Untyped Lambda Calculus

Principles of Programming Languages

CSE 526

1 Syntax
2 Variables and Substitution
3 Reductions
4 Recursion
5 Nameless Representation
Lambda Calculus

- A formal notation to study computability and programming.
- Can be considered as the smallest universal programming language.
  - **Universal**: Can be used to express any computation that can be performed on a Turing Machine
  - **Small**: Has only two constructs: abstraction and application.
- Brief History:
  - Introduced by Church and Kleene in 1930s.
  - Used by Church to study problems in computability.
  - Concepts have heavily influenced *functional programming*.
  - Used to study *types* and type systems in programming languages
Lambda Terms

Syntax of the $\lambda$-calculus

$t ::= \text{Terms}$
Lambda Terms

Syntax of the $\lambda$-calculus

\[ t \ ::= \begin{array}{ll}
  \times & \text{Terms} \\
  \times & \text{Variable}
\end{array} \]
Lambda Terms

Syntax of the $\lambda$-calculus

$$t ::= x \quad \text{Variable} \quad | \quad \lambda x. \ t \quad \text{Abstraction}$$
Lambda Terms

Syntax of the $\lambda$-calculus

$$t ::= \begin{array}{l}
\bar{x} \quad \text{Variable} \\
\lambda x. \ t \quad \text{Abstraction} \\
t \ t \quad \text{Application}
\end{array}$$
Lambda Terms

Syntax of the $\lambda$-calculus

$$t ::= \begin{cases} 
    \chi & \text{Variable} \\
    \lambda \chi. \, t & \text{Abstraction} \\
    t \, t & \text{Application}
\end{cases}$$

Textual Representation:

Use parentheses to represent trees using linear text.
Informal Semantics

λ-expressions can be considered as expressions in a functional language

- **Abstraction:** \((\lambda x. \ t)\) is a “function” with formal parameter \(x\) that returns (the value of) term \(t\).
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  - Example 1: $\lambda x. \ x$ is the identity function: one that returns the argument value itself.
  
  - Example 2: $\lambda x. \lambda y. \ x$ is a function that takes two arguments $x$ and $y$ and returns the first argument.
\textbf{Informal Semantics}

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- **Application:** \((t_1 \; t_2)\) is a “function call” where \(t_1\) is a function and \(t_2\) is the supplied argument.
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- **Application:** \((t_1 \, t_2)\) is a “function call” where \(t_1\) is a function and \(t_2\) is the supplied argument.
  - Example: \(((\lambda x. \, x) \, y)\) supplies \(y\) as the argument to the identity function.
Syntactic Conventions and Syntactic Sugar

- Parentheses can be dropped using the following conventions:
  - application is left associative
    e.g. $$((f\ f)\ x)$$ is same as $$f\ f\ x$$
  - a $$\lambda$$ binds as much as possible to its right.
    e.g. $$\lambda f.\ \lambda x.\ f\ (f\ x)$$ is same as $$(\lambda f.(\lambda x.\ f\ (f\ x)))$$
- Multiple consecutive abstractions can be combined:
  e.g. $$\lambda f.\lambda x.f\ (f\ x)$$ is same as $$\lambda f\ x.\ f\ (f\ x)$$
The Meaning of Lambda Expressions

- Recall: $\lambda x. \ t$ stands for a function with $x$ as the parameter and (the value of) $t$ as the return value.
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- $(t_1\ t_2)$ stands for “calling” the function $t_1$ with $t_2$ as the parameter.
The Meaning of Lambda Expressions

- Recall: $\lambda x. t$ stands for a function with $x$ as the parameter and (the value of) $t$ as the return value.
- $(t_1 \ t_2)$ stands for “calling” the function $t_1$ with $t_2$ as the parameter.
- Example: Consider the expression

$$((\lambda wyx. y \ (w \ y \ x)) \ (\lambda sz. z))$$

This is an instance of an application. The expression in blue is passed as an argument to the function in red.
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In the above example

1. $\lambda y x. \ y \ ((\lambda s z. \ z) \ y \ x)$
2. $\lambda y x. \ y \ ((\lambda z. \ z) \ x)$
The Meaning of Lambda Expressions

- Recall: \( \lambda x. \ t \) stands for a function with \( x \) as the parameter and (the value of) \( t \) as the return value.
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1. \( \lambda yx. \ y \ ((\lambda sz. \ z) \ y \ x) \)
2. \( \lambda yx. \ y \ ((\lambda z. \ z) \ x) \)
3. \( \lambda yx. \ y \ x \)
Encoding Booleans in the $\lambda$-Calculus

