example: take

\[ a_n = \frac{n}{(n - 2)(n - 5)} \]

Domain of the sequence \( f(n) = a_n \) is \( N - \{2, 5\} \) and

\[ f : N - \{2, 5\} \rightarrow \mathbb{R} \]

Let \( T = \{-1, -2, 3, 4\} \) be a FINITE set

\( f(n) = a_n \) for \( n \in T \) is now a finite sequence with the domain \( T \).
Families of Sets Revisited

**Family of sets** Any collection of sets is called a Family of sets. We denote it by $\mathcal{F}$

**Sequence of sets** is a sequence $f : N \to \mathcal{F}$, i.e a sequence where all its elements are SETS

We use CAPITAL letters to denote the sets, so we also use capital letters to denote sequences of sets: $\{A_n\}_{n \in N}$, or $\{A_n\}_{n \in N \setminus \{0\}}$

**Generalized Union** of a sequence of sets

$$\bigcup_{n \in N} A_n = \{ x : \exists n \in N \ x \in A_n \}$$

We write

$$x \in \bigcup_{n \in N} A_n \iff \exists n \in N \ x \in A_n$$
Families of Sets Revisited

**Generalized Intersection** of a sequence of sets

\[
\bigcap_{n \in \mathbb{N}} A_n = \{x : \forall n \in \mathbb{N} \ x \in A_n\}
\]

We write

\[x \in \bigcap_{n \in \mathbb{N}} A_n \iff \forall n \in \mathbb{N} \ x \in A_n\]

**Indexed Family** of sets

Let \( \mathcal{F} \) be a family of sets, and \( T \neq \emptyset \)

Any \( f : T \to \mathcal{F}, \ f(t) = A_t \) is called an indexed family of sets with the set of indexes \( T \)

We write it \( \{A_t\}_{t \in T} \)

**Notice** that any sequence of sets is an indexed family of sets for \( T = \mathbb{N} \)
Chapter 1
PART 2: Some Simple Questions and Problems
Simple Short Questions

Here are some short Yes/No questions. Answer them and write a short justification of your answer.

Q1  \( 2\{1,2\} \cap \{1, 2\} \neq \emptyset \)

Q2  \( \{\{a, b\}\} \in 2\{a,b,\{a,b\}\} \)

Q3  \( \emptyset \in 2\{a,b,\{a,b\}\} \)

Q4  For any function \( f \) from \( A \neq \emptyset \) onto \( A \), \( f \) has property \( f(a) \neq a \) for certain \( a \in A \)

Q5  Let \( f : N \longrightarrow \mathcal{P}(N) \) be given by formula: \( f(n) = \{m \in N : \ m < n^2\} \), then \( \emptyset \in f[\{0,1,2]\]  

Q6  Some \( R \subseteq A \times B \) are functions that map \( A \) into \( B \)
Answers to Short Questions

Q1 \[ 2\{1,2\} \cap \{1, 2\} \neq \emptyset \]

NO: \[ 2\{1,2\} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \cap \{1, 2\} = \emptyset \]

Q2 \[ \{\{a, b\}\} \in 2\{a, b, \{a, b\}\} \]

YES: \[ \{\{a, b\}\} \subseteq \{a, b, \{a, b\}\} \]

Q3 \[ \emptyset \in 2\{a, b, \{a, b\}\} \]

YES: For any set A, \( \emptyset \subseteq A \), hence \( \emptyset \in 2^A \)

Q4 For any function \( f \) from \( A \neq \emptyset \) onto \( A \), \( f \) has property \( f(a) \neq a \) for certain \( a \in A \)

NO: There is a function defined by \( f(a) = a \) for all \( a \in A \)

Obviously \( f \) is "onto" and and there is no \( a \in A \) for which \( f(a) \neq a \)
Answers to Short Questions

Q5 Let \( f : N \rightarrow \mathcal{P}(N) \) be given by formula:
\[
f(n) = \{ m \in N : m < n^2 \}, \text{ then } \emptyset \in f[\{0, 1, 2\}]\]
YES We evaluate
\[
f(0) = \{ m \in N : m < 0 \} = \emptyset
\]
\[
f(1) = \{ m \in N : m < 1 \} = \{0\}
\]
\[
f(2) = \{ m \in N : m < 2^2 \} = \{0, 1, 2, 3\}
\]
and so by definition of \( f[A] \) get that
\[
f[\{0, 1, 2\}] = \{\emptyset, \{0\}, \{0, 1, 2, 3\}\} \text{ and hence } \emptyset \in f[\{0, 1, 2\}]\]
Q6 Some \( R \subseteq A \times B \) are functions that map \( A \) into \( B \)
YES: Functions are special type of relations
Simple Short Questions

