CSE303 PRACTICE FINAL SOLUTIONS

1 YES/NO questions

Circle the correct answer (each question is worth 1pt) Write SHORT justification.

1. A is uncountable iff $\mid A \mid = \mathbf{c}$ (continuum).
   Justify: 2^R, R real numbers, is uncountable and $|2^R| > |R|$

   n

2. $(ab \cup a^*b)^*$ is a regular language
   Justify: this is a regular expression, not a language

   n

3. There are uncountably many languages over $\Sigma = \{a\}$.
   Justify: $|\{a\}^*| = \aleph_0$ and $|2^{\{a\}^*}| = \mathbf{c}$ and any set of cardinality $\mathbf{c}$ is uncountable.

   y

4. $L^* = \{w \in \Sigma^* : \exists_{q \in F}(s, w) \vdash_M^* (q, e)\}$.
   Justify: this is definition of $L(M)$, not L^*

   n

5. $L^* = L^+ - \{e\}$.
   Justify: only when $e \notin L$

   y

6. $L^* = \{w_1 \dots w_n, w_i \in L,\ i = 1, \dots, n\}$.
   Justify: $i = 0, 1, \dots, n\}$

   n

7. $((\phi^* \cap a) \cup (\phi \cup b^*)) \cap \phi^*$ represents a language $L = \{e\}$.
   Justify: $((\{e\} \cap \{a\}) \cup \{b\}^*) \cap \{e\} = \{b\}^* \cap \{e\} = \{e\}$

   y

8. If M is a FA, then $L(M) \neq \phi$.
   Justify: take M with $\Sigma = \phi$

   n

9. $L(M_1) = L(M_2)$ iff M_1 and M_2 are finite automata.
   Justify: take as M_1 any automata such that $L(M_1) \neq \phi$ and M_2 such that $L(M_2) = \phi$

   n

10. A language is regular if and only if $L = L(M)$ and M is a finite automaton
    Justify: Main Theorem

    y

11. If L is regular, there is a PDA M such that $L = L(M)$.
    Justify: FA is a PDA operating on an empty stock

    y

12. $((p, e, \beta), (q, \gamma)) \in \Delta$ means: read nothing, move from p to q
    Justify: and replace β by γ on the top of the stack

    n

13. Every subset of a regular language is a language.
Justify: subset of a set is a set and languages are sets

y

14. Any finite language is CF.
Justify: any finite language is regular and $RL \subset CFL$

y

15. Intersection of any two regular languages is CF language.
Justify: Regular languages are closed under intersection and $RL \subset CFL$

y

16. Union of a regular and a CF language is a CF language.
Justify: $RL \subseteq CFL$ and FCL are closed under union

y

17. If L is regular, there is a CF grammar G, such that $L = L(G)$.
Justify: $RL \subseteq CFL$

y

18. $L = \{a^n b^n c^n : n \geq 0\}$ is CF.
Justify: is not CF, as proved by Pumping Lemma for CF languages

n

19. $L = \{a^n b^n : n \geq 0\}$ is CF.
Justify: $L = L(G)$ for G with $R = \{S \to aSb|e\}$

y

20. Let $\Sigma = \{a\}$, then for any $w \in \Sigma^*, w^R = w$
Justify: $a^R = a$ and $w^R = w$ for $w \in \{a\}^*$

y

21. Let $G = (\{S, (,)\}, \{(,)\}, R, S)$ for $R = \{S \to SS \mid (S)\}$. $L(G)$ is regular.
Justify: $L(G) = \emptyset$ and hence regular

y

22. $L = \{a^n b^m c^n : n, m \in N\}$ is CF.
Justify: construct a gramma with rules: $S \to aSc \mid b \mid B \mid e$, and $B \to b \mid e$

