Untyped Arithmetic Expressions

Principles of Programming Languages

CSE 526

1. Syntax
2. Operational Semantics
3. Examples

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Formal Description of Programming Languages

- Formal Definition of Syntax
  - Grammars to define the set of *strings* that define a syntactically valid program
  - Inductive definitions of *abstract syntax trees*.
- Formal Definition of Semantics
  - Structural operational semantics

Syntax

Example: A language of untyped arithmetic expressions

$$
\begin{array}{lcl}
t & ::= & \texttt{true} \\
  & | & \texttt{false} \\
  & | & \texttt{if}(t, t, t) \\
  & | & \texttt{0} \\
  & | & \texttt{succ} \;\; t \\
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Inductive Definition: The set $\mathcal{T}$ of *terms* is the smallest set such that:

1. $\{\texttt{true}, \texttt{false}, \texttt{0}\} \subseteq \mathcal{T}$
2. if $t_1 \in \mathcal{T}$ then $\{\texttt{succ}\ t_1, \texttt{pred}\ t_1, \texttt{iszero}\ t_1\} \subseteq \mathcal{T}$.
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Inference Rules: The set $\mathcal{T}$ is defined by the following rules:

$$\texttt{true} \in \mathcal{T} \qquad \texttt{false} \in \mathcal{T} \qquad \texttt{0} \in \mathcal{T}$$

$$\frac{t_1 \in \mathcal{T}}{\texttt{succ}\ t_1 \in \mathcal{T}} \qquad \frac{t_1 \in \mathcal{T}}{\texttt{pred}\ t_1 \in \mathcal{T}} \qquad \frac{t_1 \in \mathcal{T}}{\texttt{iszero}\ t_1 \in \mathcal{T}}$$

$$\frac{t_1, t_2, t_3 \in \mathcal{T}}{\texttt{if}(t_1, t_2, t_3) \in \mathcal{T}}$$

Alternative Definitions of Terms (contd.)

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Constructive Definition: For each natural number i define set S_i as follows:

$$S_0 = \emptyset$$

$$S_{i+1} = \begin{cases} & \{\texttt{true}, \texttt{false}, \texttt{0}\} \\ \cup & \{\texttt{succ}\ t_1, \texttt{pred}\ t_1, \texttt{iszero}\ t_1 \mid t_1 \in S_i\} \\ \cup & \{\texttt{if}(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in S_i\} \end{cases}$$

$$\mathcal{S} = \bigcup_i S_i$$

Alternative Definitions of Terms (contd.)

Properties:

- The sets S_i are cumulative, i.e., $\forall i\ S_i \subseteq S_{i+1}$

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Properties:

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- $\mathcal{T} = \mathcal{S}$
  1. $\mathcal{S}$ satisfies the conditions on $\mathcal{T}$
  2. Let $\mathcal{S}'$ be a set that satisfies the conditions on $\mathcal{T}$. Then $\mathcal{S} \subseteq \mathcal{S}'$.

Equivalence of $\mathcal{S}$ and $\mathcal{T}$

$$S_0 = \emptyset$$
$$S_{i+1} = \begin{cases} & \{\texttt{true}, \texttt{false}, \texttt{0}\} \\ \cup & \{\texttt{succ}\ t_1, \texttt{pred}\ t_1, \texttt{iszero}\ t_1 \mid t_1 \in S_i\} \\ \cup & \{\texttt{if}(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in S_i\} \end{cases}$$

1. $\forall i\ S_i \subseteq S_{i+1}$

Proof is by *ordinary* induction on i: $P(0)$ and $\forall k.P(k) \implies P(k+1)$, where

$P(i) : S_i \subseteq S_{i+1}$

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$P(k) \implies P(k+1)$: We'll show that every $t \in S_{k+1}$ is also $\in S_{k+2}$.

Consider $t \in S_{k+1}$. Then t is of one of the following forms:

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1. $t \in \{\texttt{true}, \texttt{false}, \texttt{0}\}$. Then $t \in S_{k+2}$ by definition.
2. $t = \texttt{succ}(t_1)$ for some $t_1 \in S_k$. By ind. hyp., $t_1 \in S_{k+1}$ and hence $t \in S_{k+2}$.

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3–5. proof steps for terms of the form $\texttt{pred}(t_1)$ etc. are similar to case 2.