Q7 $\{(1, 2), (a, 1)\}$ is a binary relation on $\{1, 2\}$

Q8 For any binary relation $R \subseteq A \times A$, the inverse relation $R^{-1}$ exists

Q9 For any binary relation $f \subseteq A \times A$ that is a function, the inverse function $f^{-1}$ exists

Q10 Let $A = \{a, \{a\}, \emptyset\}$ and $B = \{\emptyset, \{\emptyset\}, \emptyset\}$

There is a function $f : A \longrightarrow B$

Q11 If $f : A \longrightarrow B$ and $g : B \longrightarrow A$, then compositions $g \circ f$ and $f \circ g$ exist

Q12 $f : N \longrightarrow P(R)$ given by the formula:

$f(n) = \{x \in R : x > \frac{\ln(n^3 + 1)}{\sqrt{n + 6}}\}$ is a sequence
Answers to Short Questions

Q7 \( \{(1, 2), (a, 1)\} \) is a binary relation on \( \{1, 2\} \)

NO \( a \notin \{(1, 2), (a, 1)\} \)

Q8 For any binary relation \( R \subseteq A \times A \), the inverse relation \( R^{-1} \) exists

YES: The set \( R^{-1} = \{(b, a) : (a, b) \in R\} \) is well defined for any relation

Q9 For any binary relation \( f \subseteq A \times A \) that is a function, the inverse function \( f^{-1} \) exists

NO: \( f \) must be 1–1 and onto
Answers to Short Questions

Q10 Let $A = \{ a, \{ a \}, \emptyset \}$ and $B = \{ \emptyset, \{ \emptyset \}, \emptyset \}$
There is a function $f : A \longrightarrow \mathbb{1}^{-1}_{onto} B$

NO The set $A$ has 3 elements and
$B = \{ \emptyset, \{ \emptyset \}, \emptyset \} = \{ \emptyset, \{ \emptyset \} \}$ has 2 elements

Q11 If $f : A \longrightarrow B$ and $g : B \longrightarrow \mathbb{onto} A$, then compositions $g \circ f$ and $f \circ g$ exist

YES By definition (the "onto" information is irrelevant)

Q12 $f : \mathbb{N} \longrightarrow \mathcal{P}(\mathbb{R})$ given by the formula:
$f(n) = \{ x \in \mathbb{R} : x > \frac{\ln(n^3+1)}{\sqrt{n+6}} \}$ is a sequence

YES By definition (the domain of $f$ is $\mathbb{N}$)
CHAPTER 1
PART 3: Special Types of Binary Relations
EQUIVALENCE RELATION

Equivalence relation  \( R \subseteq A \times A \) is an equivalence relation in \( A \) iff it is reflexive, symmetric and transitive

Symbols  We denote equivalence relation by \( \sim, \approx, \) or \( \equiv \)
In my notes we usually use the symbol \( \approx \)

Equivalence class  If \( \approx \subseteq A \times A \) is an equivalence relation then the set

\[
E(a) = \{ b \in A : a \approx b \}
\]

is called an equivalence class

Symbol  The equivalence classes are usually denoted by

\[
[a] = \{ b \in A : a \approx b \}
\]

and the element \( a \) is called a representative of the equivalence class \([a]\)
PARTITIONS

Partition A family of sets $P \subseteq \mathcal{P}(A)$ is called a partition of the set $A$ iff the following conditions hold

1. $\forall X \in P \ (X \neq \emptyset)$
   i.e. all sets in the partition are non-empty

2. $\forall X, Y \in P \ (X \cap Y = \emptyset)$
   i.e. all sets in the partition are disjoint

3. $\bigcup P = A$
   i.e sum of all sets from $P$ is the set $A$

Notation $A/\approx$ denotes the set of all equivalence classes of $\approx$, i.e.

$$A/\approx = \{[a] : a \in A\}$$
We prove the following theorem (theorem 1.3.1)

Theorem 1

Let $A \neq \emptyset$

IF $\sim$ is an equivalence relation on $A$, THEN $A/\sim$ is a partition of $A$, i.e.