y

23. If L is regular, then there is a CF grammar G, such that $L = L(G)$.
Justify: $RL \subseteq CF$

y

24. Class of context-free languages is closed under intersection.
Justify: $L_1 = \{a^n b^n c^m, n, m \geq 0\}$ is CF, $L_1 = \{a^m b^n c^n, n, m \geq 0\}$ is CF, but $L_1 \cap L_2 = \{a^n b^n c^n, n \geq 0\}$ is not CF

n

25. A CF language is a regular language.
Justify: $L = \{a^n b^n : n \geq 0\}$ is CF and not regular

n

2 PART 2: PROBLEMS

QUESTION 1

Let L_1, L_2 be the following languages over $\Sigma = \{a, b\}$:

$$L_1 = \{w \in \Sigma^\star : \quad \exists_{u \in \Sigma\Sigma}(w = uu^R u)\},$$

$$L_2 = \{w \in \Sigma^\star : \quad ww = www\}.$$

1. List elements of $\Sigma\Sigma$

Solution

Observe that Σ is a language over Σ as $\Sigma \subseteq \Sigma$

$\Sigma\Sigma$ is hence a concatenation of two languages and we evaluate

$$\Sigma\Sigma = \{a, b\} \circ \{a, b\} \ = \{aa, bb, ab, ba\}$$

2. Show that L_1 is a finite set

Solution

We have that

$$L_1 = \{w \in \Sigma^\star : \quad \exists u \in \{aa, bb, ab, ba\}(w = uu^R u)\}$$

By definition of L_1 we evaluate that

$$L_1 = \{aaaaaa, abbaab, baabba, aaaaaa, bbbbbb\}$$

This proves that L_1 is a finite set

3. Give examples of 2 words w over Σ such that $w \notin L_1$.

Solution Obviously, $a, b \notin L_1$ because $a, b \notin \Sigma\Sigma$

4. Show that $L_2 \neq \emptyset$.

Solution $e = eee$, hence $e \in L_2$

QUESTION 2

Let Σ be any alphabet, L_1, L_2 two languages over Σ such that $e \in L_1$ and $e \in L_2$. Show that

$$(L_1 \Sigma^\star L_2)^\star = \Sigma^\star$$

Solution : By definition, $L_1 \subseteq \Sigma^\star, L_2 \subseteq \Sigma^\star$ and $\Sigma^\star \subseteq \Sigma^\star$. Hence

$$(L_1 \Sigma^\star L_2)^\star \subseteq \Sigma^\star.$$

We have to show that also

$$\Sigma^\star \subseteq (L_1 \Sigma^\star L_2)^\star.$$

Let $w \in \Sigma^\star$. We have that also $w \in (L_1 \Sigma^\star L_2)^\star$ because $w = ewe$ and $e \in L_1$ and $e \in L_2$.

QUESTION 3

Use book or lecture definition (specify which are you using) to construct a non-deterministic finite automaton M, such that

$$L(M) = (ab)^*(ba)^*.$$

Draw a state diagram. Justify your construction by listing some strings accepted by the state diagram.

Solution 1 We use the lecture definition.

Components of M are: $\Sigma = \{a, b\}$, $K = \{q_0, q_1\}$, $s = q_0$, $F = \{q_0, q_1\}$.
We define Δ as follows.
$\Delta = \{(q_0, ab, q_0), (q_0, e, q_1), (q_1, ba, q_1)\}$.

Strings accepted : $ab, abab, abba, ababba, ababbaba, ...$

Solution 2 We use the book definition.

Components of M are: $\Sigma = \{a, b\}$, $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$, $F = \{q_2\}$.
We define Δ as follows.
$\Delta = \{(q_0, a, q_1), (q_1, b, q_0), (q_0, e, q_2), (q_2, b, q_3), (q_3, a, q_2)\}$.

Strings accepted : $ab, abab, abba, ababba, ababbaba, ...$

QUESTION 4 Construct a PDA M, such that

$$L(M) = \{b^n a^{2n} : n \geq 0\}.$$

Solution

$M = (K, \Sigma, \Gamma, \Delta, s, F)$ for

$$K = \{s, f\}, \Sigma = \{a, b\}, \Gamma = \{a\}, s, F = \{f\},$$

$$\Delta = \{((s, b, e), (s, aa)), ((s, e, e), (f, e)), ((f, a, a), (f, e))\}$$

Explain the construction. Write motivation.