<table>
<thead>
<tr>
<th>B</th>
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<tbody>
<tr>
<td>true</td>
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</tr>
<tr>
<td>false</td>
<td>$\lambda x. \lambda y. y$</td>
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<td>$\lambda x. \lambda y. ((x y) \text{false})$</td>
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<td>!</td>
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<td>if</td>
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This is known as the

*Church encoding of Booleans,*
or simply *Church Booleans.*
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$$ (\text{true} \&\& \text{false}) $$
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Example:

$$(true \&\& false) \equiv (\lambda x. \lambda y. ((x y) false)) (\lambda x. \lambda y. x) (\lambda x. \lambda y. y)$$

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Example:

\[(\text{true} \&\& \text{false})\]

≡  \[(\lambda x. \lambda y. ((x y) \text{false}))\]  
   \[\quad (\lambda x. \lambda y. x)\]  
   \[\quad (\lambda x. \lambda y. y)\]

→  \[\quad (\lambda y. (((\lambda x. \lambda y. x) y) \text{false}))\]  
   \[\quad (\lambda x. \lambda y. y)\]
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Example:

$$(true \ && \ false)$$

$\equiv (\lambda x. \lambda y. ((x \ y) \ false))$

$$(\lambda x. \lambda y. x)$$

$$(\lambda x. \lambda y. y)$$

$\rightarrow (\lambda y. (((\lambda x. \lambda y. x) \ y) \ false))$

$$(\lambda x. \lambda y. y)$$

$\rightarrow (((\lambda x. \lambda y. x) (\lambda x. \lambda y. y)) \ false)$$

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\[(true \ && \ false)\]

\[\equiv (\lambda x. \lambda y. ((x \ y) \ false)))\]
\[\quad (\lambda x. \lambda y. x)\]
\[\quad (\lambda x. \lambda y. y)\]

\[\rightarrow (\lambda y. (((\lambda x. \lambda y. x) \ y) \ false)))\]
\[\quad (\lambda x. \lambda y. y)\]

\[\rightarrow ( ((\lambda x. \lambda y. x) (\lambda x. \lambda y. y)) \ false)\]

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Example:

$$ (\text{true} \&\& \text{false})$$

\[
\equiv (\lambda x. \lambda y. ((x y) \text{false}))(\lambda x. \lambda y. x)(\lambda x. \lambda y. y) \\
\rightarrow (\lambda y. (((\lambda x. \lambda y. x) y) \text{false}))(\lambda x. \lambda y. y) \\
\rightarrow (((\lambda x. \lambda y. x) (\lambda x. \lambda y. y)) \text{false}) \\
\rightarrow (\lambda x. \lambda y. y)
\]
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Example:

\[(true \&\& false)\]

\[\equiv (\lambda x. \lambda y. ((x \ y) \ false)))\]

\[\equiv (\lambda x. \lambda y. x)\]

\[\equiv (\lambda x. \lambda y. y)\]

This is known as the *Church encoding of Booleans*, or simply *Church Booleans*. 

\[(\lambda x. \lambda y. x)\]

\[\equiv \text{false}\]
## Encoding Natural Numbers in the \(\lambda\)-Calculus

<table>
<thead>
<tr>
<th>N</th>
<th>(\lambda)-calculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\lambda s. \lambda z. z)</td>
</tr>
<tr>
<td>1</td>
<td>(\lambda s. \lambda z. (s\ z))</td>
</tr>
<tr>
<td>2</td>
<td>(\lambda s. \lambda z. (s\ (s\ z)))</td>
</tr>
<tr>
<td>3</td>
<td>(\lambda s. \lambda z. (s\ (s\ (s\ z))))</td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
</tr>
<tr>
<td>(inc)</td>
<td>(\lambda n. \lambda s. \lambda z. (s\ ((n\ s)\ z)))</td>
</tr>
<tr>
<td>(plus)</td>
<td>(\lambda m. \lambda n. \lambda s. \lambda z. ((m\ s)\ ((n\ s)\ z)))</td>
</tr>
<tr>
<td>(times)</td>
<td>(\lambda m. \lambda n. ((m\ (plus\ n))\ 0))</td>
</tr>
<tr>
<td>(iszero)</td>
<td>(\lambda m. ((m\ (\lambda x. false))\ true))</td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
</tr>
</tbody>
</table>

