Prolog

Principles of Programming Languages

CSE 526

Compiled at 08:55 on 2019/02/18

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  - Memoization: Tamaki & Sato (Tokyo); Warren et al (Stony Brook)

Prolog Systems

- SWI Prolog (`www.swi-prolog.org`)
  - Can be obtained for free and installed on Windows, Linux, Mac.
  - Has a good development environment (command completion, help, graphical debugger, etc.)
  - On compute* (Unix) servers: `~cram/bin/swipl`
- XSB Prolog (`xsb.sourceforge.net`)
  - Can be obtained for free and installed on Windows, Linux, Mac.
  - Supports a powerful extension (memoization) to Prolog
  - Command-line interface (e.g. no graphical debugger)
  - On compute* (Unix) servers: `~cram/bin/xsb`

Using Prolog Systems

- Prolog programs are in files with "`.pl`" extension ("`.P`" for XSB)
- Prolog systems typically support an interactive mode.
- "`[filename].`" to compile and load a program in `filename.pl` (`filename.P` in XSB).
- "`halt.`" to exit the system.

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- Example:

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inc(X,Y) :- Y is X+1.
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  - `X` and `Y` are *variables*.
  - `inc` is a predicate.
  - The predicate is defined using a single rule.

Logic Programs (contd.)

```
inc(X,Y) :- Y is X+1.
```

- "`:-`" separates the *body* of the rule from its head.
- "`X`" and "`Y`" are also "parameters" of the predicate.

  In this case, `X` is the input parameter, and `Y` is the return parameter (where the return values are stored).
- "`Y is X+1`" defines `Y` in terms of `X`.
- The period ("`.`") marks the end of a rule.
- The predicate is *called* by giving values to its parameters. e.g.

  `inc(6, B)` returns with B=7.

  `inc(11, B)` returns with B=12.

Syntax of Prolog

- *Variables* are identifiers that begin with an upper case letter or underscore.
    - An underscore, by itself, represents an *anonymous variable*.
- *Predicate* names (and later, data structure symbols) are identifiers that begin with a lower case letter.
- All variables are *local* to the clause in which they occur.
- Different occurrences of the same variable in a clause denote the same data.
- Variables need not be declared, and have no type.

How Prolog Works (An Example)

```
big(bear).
big(elephant).

brown(bear).

black(cat).

small(cat).

gray(elephant).

dark(Z) :- black(Z).
dark(Z) :- brown(Z).

dangerous(X) :- dark(X), big(X).
```