1. $\forall [a] \in A/\sim ([a] \neq \emptyset)$
   i.e. all equivalence classes are non-empty
2. $\forall [a] \neq [b] \in A/\sim ([a] \cap [b] = \emptyset)$
   i.e. all different equivalence classes are disjoint
3. $\bigcup A/\sim = A$
   i.e generalize union of all equivalence classes (sets from $A/\sim$) is equal to the set $A$
We also prove a following:

**Theorem 2** For any partition $P \subseteq \mathcal{P}(A)$ of $A$, one can construct a binary relation $R$ on $A$ such that $R$ is an equivalence and its equivalence classes are exactly the sets of the partition $P$.

**Observe** that we can consider, for ANY binary relation $R$ on $A$ the sets that "look" like equivalence classes i.e. are defined as follows:

$$R(a) = \{ b \in A; aRb \} = \{ b \in A; (a, b) \in R \}$$

**R(a) Fact 1** If the relation $R$ is an equivalence on $A$, then the family $\{ R(a) \}_{a \in A}$ is a partition of $A$.

**R(a) Fact 2** If the family $\{ R(a) \}_{a \in A}$ is not a partition of $A$, then $R$ is not an equivalence on $A$. 
Proof of Theorem 1

Theorem 1
Let $A \neq \emptyset$
IF $\approx$ is an equivalence relation on $A$, THEN $A/\approx$ is a partition of $A$

Proof
Let $A/\approx = \{[a] : a \in A\} = P$. We must show that all sets in $P$ are nonempty, disjoint, and together exhaust $A$.

1. All equivalence classes are nonempty, since $a \in [a]$ for all $a \in A$, by reflexivity of equivalence relation.
2. To show that all different equivalence classes are disjoint let's consider two different equivalence classes \([a] \neq [b]\) and assume that \([a] \cap [b] \neq \emptyset\).

We have that \([a] \neq [b]\), thus there is an element \(c\) such that \(c \in [a]\) and \(c \in [b]\).

Hence \((a, c) \in \approx\) and \((c, b) \in \approx\); since \(\approx\) is transitive, we get \((a, b) \in \approx\).

Since \(\approx\) is symmetric, we have that also \((a, b) \in \approx\).

Now take any element \(d \in [a]\); then \((d, a) \in \approx\), and by transitivity, \((d, b) \in \approx\).

Hence \(d \in [b]\), so that \([a] \subseteq [b]\).

Likewise \([b] \subseteq [a]\) and \([a] = [b]\) what contradicts the assumption that \([a] \neq [b]\).
Proof of Theorem 1

3. To see that $\bigcup P = \bigcup A/ \sim = A$, we simply notice that each element $a \in A$ is in some set in $P$, namely we have by reflexivity that always $a \in [a]$.

This ends the proof of Theorem 1.

Now we are going to prove that the Theorem 1 can be reversed, namely that the following is also true.
Proof of Theorem 1

**Theorem 2** For any partition $\mathbf{P} \subseteq \mathcal{P}(A)$ of $A$, one can construct a binary relation $R$ on $A$ such that $R$ is an equivalence and its equivalence classes are exactly the sets of the partition $\mathbf{P}$

**Proof**

We define a binary relation $R$ as follows

$$R = \{(a, b) : a, b \in X \text{ for some } X \in \mathbf{P}\}$$
Simple Short Questions

Q1 Let $R \subseteq A \times A$ then the set $[a] = \{ b \in A : (a, b) \in R \}$ is an equivalence class with a representative $a$

Q2 The set $\{(\emptyset, \emptyset), (\{a\}, \{a\}), (3, 3)\}$ represents a transitive relation

Q3 There is an equivalence relation on $A = \{\{0\}, \{0, 1\}, 1, 2\}$ with 3 equivalence classes

Q4 Let $A \neq \emptyset$ such that there are exactly 25 partitions of $A$ It is possible to define 20 equivalence relations on $A$
Short Questions Answers

Q1  Let \( R \subseteq A \times A \) then the set \([a] = \{b \in A : (a, b) \in R\}\) is an equivalence class with a representative \(a\)