Equivalence of $\mathcal{S}$ and $\mathcal{T}$

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1. $\{\texttt{true}, \texttt{false}, \texttt{0}\} \subseteq \mathcal{T}$ 2. if $t_1 \in \mathcal{T}$ then $\{\texttt{succ}\ t_1, \texttt{pred}\ t_1, \texttt{iszero}\ t_1\} \subseteq \mathcal{T}$. 3. if $t_1, t_2, t_3 \in \mathcal{T}$ then $\texttt{if}(t_1, t_2, t_3) \in \mathcal{T}$.	$S_0 = \emptyset$ $S_{i+1} = \begin{cases} \{\texttt{true}, \texttt{false}, \texttt{0}\} \\ \cup\ \{\texttt{succ}\ t_1, \texttt{pred}\ t_1, \texttt{iszero}\ t_1 \mid t_1 \in S_i\} \\ \cup\ \{\texttt{if}(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in S_i\} \end{cases}$ $\mathcal{S} = \bigcup_{i \geq 0} S_i$

2a. $\mathcal{S}$ satisfies the conditions 1, 2, and 3 on $\mathcal{T}$

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3. If $t_1, t_2, t_3 \in \mathcal{S}$ then there are k_1, k_2, k_3 such that $t_1 \in S_{k_1}$, $t_2 \in S_{k_2}$, and $t_3 \in S_{k_3}$. From Lemma 1, t_1, t_2, t_3 are all $\in S_k$ for $k \geq \max(k_1, k_2, k_3)$. Hence `if`$(t_1, t_2, t_3) \in S_{k+1}$ and consequently $\in \mathcal{S}$.

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2b. If $\mathcal{S}'$ satisfies the conditions 1, 2, and 3 on $\mathcal{T}$, then $\mathcal{S} \subseteq \mathcal{S}'$

We will show this by proving that S_i is a subset of $\mathcal{S}'$ by *complete* induction: $(\forall j < i\ P(j)) \implies P(i)$.

We get two cases from the definition of S_i:

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Equivalence of $\mathcal{S}$ and $\mathcal{T}$ (Contd.)

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3–5. Proof steps are similar to case 2 for $t = \texttt{pred}(t_1)$ etc.

Inductive Definitions

The following recursive definition is "well-defined" since the function on a term is defined based on that on *smaller* terms.

$$
\begin{array}{rcl}
Const(\texttt{true}) & = & \{\texttt{true}\} \\
Const(\texttt{false}) & = & \{\texttt{false}\} \\
Const(\texttt{0}) & = & \{\texttt{0}\} \\
Const(\texttt{succ}\ t_1) & = & Const(t_1) \\
Const(\texttt{pred}\ t_1) & = & Const(t_1) \\
Const(\texttt{iszero}\ t_1) & = & Const(t_1) \\
Const(\texttt{if}(t_1, t_2, t_3)) & = & Const(t_1) \cup Const(t_2) \cup Const(t_3)
\end{array}
$$

Inductive Definitions (contd.)

The size of a term is also defined inductively:

$$\begin{aligned}
\mathit{size}(\texttt{true}) &= 1\\
\mathit{size}(\texttt{false}) &= 1\\
\mathit{size}(\texttt{0}) &= 1\\
\mathit{size}(\texttt{succ}\ t_1) &= \mathit{size}(t_1) + 1\\
\mathit{size}(\texttt{pred}\ t_1) &= \mathit{size}(t_1) + 1\\
\mathit{size}(\texttt{iszero}\ t_1) &= \mathit{size}(t_1) + 1\\
\mathit{size}(\texttt{if}(t_1, t_2, t_3)) &= \mathit{size}(t_1) + \mathit{size}(t_2) + \mathit{size}(t_3) + 1
\end{aligned}$$

The *depth* of terms can be defined similarly.

Induction on terms

- **Structural Induction:**

  If, for each term s,
  given $P(r)$ for all immediate sub-terms r of s
  we can show $P(s)$
  then $P(s)$ holds for all s.

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  we can show $P(s)$

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- **Induction on size:**

  If, for each term s,

  given $P(r)$ for all terms r such that $size(r) < size(s)$

  we can show $P(s)$

  then $P(s)$ holds for all s.