Derivations

```
big(bear).          brown(bear).          dark(Z) :- black(Z).
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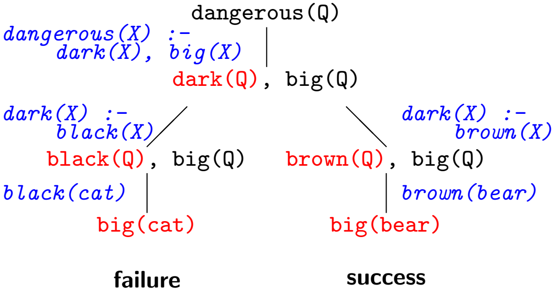

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  - Under that substitution for variables, prove $B_1, B_2, \ldots, B_k, G_2, \ldots, G_n$.
  - If nothing is left to prove then the proof is complete. If there are no more clauses to match, the proof attempt fails.

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5. Now select `brown(bear)` and prove `big(bear)`.
6. Select the fact `big(bear)`.

   There is nothing left to prove, so the proof is complete

Data Representation in Prolog

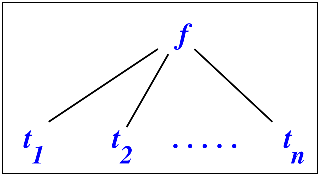
- Prolog has no notion of data types
- All data is represented as *terms*, which can be:
  - Variables
  - Non-variable Terms
    - Atomic data (Integers, floating point numbers, constants, ...)
    - Compound Terms (Structures)

Atomic Data

- **Numeric constants:** Integers, floating point numbers (e.g. `1024`, `-42`, `3.1415`, `6.023e23` ...)
- **Atoms:**
  - Strings of characters enclosed in single quotes (e.g. `'cram'`, `'Stony Brook'`)
  - Identifiers: sequence of letters, digits, underscore, beginning with a letter (e.g. `cram`, `r2d2`, `x_24`).

Data Structures

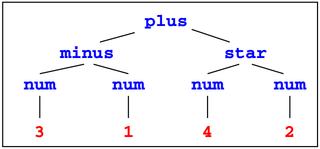
- If f is an identifier and $t_1, t_2, \ldots t_n$ are terms, then $f(t_1, t_2, \ldots t_n)$ is a term.



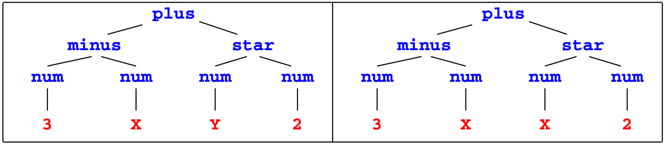
- In the above, f is called a *function symbol* (or *functor*) and t_i is an *argument*.
- Structures are used to group related data items together (in some ways similar to `struct` in C and objects in Java).
- Structures are used to construct trees (and, as a special case, lists).

Trees

- Example: expression trees:
`plus(minus(num(3), num(1)), star(num(4), num(2)))`



- Data structures may have variables. And the same variable may occur multiple times in a data structure.



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(We'll extend this to *unification* later)

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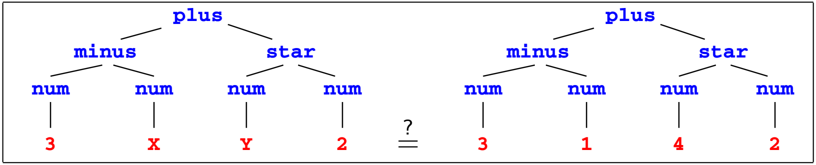
- 

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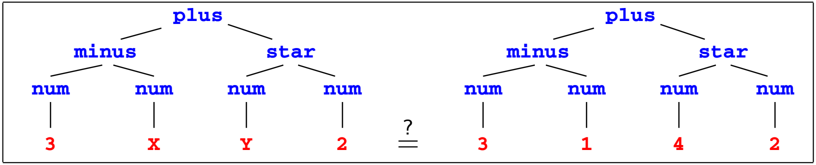
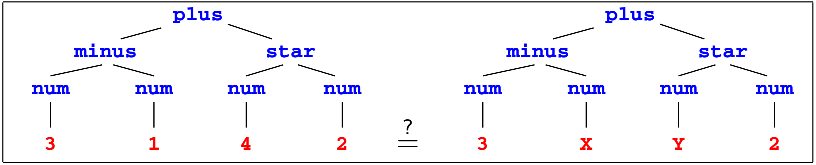
- 

  Yes, with $X = 1$, $Y = 4$.

Matching (contd.)



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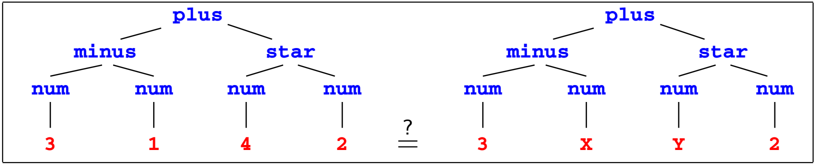

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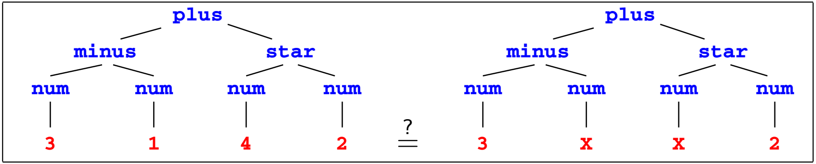

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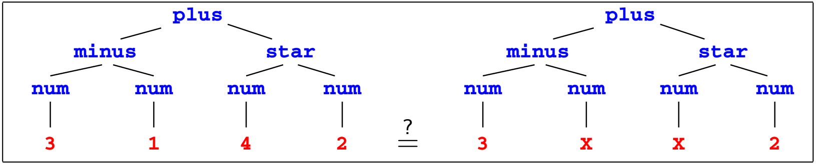


No! X cannot be 1 and 4 at the same time

Accessing arguments of a structure

- Matching is the common way to access a structure's arguments.
- Let `date('Sep', 1, 2005)` be a structure used to represent dates, with the month, day and year as the three arguments (in that order).
- Then `date(M, D, Y) = date('Sep', 1, 2005)` makes $M =$ `'Sep'`, $D = 1$, $Y = 2005$.
- If we want to get only the day, we can write `date(_, D, _) = date('Sep', 1, 2005)`. Then we get $D = 1$.

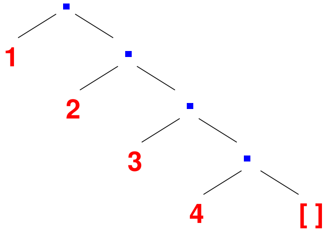
Lists

Prolog uses a special syntax to represent and manipulate lists.

- `[1,2,3,4]`: represents a list with 1, 2, 3 and 4, respectively.
- This can also be written as `[1 | [2,3,4]]`: a list with 1 as the head (its first element) and `[2,3,4]` as its tail (the list of remaining elements).
- If `X = 1` and `Y = [2,3,4]` then `[X|Y]` is same as `[1,2,3,4]`.
- The empty list is represented by `[ ]`.
- The symbol "|" (called *cons*) and is used to separate the beginning elements of a list from its tail.
  For example: `[1,2,3,4]` = `[1 | [2,3,4]]`
  = `[1 | [2 | [3,4]]]`
  = `[1,2 | [3,4]]`

Lists (contd.)

- Lists are special cases of trees.
  For instance, the list `[1,2,3,4]` is represented by the following structure:



- The function symbol `./2` is the list constructor.
  `[1,2,3,4]` is same as `.(1, .(2, .(3, .(4, []))))`

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Example queries:

```
member(s, [l,i,s,t])
member(X, [l,i,s,t])
member(f(X), [f(1), g(2), f(3), h(4), f(5)])
```

Programming with Lists — II

`append/3`: concatenate two lists to form the third list.

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The program:

```
append([], L, L).
append([X|Xs], Ys, [X|Zs]) :- append(Xs, Ys, Zs).
```

Programming with Lists — II

append/3: concatenate two lists to form the third list.

The program:

```
append([], L, L).
append([X|Xs], Ys, [X|Zs]) :- append(Xs, Ys, Zs).
```

Example queries:

```
append([f,i,r], [s,t], L)
append(X, Y, [s,e,c,o,n,d])
append(X, [t,h], [f,o,u,r,t,h])
```

Programming with Lists — III

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Define a predicate, `len/2` that finds the length of a list (first argument).

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len([_|Xs], N+1) :- len(Xs, N).
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Example queries:

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len([], X)
len([l,i,s,t], 4)
len([l,i,s,t], X)
```

Arithmetic

```
| ?- 1+2 = 3.
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  - General form: R `is` E where E is an expression to be evaluated and R is matched with the expression's value.

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  - General form: R `is` E where E is an expression to be evaluated and R is matched with the expression's value.
  - `Y is X + 1` will give an error if `X` does not (yet) have a value.

The list length example revisited

Define a predicate, `length/2` that finds the length of a list (first argument).

The program:

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length([], 0).
length([_|Xs], M) :- length(Xs, N), M is N+1.
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Example queries:

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length([], X)
length([l,i,s,t], 4)
length([l,i,s,t], X)
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Conditional Evaluation

Consider the computation of $n!$, i.e. the factorial of n.

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  - $N <= 0$: $F = 1$
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-
```
factorial(N, F) :-
    (N > 0
        -> N1 is N-1, factorial(N1, F1), F is N*F1
        ;  F = 1
    ).
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More Prolog Syntax

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- If more than one action needs to be performed in a rule, they are written one after another, separated by a comma.
- Arithmetic expressions are not directly used as arguments when calling a predicate; they are first evaluated, and then passed to the called predicate.