**NO**  Only when \(R\) is an equivalence

Q2  The set \(\{(\emptyset, \emptyset), (\{a\}, \{a\}), (3, 3)\}\) represents a transitive relation

**YES**  Transitivity condition is vacuously true

Q3  There is an equivalence relation on \(A = \{\{0\}, \{0, 1\}, 1, 2\}\) with 3 equivalence classes

**YES**  For example \(P = \{\{0\}\}, \{\{0, 1\}\}, \{1, 2\}\)  

Q4  Let \(A \neq \emptyset\) such that there are exactly 25 partitions of \(A\)

It is possible to define 20 equivalence relations on \(A\)

**YES**  By Theorem 2 one can define up to 25 (as many as partitions) of equivalence classes
Long Questions

Q1 Consider a function $f : A \rightarrow B$
Show that the following relation $R$ is an equivalence relation on $A$

\[ R = \{ (a, b) \in A \times A : f(a) = f(b) \} \]

Q2 Let $f : N \rightarrow N$ be such that

\[
f(n) = \begin{cases} 
1 & \text{if } n \leq 6 \\
2 & \text{if } n > 6 
\end{cases}
\]

Find equivalence classes of $R$ from Q1 for this particular $f$
Q1 Consider a function $f: A \rightarrow B$

Show that the following relation $R$ is an equivalence relation on $A$

$$R = \{(a, b) \in A \times A : f(a) = f(b)\}$$

Solution

1. $R$ is reflexive
   
   $(a, a) \in R$ for all $a \in A$ because $f(a) = f(a)$

2. $R$ is symmetric: let $(a, b) \in R$, by definition it means that $f(a) = f(b)$, but hence $f(b) = f(a)$, and consequently $(b, a) \in R$

3. $R$ is transitive, because for any $a, b, c \in A$ we get that $f(a) = f(b)$ and $f(b) = f(c)$ implies that $f(a) = f(c)$
Q2 Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be such that

\[
f(n) = \begin{cases} 
1 & \text{if } n \leq 6 \\
2 & \text{if } n > 6 
\end{cases}
\]

Find equivalence classes of \( R \) from Q1 for this particular \( f \)

Solution: we evaluate

\[
[0] = \{ n \in \mathbb{N} : f(0) = f(n) \} = \{ n \in \mathbb{N} : f(n) = 1 \} = \{ n \in \mathbb{N} : n \leq 6 \}
\]

\[
[7] = \{ n \in \mathbb{N} : f(7) = f(n) \} = \{ n \in \mathbb{N} : f(n) = 2 \} = \{ n \in \mathbb{N} : n > 6 \}
\]

There are two equivalence classes:

\[
A_1 = \{ n \in \mathbb{N} : n \leq 6 \}, \quad A_2 = \{ n \in \mathbb{N} : n > 6 \}
\]
ORDER RELATIONS

We introduce now of another type of important binary relations: the order relations

Definition

$R \subseteq A \times A$ is an order relation on $A$ iff $R$ is 1. Reflexive, 2. Antisymmetric, and 3. Transitive, i.e. the following conditions are satisfied

1. $\forall a \in A (a, a) \in R$
2. $\forall a, b \in A ((a, b) \in R \cap (b, a) \in R \Rightarrow a = b)$
3. $\forall a, b, c \in A ((a, b) \in R \cap (b, c) \in R \Rightarrow (a, c) \in R)$
ORDER RELATIONS

Definition

$R \subseteq (A \times A)$ is a total order on $A$ iff $R$ is an order and any two elements of $A$ are comparable, i.e. additionally the following condition is satisfied

4. $\forall a, b \in A \ ((a, b) \in R \cup (b, a) \in R)$

Names: order relation is also called historically a partial order

total order is also called historically a linear order
ORDER RELATIONS

Notations

order relations are usually denoted by \( \leq \), or when we want to make a clear distinction from the natural order in sets of numbers we denote it by \( \preceq \).

Remember, that even if we use \( \leq \) as the order relation symbol, it is a SYMBOL for ANY order relation and not only a symbol for a natural order \( \leq \) in sets of numbers.
POSETS

A set \( A \neq \emptyset \) ordered by an order relation \( R \) is called a poset. We write it as a tuple (depending on symbols used) \((A, R), (A, \leq), (A, \preceq)\). Name poset stands historically for "partially ordered set".

