EQUIVALENcy of AUTOMATA

\[ M_1 \cong M_2 \iff L(M_1) = L(M_2) \]

THEOREM (We use book Def. (P69))

For any nondeterministic automata there is an equivalent deterministic finite automata.

FACT

Any deterministic automata is a a non-deterministic

PROOF: \( \delta: K \times \Sigma \rightarrow K \), \( \delta \in K \times \Sigma \rightarrow K \). Function is a relation.

FACT says: CLASS OF DFA is a subset of the class of NDFSA

Thus says: \( L(\text{NDFSA}) \supseteq L(\text{DFA}) \) i.e

\( L(\text{DFA}) = L(\text{NFA}) \)
Theorem

For any non-deterministic

\[ M = (K, \Sigma, \delta, s, F) \]

there is (we give an algorithm for its construction)

a deterministic

\[ M' = (K', \Sigma, \delta' = \delta, s', F') \]

such that

\[ M \cong M' \quad \text{i.e.} \quad L(M) = L(M') \]
NON-det $\rightarrow$ deterministic

- $\Delta$ is not a function
- $\Delta$ contains $(q, e, q')$
  i.e. $M$ reads "e".

To transform a non-det $M$ into deterministic $M'$, we have to eliminate

1. $\Delta$
2. $(q, e, q')$

Let's look at 1 FIRST

**Example:**

$M = (K, \Sigma, \Delta, s_1, F)$

$S = \{q_0, q_1, q_2\}$

$\Delta = \{(q_0, a, q_1), (q_1, b, q_2), (q_2, a, q_2)\}$

$\Delta$ NOT a function
NON-function part (NON-DFT)

HOW TO CHANGE IT TO DETERMINISTIC FUNCTION?

IDEAL: states are SETS

\[
\begin{align*}
\delta \left( \{q_2, q_3\}, b \right) &= \{q_2, q_3\} \\
\delta \left( \{q_2\}, a \right) &= \{q_2\} \\
\delta \left( \{q_3\}, a \right) &= \{q_3\}
\end{align*}
\]
We construct $M' = (K', \Sigma, s', \delta, f)$.

**Idea One**

- $K' = 2^K$
- $s'$ initial
- $s' = \{9, 15\}$
- $20$ - initial
  - In this case only

$F'$ final

$F' = \{Q \in K : Q \cap F \neq \emptyset\}$

Here: $\{\text{I25}\}$, $\{\text{20, 22}\}$

\[
\delta : 2^K \times \Sigma \rightarrow 2^K
\]

+ Trap States

Not finished
IDEA TWO

eliminate \( e \) transitions

\[ L(M) = a \]

\( a \in L(M) \)

compute \( a \)

\((q_0, e, q_1), (q_1, a, q_2)\)

compute \( b \)

\((q_0, b, q_2), (q_2, e, q_3)\)

\((q_0, b, q_1), (q_1, e, q_3)\)

Pictures

1. \( e \rightarrow o \rightarrow a \rightarrow o \)
   \( q_0 \rightarrow q_1 \rightarrow q_2 \)

2. \( a \rightarrow o \rightarrow e \rightarrow o \)
   \( q_0 \rightarrow q_1 \rightarrow q_2 \)

3. \( b \rightarrow o \rightarrow e \rightarrow o \rightarrow q \)
   \( q_0 \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \)
In all examples we had computations of $a, b$ that went through states without reading any output (e.g., reading).

**Introduce notion $E(q)$**

\[ E(q) = \{ p \in K : \langle q, e \rangle \xrightarrow{\star} \underset{M}{\langle p, e \rangle} \} \]

$E(q)$ is the set of all states of $M$ that are reachable from $q$ without reading any output.
FACT  For any state \( q \)
\[ q \in E(q) \]

became
\[ (2, e) \xrightarrow{m} (2, e) \]
always. Path-computation \((q, e)\) of length \( m \) one. always except one case \( n = 1 \)

**Basic Idea 2** of deterministic \( M' \) construction is that a move \( M' \) imitates moves \( M \) on reading an input symbol at \( E \) imitates a move of \( M \) on \( a \), possibly followed by any number of \( e \)-moves of \( M \). \[ \rightarrow \text{RCE} (G2) \]
\[ E(q) = \{ p \in K^* : (q, e) \overset{*}{\rightarrow}^M (p, e) \} \]

\( E(q) \) is a closure of the set \( \{ q \} \) under the relation

\[ \{ (p, r) : \text{there is a transition} \quad (p, e, r) \in \Delta' \} \]