Arithmetic Operators

- Integer/Floating Point operators: `+`, `-`, `*`, `/`
- Integer operators: `mod`, `//` (div)
- Int $\leftrightarrow$ Float operators: `floor`, `ceiling`
- Comparison operators: `<`, `>`, `=<`, `>=`, `=:=`, `=\=`

Sequences, revisited

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- `append(X, [t,h], [f,o,u,r,t,h])`

Mystery Program

```
m(X, X).
m(X1, X5) :- a(X1, X2), m(X2, X3), b(X3, X4), m(X4, X5).

a([0|Y], Y).
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?- m([0,0,1,1,1,0], L).
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Definite Clause Grammars

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`L=[0,1,0,0,1,1], ...`

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`yes`

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L=[]
L=[0,1]
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        ⋮
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Definite Clause Grammars (Magic?)

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Translated to:

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r([], X, X).
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```

- A way to reverse a list in polynomial time!

Unification

- Operation done to “match” the goal atom with the head of a clause in the program.
- Forms the basis for the *matching* operation we used for Prolog evaluation.
  - `f(a,Y)` and `f(X,b)` unify when `X=a` and `Y=b`.
  - `f(a,X)` and `f(X,b)` do not unify.
  - `X` and `f(X)` do not unify
    (but they “match” in Prolog!)

Substitutions

A substitution is a mapping between variables and values (terms).

- Denoted by $\{X_1 \mapsto t_1, X_2 \mapsto t_2, \ldots, X_n \mapsto t_n\}$ such that
  - $X_i \neq t_i$, and
  - X_i and X_j are distinct variables when $i \neq j$.
- Empty subsititution is denoted by ϵ.
- A substition is said to be a **renaming** if it is of the form $\{X_1 \mapsto Y_1, \ldots, X_n \mapsto Y_n\}$ and $Y_1, \ldots, Y_n$ is a permutation of $X_1, \ldots, X_n$.
- Example: $\{X \mapsto Y, Y \mapsto X\}$ is a renaming substitution.

Substitutions and Terms

- Application of a substitution:
  - $X\theta = t$ if $X \mapsto t \in \theta$.
  - $X\theta = X$ if $X \mapsto t \notin \theta$ for any term t.
- Application of a substitution $\{X_1 \mapsto t_1, \ldots, X_n \mapsto t_n\}$ to a *term* s:
  - is a term obtained by *simultaneously* replacing *every* occurrence of X_i in s by t_i.
  - Denoted by $s\theta$
    and $s\theta$ is said to be an *instance* of s
- Example:

$$
\begin{array}{rl}
 & p(f(X,Z), f(Y,a))\ \{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\} \\
= & p(f(g(Y),a), f(Z,a))
\end{array}
$$

Composition of Substitutions

- Composition of substitutions $\theta = \{X_1 \mapsto s_1, \ldots, X_m \mapsto s_m\}$ and $\sigma = \{Y_1 \mapsto t_1, \ldots, Y_n \mapsto t_n\}$:
  - First form the set $\{X_1 \mapsto s_1\sigma, \ldots, X_m \mapsto s_m\sigma, Y_1 \mapsto t_1, \ldots, Y_n \mapsto t_n\}$
  - Remove from the set $X_i \mapsto s_i\sigma$ if $s_i\sigma = X_i$
  - Remove from the set $Y_j \mapsto t_j$ if Y_j is identical to some variable X_i
- Example: Let $\theta = \sigma = \{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\}$. Then $\theta\sigma =$

$$\begin{aligned} & \{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\}\{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\} \\ = & \{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\} \end{aligned}$$

- More examples: Let $\theta = \{X \mapsto f(Y)\}$ and $\sigma = \{Y \mapsto a\}$
  - $\theta\sigma = \{X \mapsto f(a), Y \mapsto a\}$
  - $\theta\sigma = \{X \mapsto f(Y), Y \mapsto a\}$
- Composition is not *commutative* but is *associative*: $\theta(\sigma\gamma) = (\theta\sigma)\gamma$
- Also, $E(\theta\sigma) = (E\theta)\sigma$

Idempotence

- A subsitution θ is **idempotent** iff $\theta\theta = \theta$.
- Examples:
  - $\{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\}$ is not idempotent since

$$\begin{array}{ll} & \{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\}\{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\} \\ = & \{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\} \end{array}$$

