CSE303 PRACTICE Q1 SOLUTIONS

PART 1: Yes/No Questions

1. $\{\emptyset\} \subseteq \{a, b, c\}$
   Justify: $\emptyset \notin \{a, b, c\}$ **n**
2. Set A is countable iff $N \subseteq A$ (N is the set of natural numbers).
   Justify: $A = \{\emptyset\}$ is countable (finite), but N is not a subset of $\{\emptyset\}$, i.e. $N \not\subseteq \{\emptyset\}$.
   In fact A can be ANY finite set, or any infinite set that does not include N, for example $A = \{\{n\} : \ n \in N\}$.
   In this case $|A| = |N|$, but N is not a subset of A, i.e. $N \not\subseteq A$. **n**
3. 2^N is infinitely countable.
   Justify: $|2^N| = |R| = \mathcal{C}$ and R are uncountable. **n**
4. Let $A = \{\{n\} \in 2^N : n^2 + 1 \leq 15\}$. A is infinite.
   Justify: $\{n \in N : \ n^2 + 1 \leq 15\} = \{0, 1, 2, 3\}$,
   hence $A = \{\{0\}, \{1\}, \{2\}, \{3\}\}$ is a finite set. **n**
5. Let $\Sigma = \{a\}$. There are countably many languages over Σ.
   Justify: There is any many languages as subsets of $\Sigma^\star$, i.e. uncountably many and exactly as many as real numbers. **n**
6. For any L, $L^+ = L^\star - \{e\}$.
   Justify: Only when $e \notin L$. **n**
7. $L^* = \{w_1 ... w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\}$.
   Justify: $n \geq 0$. **n**
8. For any language L over an alphabet Σ, $L^+ = L \cup L^*$.
   Justify: Take L such that $e \notin L$. We get that $e \in L \cup L^*$ as $e \in L^*$ and $e \notin L^+$. **n**

PART 2: PROBLEMS

QUESTION 1 Let Σ be any alphabet, L_1, L_2 two languages over Σ such that $e \in L_1$ and $e \in L_2$. Show that

$$(L_1 \Sigma^\star L_2)^\star = \Sigma^\star$$

Solution: By definition, $L_1 \subseteq \Sigma^\star, L_2 \subseteq \Sigma^\star$ and $\Sigma^\star \subseteq \Sigma^\star$. Hence

$$(L_1 \Sigma^\star L_2) \subseteq \Sigma^\star.$$

Now we use the following property:

Property For any languages $L_1.L_2$,

if $L_1 \subseteq L_2$, then ${L_1}^* \subseteq {L_2}^*$

and obtain that

$$(L_1\Sigma^\star L_2)^\star \subseteq \Sigma^{\star\star} = \Sigma^\star.$$

We have to show that also

$$\Sigma^\star \subseteq (L_1\Sigma^\star L_2)^\star.$$

Let $w \in \Sigma^\star$ we have that also $w \in (L_1\Sigma^\star L_2)^\star$ because $w = ewe$ and $e \in L_1$ and $e \in L_2$.

QUESTION 2 Let $\mathcal{L}$ be a function that associates with any regular expression α the regular language $\mathcal{L}(\alpha)$.

1. Evaluate $\mathcal{L}(((a \cup b)^\star a))$.

Solution: $\mathcal{L}(((a\cup b)^\star a)) = \mathcal{L}((a\cup b)^\star)\mathcal{L}(a) = (\mathcal{L}(a\cup b))^\star\{a\} = (\mathcal{L}(a)\cup\mathcal{L}(b))^\star\{a\} = (\{a\} \cup \{b\})^\star\{a\} = \{a, b\}^\star\{a\}$.

2. Describe a property that defines the language $\mathcal{L}(((a \cup b)^\star a))$.

Solution $\{a, b\}^\star\{a\} = \Sigma^\star\{a\} = \{w \in \{a, b\}^\star : \quad w \text{ ends with } a\ \}$.

QUESTION 3 Let $\Sigma = \{a, b\}$. Let $L_1 \subseteq \Sigma^\star$ be defined as follows:

$$L_1 = \{w \in \Sigma^\star : \text{ number of } b \text{ in } w \text{ is divisible three}\}$$

Write a regular expression α, such that $\mathcal{L}(\alpha) = L_1$. You can use shorthand notation. Explain shortly your answer.

Solution:

$$\alpha = a^\star(a^*ba^\star ba^\star ba^\star)^\star = a^\star(ba^\star ba^\star ba^\star)^\star.$$

Explanation: the part $a^*ba^\star ba^\star ba^\star$ says that there must be 3 occurrences of b in L_1. The part $(a^*ba^\star ba^\star ba^\star)^\star$ says that we the number of b's is $3n$ for $n \geq 1$.

Observe that 0 is divisible by 3, so we need to add the case of 0 number of b's ($n = 0$), i.e. words $e, a, aa, aaa, , ...$ We do so by concatenating $(a^*ba^\star ba^\star ba^\star)^\star$ with a^*.