\( E(q) \) can be computed by the following algorithm:

**Initial set** \( E(q) := \{ q \} \)

**While** there is \( (p, e, r) \in \Delta \)

with \( p \in E(q) \)

and \( r \notin E(q) \)

**Do**

\[ E(q) := E(q) \cup \{ r \} \]

\*Role of \( E(q) \): get rid of transitions that contain \( e \) without changing \( L(M) \)
back to our examples

\[\begin{array}{c}
0 \\
\xrightarrow{e} 1 \\
\xrightarrow{a} 0 \\
0 \\
\end{array}\]

\[E(0_0) = \{0_0, 0_1, 0_2\}\]
\[E(0_1) = \{0_1\}\]
\[E(0_2) = \{0_2, 0_3\}\]

We define \(\delta : 2^K \times \Sigma \rightarrow 2^K\) always

\[\delta(Q, a) = \bigcup_{p \in K} \{E(p) : (0_1, a, p) \in \Delta\} \text{ for some } q \in Q\]

\(S = q_0 = E(0_0)\)

\[\delta(\{0_0, 0_1, 0_2\}) = E(0_2) = \{0_2, 0_3\}\]

\[\delta(\{0_1, 0_2\}, a) = \{0_2, 0_3\}\]

\[\delta(\{0_2\}, a) = \{0_2, 0_3\}\]

\(c(0_0) = p\)

\((0_0, e, p) \in \Delta\) initial

\((0_1, a, p) \in \Delta\) none

\((0_2, a, 0_2) \in \Delta\) only one
Compute

$$\delta(\{q_2\}, a) = \bigcup_{p \in K} \{E(p) : (q_2, a, p) \in E\}$$

\[\{q_2\} \in F'\]

$$\delta(\phi, a) = \phi$$

Picture of $M'$

$$L(M') = a$$

$K = 2$

$$K' = 2, \{q_0, q_2\}$$

\(\text{TRAP states?}\)
\[ K' = \{ \emptyset, \{q_0\}, \{q_0, q_2\}, \{q_0, q_2, 5\}, \{20, 2, 5\} \} \]

\[ \sum = \{a\} \]

\[ \Sigma = E(20) = \{20, 2\} \]

\[ F' = \{ Q \in 2^K : F \cap Q \neq \emptyset \} \]

\[ F' = \{ \{q_2\}, \{q_0, q_2\}, \{q_1, q_2\}, \{20, 2, 5\} \} \]

All final states

Why our picture was correct?

I.e. why all other states then \( \{q_0, 2, 5\} = S \) and \( \{q_2\} \) are trap states?
\[ \delta(Q,a) = \bigcup_{p \in K} \{ E(p) : (q, a, p) \in \Delta \} \text{ for } q \in Q \]

Evaluate

\[ \delta(\{q_1, q_2\}, a) = \]

\[ E(q_2) = \{q_2\} \]

Look for \((q_1, a, p) \in \Delta\) such that \((q_1, a_1, q_2)\):

\[ \begin{array}{ccc}
q_0 & \xrightarrow{a} & q_1 \\
\{q_0, q_1\} & \xrightarrow{a} & \{q_2\} \\
 & \xrightarrow{\phi} & \emptyset
\end{array} \]

Exercise: Complete diagram of \(M\)!
Look at our example

We wanted

$$\delta(120, b) = \{q_1, q_2, q_3\}$$

Let's check if our definition

$$\delta^*(Q, b)^{\kappa_0} = \bigcup_{p \in K} \{E(p) : (q, b, p) \in \Delta \}$$

returns

$$Q = \{q_0\} = E(q_0)$$

$$\delta^*(\{q_0\}, b) = \{q_1, q_2, q_3\}$$

Evaluate:

$$E(q_0) = q_0, E(q_1) = q_1, E(q_2) = q_2, E(q_3) = q_3$$

Evaluate:

$$(q_0, b, q_1), (q_0, b, q_2), (q_0, b, q_3) \in \Delta$$

$$\delta^*(\{q_0\}, b) = \text{def} \quad E(q_1) \cup E(q_2) \cup E(q_3)$$

$$= \{q_1, q_2, q_3\}$$

$$= \{q_1, q_2, q_3\}$$

YES!
Given non-deterministic

\[ M = (K, \Sigma, \Delta, s, F) \]

we define deterministic

\[ M' = (K', \Sigma, \delta, s', F') \]

as follows:

1. \( K' = 2^K \)
2. \( s' = E(s) \)
3. \( F' = \{ Q \subseteq K : Q \cap F \neq \emptyset \} \)
4. \( \delta : 2^K \times \Sigma \rightarrow 2^K \), i.e., \( \delta : K' \times \Sigma \rightarrow K' \)

\[ \delta(q, \sigma) = \bigcup_{p \in K} \{ E(p) : \exists z \in Q \text{ s.t. } (2, \sigma, p) \in \Delta \} \]

\[ \delta(q, \sigma) = \bigcup \{ E(p) : p \in K \land (2, \sigma, p) \in \Delta \text{ for some } z \in Q \} \]
Theorem

For any $M = (K, \Sigma, \delta, s, F)$ and $M' = (K', \Sigma, \delta', s', F')$ $M'$ is deterministic and

$L(M) = L(M')$ i.e.

$M \cong M'$

Proof: for any $w \in \Sigma^*$, $p, q \in K$

$(q, w) \xrightarrow{1}^* M (p, e)$ iff

$(E(q), w) \xrightarrow{1}^* M' (p, e)$ where

$p \in \mathcal{P}$
In particular when \( q = s \) or \( p \in F \) we get

\[(s,w) \xrightarrow{M} (p,e)\]

iff \((E(s),w) \xrightarrow{M'} (P,e))\]

where \( p \in P \)

i.e.

\[w \in L(M) \iff w \in L(M')\]

because

\[E(s) = S', \quad P \cap F = \emptyset\]

as \( p \in P \) and \( p \in F \)

Proof by mathematical induction over \( \text{length} \).
Back to our first example $M$ with $S = \{q_0\}$, $F = \{q_2\}$, $K = \{q_0, q_1, q_2\}$.

Let's construct $M'$. Always follow steps 1-3 for any $M'$.

**Step 1:** Evaluate $E(q)$ for all $q \in K$.

- $E(q_0) = \{q_0\}$
- $E(q_1) = \{q_1, q_3\}$ (no e-transitions)
- $E(q_2) = \{q_2\}$

**Step 2:** Find $S'$, $F'$ (final)

- $S' = E(q_0) - \{q_0\}$
- $F' = \{Q \cup S \cup F : Q \cap F' = \emptyset\}$

$S' = \{q_0\}$

$F' = \{\{q_2\}\}$
$b_0$ to example $M$

$S' = E(2_0) = \{2_0\}$
$F' = \{\{q_2, 1\}\}$
$E(2_0) = \{2_0\}$
$E(2_1) = \{2_1\}$
$E(2_2) = \{2_2\}$

Transitions from $2_0$

$\delta(\{2_0\}, a) = E(2_1)$

$\delta(\{2_0\}, b) = \phi$

STAGE ONE

Evaluate

$\delta(\{2_1\}, a)$

$\delta(\{2_1\}, b)$

STAGE TWO

Evaluate

$\delta(\phi, a) = \phi$

$\delta(\phi, b) = \phi$
\[ \delta(Q,6) = \bigcup \{ E(p) : (q_1, 6, p) \in \Delta \land q_6 \in Q \} \]

\[ \delta([q_1], a) = \emptyset \]

\[ \delta([q_2], b) = E(q_2) \cup E(q_0) \]

\[ \delta([q_1], b) = \{ q_2 \} \cup \{ q_0 \} \]

\[ \delta([q_2], b) = \{ q_0, q_2 \} \in F^1 \]

Next Stage 3:
\[ \delta(\emptyset, 6) = \emptyset \]

Evaluate \[ \delta([q_0, q_2], 6) \]
Stage 3
Evaluate
\[ \delta(\{s_0, s_2\}, a) = \{s_2\} \]
\[ \delta(\{s_0, s_2\}, b) = \{s_2\} \]
Stage 4:
\[ \delta(\{s_0, s_1\}, a) = \{s_0, s_1\} \]
\[ \delta(\{s_0, s_1\}, b) = \{s_0\} \]
\[ \delta(\{s_0\}, a) = \{s_0\} \]
\[ \delta(\{s_0\}, b) = \emptyset \]

\[ \delta(Q, \emptyset) = \bigcup \{ E(p) : (Q, \delta(p) \in \Delta \} \]
\[ \forall Q \subseteq \{ s_0, s_1, s_2 \} \]
List:
\( (s_0, a, p) \quad (s_2, a, p) \quad E(20) \)
we have
\( E(20) \cup E(21) \)
\( (s_0, b, p) \quad (s_1, b, p) \quad \emptyset \)
Need to evaluate (stage 4)

\[ \delta ( 320, 2, 3, a ) \]

\[ \delta ( 320, 2, 3, b ) \]