  - $\{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\}$ is not idempotent either since

$$\begin{array}{ll} & \{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\}\{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\} \\ = & \{X \mapsto g(a), Y \mapsto a, Z \mapsto a\} \end{array}$$

  - $\{X \mapsto g(a), Y \mapsto a, Z \mapsto a\}$ is idempotent
- For a substitution $\theta = \{X_1 \mapsto t_1, \ldots, X_n \mapsto t_n\}$,
  - $Dom(\theta) = \{X_1, X_2, \ldots X_n\}$
  - $Range(\theta) =$ set of all variables in $t_1, \ldots t_n$
- A substitution θ is idempotent iff $Dom(\theta) \cap Range(\theta) = \emptyset$

Unifiers

- A substitution θ is a *unifier* of two terms s and t if $s\theta$ is identical to $t\theta$.
- θ is a unifier of a set of equations $\{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$, if for all i, $s_i\theta = t_i\theta$.
- A substitution θ is more general than σ (written as $\theta \succeq \sigma$) if there is a substitution ω such that $\sigma = \theta\omega$
- A substitution θ is a *most general unifier* (mgu) of two terms (or a set of equations) if for every unifer σ of the two terms (or equations) $\theta \succeq \sigma$
- Example: Consider two terms $f(g(X), Y, a, b)$ and $f(Z, W, X, b)$.
  - $\theta_1 = \{X \mapsto a, Y \mapsto b, Z \mapsto g(a), W \mapsto b\}$ is a unifier
  - $\theta_2 = \{X \mapsto a, Y \mapsto W, Z \mapsto g(a)\}$ is also a unifier
  - θ_2 is a most general unifier

Equations and Unifiers

- A set of equations $\mathcal{E}$ is in *solved form* if it is of the form $\{X_1 \doteq t_1, \ldots, X_n \doteq t_n\}$ iff
  - all X_i's are distinct, and
  - no X_i appears in any t_j.
- Given a set of equations in solved form $\mathcal{E} = \{X_1 \doteq t_1, \ldots, X_n \doteq t_n\}$ the substitution $\{X_1/t_1, \ldots X_n/t_n\}$ is an idempotent mgu of $\mathcal{E}$.
- Two sets of equations $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be *equivalent* iff they have the same set of unifiers.
- To find the mgu of two terms s and t, find a set of equations in solved form that is equivalent to $\{s \doteq t\}$.
  If there is no equivalent solved form, there is no mgu.

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\{f(X, g(Y)) \doteq f(g(Z), Z)\} \quad \Rightarrow$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\{f(X, g(Y)) \doteq f(g(Z), Z)\} \quad \Rightarrow \quad \{X \doteq g(Z), g(Y) \doteq Z\}$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{array}{rcl} \{f(X,g(Y)) \doteq f(g(Z),Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} \\ & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} \end{array}$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$
\begin{array}{rcl}
\{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} \\
& \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} \\
& \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\}
\end{array}
$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{array}{rcl} \{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} \\ & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} \\ & \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} \end{array}$$

- Example 2: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

$$\{f(X, g(X), b) \doteq f(a, g(Z), Z)\} \quad \Rightarrow$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{aligned}\{f(X,g(Y)) \doteq f(g(Z),Z)\} &\Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\}\\ &\Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\}\\ &\Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}\end{aligned}$$

- Example 2: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

$$\{f(X,g(X),b) \doteq f(a,g(Z),Z)\} \Rightarrow \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\}$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$
\begin{aligned}
\{f(X, g(Y)) \doteq f(g(Z), Z)\} &\Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\} \\
&\Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\} \\
&\Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}
\end{aligned}
$$

- Example 2: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

$$
\begin{aligned}