Diagram of order relation is a graphical representation of a poset. It is a simplified graph constructed as follows.

1. As the order relation is reflexive, i.e. \((a, a) \in R\) for all \( a \in A \), we draw a point with symbol \( a \) instead of a point with symbol \( a \) and the loop.
2. As the order relation is antisymmetric we draw a point \( b \) above point \( a \) (connected, but without the arrow) to indicate that \((a, b) \in R\).
3. As the order relation is transitive, we connect points \( a, b, c \) without arrows.
Posets Special Elements

Special elements in a poset \((A, \leq)\) are: maximal, minimal, greatest (largest) and smallest (least) and are defined below.

Smallest (least) \(a_0 \in A\) is a smallest (least) element in the poset \((A, \leq)\) iff \(\forall a \in A (a_0 \leq a)\)

Greatest (largest) \(a_0 \in A\) is a greatest (largest) element in the poset \((A, \leq)\) iff \(\forall a \in A (a \leq a_0)\)
Maximal (formal) \( a_0 \in A \) is a maximal element in the poset \((A, \leq)\) iff \( \neg \exists a \in A (a_0 \leq a \land a_0 \neq a) \)

Maximal (informal) \( a_0 \in A \) is a maximal element in the poset \((A, \leq)\) iff on a diagram of \((A, \leq)\) there is no element placed above \( a_0 \)

Minimal (formal) \( a_0 \in A \) is a minimal element in the poset \((A, \leq)\) iff \( \neg \exists a \in A (a \leq a_0 \land a_0 \neq a) \)

Minimal (informal) \( a_0 \in A \) is a minimal element in the poset \((A, \leq)\) iff on the diagram of \((A, \leq)\) there is no element placed below \( a_0 \)
Some Properties of Posets

Use Mathematical Induction to prove the following property of finite posets

**Property 1** Every non-empty finite poset has at least one maximal element

**Proof**

Let \((A, \leq)\) be a finite, not empty poset (partially ordered set by \(\leq\), such that \(A\) has \(n\)-elements, i.e. \(|A| = n\)

We carry the Mathematical Induction over \(n \in \mathbb{N} - \{0\}\)

**Reminder:** an element \(a_o \in A\) ia a maximal element in a poset \((A, \leq)\) iff the following is true.

\[\neg \exists_{a \in A}(a_o \neq a \land a \leq a_0)\]
Inductive Proof

**Base case:** \( n = 1 \), so \( A = \{a\} \) and \( a \) is maximal (and minimal, and smallest, and largest) in the poset \((\{a\}, \leq)\)

**Inductive step:** Assume that any set \( A \) such that \(|A| = n\) has a maximal element;
Denote by \( a_0 \) the maximal element in \((A, \leq)\)
Let \( B \) be a set with \( n + 1 \) elements; i.e. we can write \( B \) as \( B = A \cup \{b_0\} \) for \( b_0 \notin A \), for some \( A \) with \( n \) elements
Inductive Proof

By **Inductive Assumption** the poset \((A, \leq)\) has a maximal element \(a_0\).

To show that \((B, \leq)\) has a maximal element we need to consider 3 cases.

1. \(b_0 \leq a_0\); in this case \(a_0\) is also a maximal element in \((B, \leq)\).
2. \(a_0 \leq b_0\); in this case \(b_0\) is a new maximal in \((B, \leq)\).
3. \(a_0, b_0\) are not compatible; in this case \(a_0\) remains maximal in \((B, \leq)\).

By Mathematical Induction we have proved that
\[\forall n \in \mathbb{N} - \{0\}(|A| = n \Rightarrow A \text{ has a maximal element})\]
Some Properties of Posets

We just proved

**Property 1** Every non-empty finite poset has at least one maximal element

Show that the **Property 1** is not true for an infinite set

**Solution:** Consider a poset \((\mathbb{Z}, \leq)\), where \(\mathbb{Z}\) is the set on integers and \(\leq\) is a natural order on \(\mathbb{Z}\). Obviously no maximal element!

**Exercise:** Prove

**Property 2** Every non-empty finite poset has at least one minimal element

Show that the **Property 2** is not true for an infinite set