Operational Semantics

Example: A language of untyped boolean expressions $\mathcal{B}$:

$$t \quad \texttt{::=} \quad \texttt{true} \mid \texttt{false} \mid \texttt{if}(t, t, t) \qquad \text{Terms}$$

Operational Semantics

Example: A language of untyped boolean expressions $\mathcal{B}$:

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$$\frac{t_1 \to t_1'}{\texttt{if}(t_1, t_2, t_3) \to \texttt{if}(t_1', t_2, t_3)} \qquad \text{E-If}$$

The Inference Rule Notation

$$\frac{\textit{Premises}}{\textit{Conclusion}} \quad \textsc{Name}$$

- Inference rules without premises are called *axioms*.
- Inference rules (more precisely *rule schema*) may have meta-variables. E.g., t_1, t_2, t_3, t_1' in:

$$\frac{t_1 \rightarrow t_1'}{\texttt{if}(t_1, t_2, t_3) \rightarrow \texttt{if}(t_1', t_2, t_3)} \quad \textsc{E-If}$$

A *rule instance* is obtained by consistently replacing each meta-variable by the same term in the premises as well as the conclusion.

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- **Determinacy:** If $t \rightarrow t'$ and $t \rightarrow t''$ then $t' = t''$.

Example Evaluations

$\texttt{if}(\texttt{if}(\texttt{true}, \texttt{false}, \texttt{true}), \texttt{true}, \texttt{false})$

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Step 1:

$$\dfrac{\dfrac{}{\texttt{if}(\texttt{true},\texttt{false},\texttt{true}) \to \texttt{false}}\ \textsc{E-IfTrue}}{\texttt{if}(\texttt{if}(\texttt{true},\texttt{false},\texttt{true}),\texttt{true},\texttt{false}) \to \texttt{if}(\texttt{false},\texttt{true},\texttt{false})}\ \textsc{E-If}$$

$$\texttt{if}(\texttt{if}(\texttt{true},\texttt{false},\texttt{true}),\texttt{true},\texttt{false})$$
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Step 2:

$$\dfrac{}{\texttt{if}(\texttt{false},\texttt{true},\texttt{false}) \to \texttt{false}}\ \text{E-IfFalse}$$

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Small-Step Semantics

Properties of (previously defined) operational semantics

- **Determinacy:** If $t \to t'$ and $t \to t''$ then $t' = t''$.

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- Proof: by induction on the derivation of $t \rightarrow t'$.
- This proof is also identical to induction on structure of t
- The operational semantics defined previously is said to be "*Structural Operational Semantics* (SOS)", where the evaluation derivation follows the structure of the term being reduced.

Proof of Determinacy

$\texttt{if}(\texttt{true}, t_2, t_3) \to t_2$	E-IfTrue	$\dfrac{t_1 \to t_1'}{\texttt{if}(t_1, t_2, t_3) \to \texttt{if}(t_1', t_2, t_3)}$	E-If
$\texttt{if}(\texttt{false}, t_2, t_3) \to t_3$	E-IfFalse		

If $t \to t'$ and $t \to t''$ then $t' = t''$.

Consider the last rule used in the derivation of $t \to t'$.

- E-IfTrue: Then $t = \texttt{if}(\texttt{true}, t_2, t_3)$ for some terms t_2 and t_3, and $t' = t_2$. Consider the derivation $t \to t''$. The last rule used here cannot be E-IfFalse (does not match) or E-If (premise does not hold). Hence the last rule used in $t \to t''$ must be E-IfTrue, and $t'' = t_2 = t'$.

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- E-IfFalse: Similar to above case.
- E-If: Then $t = \texttt{if}(t_1, t_2, t_3)$, $t_1 \to t_1'$ (premise), and $t' = \texttt{if}(t_1', t_2, t_3)$. The last rule used in derivation of $t \to t''$ must also be E-If, with premise $t_1 \to t_1''$, resulting in $t'' = \texttt{if}(t_1'', t_2, t_3)$. By induction hypotheses, we know $t_1' = t_1''$. Consequently, $t' = t''$.

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  If $t \to^* u$, and u is in normal form, we say u is the normal form of t.
- **Termination:** For every term t, there is some normal form t' such that $t \to^* t'$.

Untyped Arithmetic Expressions

$$
\begin{array}{llll}
t & ::= & \texttt{true} & \text{Terms} \\
 & | & \texttt{false} & \\
 & | & \texttt{if}(t, t, t) & \\
 & | & \texttt{0} & \\
 & | & \texttt{succ}\ \ t & \\
 & | & \texttt{pred}\ \ t & \\
 & | & \texttt{iszero}\ \ t & \\
v & ::= & \texttt{true} \mid \texttt{false} \mid nv & \text{Values} \\
nv & ::= & \texttt{0} \mid \texttt{succ}\ nv & \text{Numeric Values}
\end{array}
$$