\{f(X, g(X), b) \doteq f(a, g(Z), Z)\} &\Rightarrow \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\} \\
&\Rightarrow \{X \doteq a, g(a) \doteq g(Z), b \doteq Z\}
\end{aligned}
$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{array}{rcl}\{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} \\ & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} \\ & \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\}\end{array}$$

- Example 2: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

$$\begin{array}{rcl}\{f(X, g(X), b) \doteq f(a, g(Z), Z)\} & \Rightarrow & \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, g(a) \doteq g(Z), b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, a \doteq Z, b \doteq Z\}\end{array}$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{array}{rcl} \{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} \\ & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} \\ & \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} \end{array}$$

- Example 2: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

$$\begin{array}{rcl} \{f(X, g(X), b) \doteq f(a, g(Z), Z)\} & \Rightarrow & \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, g(a) \doteq g(Z), b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, a \doteq Z, b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, Z \doteq a, b \doteq Z\} \end{array}$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{array}{rcl} \{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} \\ & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} \\ & \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} \end{array}$$

- Example 2: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

$$\begin{array}{rcl} \{f(X, g(X), b) \doteq f(a, g(Z), Z)\} & \Rightarrow & \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, g(a) \doteq g(Z), b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, a \doteq Z, b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, Z \doteq a, b \doteq Z\} \\ & \Rightarrow & \{X \doteq a, Z \doteq a, b \doteq a\} \end{array}$$

A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{aligned}
\{f(X, g(Y)) \doteq f(g(Z), Z)\} &\Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\} \\
&\Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\} \\
&\Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}
\end{aligned}$$

- Example 2: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

$$\begin{aligned}
\{f(X, g(X), b) \doteq f(a, g(Z), Z)\} &\Rightarrow \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\} \\
&\Rightarrow \{X \doteq a, g(a) \doteq g(Z), b \doteq Z\} \\
&\Rightarrow \{X \doteq a, a \doteq Z, b \doteq Z\} \\
&\Rightarrow \{X \doteq a, Z \doteq a, b \doteq Z\} \\
&\Rightarrow \{X \doteq a, Z \doteq a, b \doteq a\} \\
&\Rightarrow \textbf{fail}
\end{aligned}$$

A Simple Unification Algorithm

Given a set of equations $\mathcal{E}$:

repeat

select $s \doteq t \in \mathcal{E}$;

case $s \doteq t$ **of**

1. $f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n)$:
   replace the equation by $s_i \doteq t_i$ for all i
2. $f(s_1, \ldots, s_n) \doteq g(t_1, \ldots, t_m)$, $f \neq g$ or $n \neq m$:
   halt with **failure**
3. $X \doteq X$: remove the equation
4. $t \doteq X$: where t is not a variable
   replace equation by $X \doteq t$
5. $X \doteq t$: where $X \neq t$ and X occurs more than once in $\mathcal{E}$:
   if X is a proper subterm of t
   then halt with **failure** (5a)
   else replace all other X in $\mathcal{E}$ by t (5b)

until no action is possible for any equation in $\mathcal{E}$

return $\mathcal{E}$

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$
  $\{f(X, g(Y)) \doteq f(g(Z), Z)\} \quad \Rightarrow$

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$
  $\{f(X, g(Y)) \doteq f(g(Z), Z)\} \quad \Rightarrow \quad \{X \doteq g(Z), g(Y) \doteq Z\}$ case 1

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$
\begin{array}{rcll}
\{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} & \text{case 1} \\
 & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} & \text{case 4}
\end{array}
$$

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$
\begin{array}{llll}