Solution M operates as follows: Δ pushes aa on the top of the stock while M is reading b, switches to f (final state) non-deterministically; and pops a while reading a (all in final state). M puts on the stock two a's for each b, and then remove all a's from the stock comparing them with a's in the word while in the final state.

Trace a transitions of M that leads to the acceptance of the string $bbaaaa$.

Solution The accepting computation is:

$$(s, bbaaaa, e) \vdash_M (s, baaaa, aa) \vdash_M (s, aaaa, aaaa) \vdash_M (f, aaaa, aaaa)$$
$$\vdash_M (f, aaa, aaa) \vdash_M (f, aa, aa) \vdash_M (f, a, a) \vdash_M (f, e, e)$$

Solution 2 $M = (K, \Sigma, \Gamma, \Delta, s, F)$ for

$$K = \{s, f\}, \Sigma = \{a, b\}, \Gamma = \{b\}, s, F = \{f\},$$

$$\Delta = \{((s, b, e), (s, b)), ((s, e, e), (f, e)), ((f, aa, b), (f, e))\}$$

QUESTION 5 Given a **Regular grammar** $G = (V, \Sigma, R, S)$, where

$$V = \{a, b, S, A\}, \quad \Sigma = \{a, b\},$$

$$R = \{S \to aS \mid A \mid e, \quad A \to abA \mid a \mid b\}.$$

Part 1

Use the construction in the proof of **L-GTheorem:**

Language L is regular if and only if there exists a regular grammar G such that L = L(G)

to construct a **finite automaton** M, such that $L(G) = L(M)$.

Draw a **diagram** of M

Solution We construct a non-deterministic finite automata

$$M = (K, \Sigma, \Delta, s, F)$$

as follows:

$$K = (V - \Sigma) \cup \{f\}, \ \Sigma = \Sigma, s = S, \quad F = \{f\},$$
$$\Delta = \{(S, a, S), (S, e, A), (S, e, f), (A, ab, A), (A, a, f), (A, b, f)\}$$

2. Trace a **transition** of M that leads to the acceptance of the string $aaaababa$, and compare with a **derivation** of the same string in G.

Solution

The accepting computation is:

$$(S, aaaababa) \vdash_M (S, aaababa) \vdash_M (S, aababa) \vdash_M (S, ababa) \vdash_M (A, ababa)$$
$$\vdash_M (A, aba) \vdash_M (A, a) \vdash_M (f, e)$$

G derivation is:

$$S \Rightarrow aS \Rightarrow aaS \Rightarrow aaaS \Rightarrow aaaA \Rightarrow aaaabA \Rightarrow aaaababA \Rightarrow aaaababa$$

QUESTION 6

Prove that the Class of context-free languages is NOT closed under intersection

Proof

Assume that the context-free languages are **are closed** under **intersection**

Observe that both languages

$$L_1 = \{a^n b^n c^m : \quad m, n \geq 0\} \quad \text{and} \quad L_2 = \{a^m b^n c^n : \quad m, n \geq 0\}$$

are **context-free**

So the language

$$L_1 \cap L_2$$

must be **context-free**, but

$$L_1 \cap L_2 = \{a^n b^n c^n : \quad n \geq 0\}$$

and we have proved that $L = \{a^n b^n c^n : \quad n \geq 0\}$ is **not** context-free. **Contradiction**

EXTRA CREDIT

Use closure under union for CF languages to show that

$$L = \{a^n b^n : \ n \neq m\}$$

is a CF language

Solution

$L = L_1 \cup L_2$ for $L_1 = \{a^n b^m : \ n > m\}$ and $L_2 = \{a^n b^m : \ n < m\}$ and $L = L_1, L_2$ are both CF