Operational Semantics of Untyped Arithmetic Expressions

$$\frac{t_1 \to t_1'}{\texttt{succ}\ t_1 \to \texttt{succ}\ t_1'} \quad \text{E-Succ}$$

$$\texttt{pred 0} \to \texttt{0} \quad \text{E-PredZero}$$

$$\texttt{pred succ}\ nv_1 \to nv_1 \quad \text{E-PredSucc}$$

$$\frac{t_1 \to t_1'}{\texttt{pred}\ t_1 \to \texttt{pred}\ t_1'} \quad \text{E-Pred}$$

$$\texttt{iszero 0} \to \texttt{true} \quad \text{E-IsZeroZero}$$

$$\texttt{iszero succ}\ nv_1 \to \texttt{false} \quad \text{E-IsZeroSucc}$$

$$\frac{t_1 \to t_1'}{\texttt{iszero}\ t_1 \to \texttt{iszero}\ t_1'} \quad \text{E-IsZero}$$

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Stuck terms correspond to “run-time errors”.

Reflexive Transitive Closure

Let $R \subseteq D \times D$ be a binary relation. The reflexive transitive closure R^* of R is the smallest relation such that

- $\forall d \in D \quad (d, d) \in R^*$
- $R \subseteq R^*$
- $\forall x, y, z \in D \quad (x, y) \in R^* \wedge (y, z) \in R^* \Rightarrow (x, z) \in R^*$

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Big-Step Semantics

Small-Step Semantics for **B**:

$$\texttt{if}(\texttt{true}, t_2, t_3) \to t_2 \qquad \text{E-IfTrue}$$

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$\forall t\ \exists v.\ t \to^* v$ (Uniqueness of N.F. & Termination)

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Soundness of Big-Step Semantics

If $\boldsymbol{t} \Downarrow \boldsymbol{v}$ then $\boldsymbol{t} \longrightarrow^* \boldsymbol{v}$

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Stated as a Lemma and proved separately

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3. B-IFFALSE (similar to the above case)

Soundness of Big-Step Semantics (contd.)

Lemma needed for soundness proof:

If $t_1 \rightarrow^* t_1'$ then $\texttt{if}(t_1, t_2, t_3) \rightarrow^* \texttt{if}(t_1', t_2, t_3)$

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Proof: by induction on the length of evaluation sequence $t_1 \to^* t_1'$.

Soundness of Big-Step Semantics (contd.)

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- $t_1 \rightarrow \hat{t_1} \rightarrow^* t_1'$: Then t_1 is not a value (by defn of small-step semantics)

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  - By induction hypothesis, $\hat{t_1} \rightarrow^* t_1'$ means $\texttt{if}(\hat{t_1}, t_2, t_3) \rightarrow^* \texttt{if}(t_1', t_2, t_3)$

Soundness of Big-Step Semantics (contd.)

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  - Hence $\texttt{if}(t_1, t_2, t_3) \rightarrow \texttt{if}(\hat{t_1}, t_2, t_3) \rightarrow^* \texttt{if}(t_1', t_2, t_3)$

Completeness of Big-Step Semantics

If $t \longrightarrow^* v$ then $t \Downarrow v$

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Proof: by induction on length of evaluation of $t \rightarrow^* v$

1. $t = v$: $v \Downarrow v$ by B-VALUE.
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   Use the following lemma:

   If $\texttt{if}(t_1, t_2, t_3) \rightarrow^* v$ then

   $t_1 \rightarrow^* \texttt{true}$ and $t_2 \rightarrow^* v$, or

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   If $t_1 \rightarrow^* \texttt{true}$, then by induction hypothesis, $t_1 \Downarrow \texttt{true}$ and $t_2 \Downarrow v$

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   If $t_1 \rightarrow^* \texttt{true}$, then by induction hypothesis, $t_1 \Downarrow \texttt{true}$ and $t_2 \Downarrow v$
   Hence by applying B-IFTRUE, we get $\texttt{if}(t_1, t_2, t_3) \Downarrow v$.

Completeness of Big-Step Semantics

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Proof: by induction on length of evaluation of $t \rightarrow^* v$

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   If $t_1 \rightarrow^* \texttt{true}$, then by induction hypothesis, $t_1 \Downarrow \texttt{true}$ and $t_2 \Downarrow v$

   Hence by applying B-IFTRUE, we get $\texttt{if}(t_1, t_2, t_3) \Downarrow v$.

   Proof if $t_1 \rightarrow^* \texttt{false}$ is similar.