\{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} & \text{case 1} \\
& \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} & \text{case 4} \\
& \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} & \text{case 5b}
\end{array}
$$

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{array}{rcll} \{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} & \text{case 1} \\ & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} & \text{case 4} \\ & \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} & \text{case 5b} \end{array}$$

- Example 3: Find the mgu of $f(X, g(X))$ and $f(Z, Z)$

$$\{f(X, g(X)) \doteq f(Z, Z)\} \quad \Rightarrow$$

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{array}{llll} \{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} & \text{case 1} \\ & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} & \text{case 4} \\ & \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} & \text{case 5b} \end{array}$$

- Example 3: Find the mgu of $f(X, g(X))$ and $f(Z, Z)$

$$\{f(X, g(X)) \doteq f(Z, Z)\} \Rightarrow \{X \doteq Z, g(X) \doteq Z\} \quad \text{case 1}$$

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$
\begin{array}{lllr}
\{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} & \text{case 1} \\
& \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} & \text{case 4} \\
& \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} & \text{case 5b}
\end{array}
$$

- Example 3: Find the mgu of $f(X, g(X))$ and $f(Z, Z)$

$$
\begin{array}{lllr}
\{f(X, g(X)) \doteq f(Z, Z)\} & \Rightarrow & \{X \doteq Z, g(X) \doteq Z\} & \text{case 1} \\
& \Rightarrow & \{X \doteq Z, g(Z) \doteq Z\} & \text{case 5b}
\end{array}
$$

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\begin{array}{rcll} \{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} & \text{case 1} \\ & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} & \text{case 4} \\ & \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} & \text{case 5b} \end{array}$$

- Example 3: Find the mgu of $f(X, g(X))$ and $f(Z, Z)$

$$\begin{array}{rcll} \{f(X, g(X)) \doteq f(Z, Z)\} & \Rightarrow & \{X \doteq Z, g(X) \doteq Z\} & \text{case 1} \\ & \Rightarrow & \{X \doteq Z, g(Z) \doteq Z\} & \text{case 5b} \\ & \Rightarrow & \{X \doteq Z, Z \doteq g(Z)\} & \text{case 4} \end{array}$$

A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$
\begin{array}{llll}
\{f(X, g(Y)) \doteq f(g(Z), Z)\} & \Rightarrow & \{X \doteq g(Z), g(Y) \doteq Z\} & \text{case 1} \\
 & \Rightarrow & \{X \doteq g(Z), Z \doteq g(Y)\} & \text{case 4} \\
 & \Rightarrow & \{X \doteq g(g(Y)), Z \doteq g(Y)\} & \text{case 5b}
\end{array}
$$

- Example 3: Find the mgu of $f(X, g(X))$ and $f(Z, Z)$

$$
\begin{array}{llll}
\{f(X, g(X)) \doteq f(Z, Z)\} & \Rightarrow & \{X \doteq Z, g(X) \doteq Z\} & \text{case 1} \\
 & \Rightarrow & \{X \doteq Z, g(Z) \doteq Z\} & \text{case 5b} \\
 & \Rightarrow & \{X \doteq Z, Z \doteq g(Z)\} & \text{case 4} \\
 & \Rightarrow & \textbf{fail} & \text{case 5a}
\end{array}
$$

Complexity of the unification algorithm

Consider
$\mathcal{E} = \{g(X_1, \ldots, X_n) \doteq g(f(X_0, X_0), f(X_1, X_1), \ldots, f(X_{n-1}, X_{n-1})\}$.

- By applying **case 1** of the algorithm, we get

$$\{X_1 = f(X_0, X_0), X_2 = f(X_1, X_1), \ldots, X_n = f(X_{n-1}, X_{n-1})\}$$

- If terms are kept as *trees*, the final value for X_n is a tree of size $O(2^n)$.
- Recall that for **case 5** we need to first check if a variable appears in a term, and this could now take $O(2^n)$ time.
- There are *linear-time* unification algorithms that share structures (terms as DAGs).
- $X = t$ is the most common case for unification in Prolog. The fastest algorithms are linear in t.
- Prolog cuts corners by omitting *case 5a* (the occur check), thereby doing $X = t$ in *constant time*.

Most General Unifiers

- Note that mgu stands for **a** most general unifier.
- There may be more than one mgu. E.g. $f(X) \doteq f(Y)$ has two mgus:
  - $\{X \mapsto Y\}$
  - $\{Y \mapsto X\}$
- If θ is an mgu of s and t, and ω is a *renaming*, then $\theta\omega$ is an mgu of s and t.
- If θ and σ are mgus of s and t, then there is a renaming ω such that $\theta = \sigma\omega$.