This is known as the *Church encoding of Naturals*, or simply *Church Numerals*. 
Encoding Data Structures in the $\lambda$-Calculus

<table>
<thead>
<tr>
<th>Function</th>
<th>Encoding</th>
</tr>
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<tbody>
<tr>
<td>$\text{pair}$</td>
<td>$\lambda f. \lambda s. \lambda c. ((c f) s)$</td>
</tr>
<tr>
<td>$\text{fst}$</td>
<td>$\lambda p. (p \text{ true})$</td>
</tr>
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<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
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<td>$\lambda f. \lambda s. \lambda c. ((c f) s)$</td>
</tr>
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Encoding Data Structures in the \( \lambda \)-Calculus

<table>
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<tr>
<th>pair</th>
<th>( \lambda f. \lambda s. \lambda c. ((c , f) , s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{fst} )</td>
<td>( \lambda p. ,(p , \text{true}) )</td>
</tr>
<tr>
<td>( \text{snd} )</td>
<td>( \lambda p. ,(p , \text{false}) )</td>
</tr>
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</table>

Example: Let \( \phi_1 \) and \( \phi_2 \) be two arbitrary expressions.

\[
\text{pair} \ \phi_1 \ \phi_2 \\
\equiv \ ( \ (\lambda f. \lambda s. \lambda c. \((c \ f) \ s) \ \phi_1) \ \phi_2)
\]
Encoding Data Structures in the $\lambda$-Calculus

| $pair$ | $\lambda f. \lambda s. \lambda c. ((c \ f) \ s)$ |
| $fst$ | $\lambda p. (p \ true)$ |
| $snd$ | $\lambda p. (p \ false)$ |

Example: Let $\varphi_1$ and $\varphi_2$ be two arbitrary expressions.

$$pair \ \varphi_1 \ \varphi_2$$

$$\equiv ( (\lambda f. \lambda s. \lambda c. ((c \ f) \ s) \ \varphi_1) \ \varphi_2)$$

$$\rightarrow^* \ \lambda c. ((c \ \varphi_1) \ \varphi_2)$$
## Encoding Data Structures in the $\lambda$-Calculus

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\[
\text{pair } \varphi_1 \varphi_2
\equiv
( (\lambda f. \lambda s. \lambda c. ((c f) s) \varphi_1) \varphi_2)
\rightarrow^* \lambda c. ((c \varphi_1) \varphi_2)
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\[
fst (\text{pair } \varphi_1 \varphi_2)
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$$ pair \ \varphi_1 \ \varphi_2$$

$$ \equiv \ ( (\lambda f. \ \lambda s. \ \lambda c. \ ((c \ f) \ s) \ \varphi_1) \ \varphi_2) $$

$$ \rightarrow^* \ \lambda c. \ ((c \ \varphi_1) \ \varphi_2) $$

$$ fst \ (pair \ \varphi_1 \ \varphi_2) $$

$$ \equiv \ (\lambda p. (p \ true)) \ (pair \ \varphi_1 \ \varphi_2) $$
## Encoding Data Structures in the λ-Calculus

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\[
\text{pair} \ \varphi_1 \ \varphi_2 \\
\equiv \ \left( (\lambda f . \lambda s . \lambda c . ((c f) s) \varphi_1) \ \varphi_2 \right) \\
\rightarrow^* \ \lambda c . ((c \ \varphi_1) \ \varphi_2)
\]

\[
\text{fst} (\text{pair} \ \varphi_1 \ \varphi_2) \\
\equiv \ \left( \lambda p . (p \text{ true}) \right) (\text{pair} \ \varphi_1 \ \varphi_2) \\
\rightarrow \ (\text{pair} \ \varphi_1 \ \varphi_2) \ \text{true}
\]
Encoding Data Structures in the \( \lambda \)-Calculus

<table>
<thead>
<tr>
<th>\text{pair}</th>
<th>\text{fst}</th>
<th>\text{snd}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda f. \lambda s. \lambda c. ((c \ f) \ s) )</td>
<td>( \lambda p. (p \ true) )</td>
<td>( \lambda p. (p \ false) )</td>
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</tbody>
</table>

Example: Let \( \varphi_1 \) and \( \varphi_2 \) be two arbitrary expressions.

\[
\begin{align*}
\text{pair} \ \varphi_1 \ \varphi_2 & \\
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\text{fst} (\text{pair} \ \varphi_1 \ \varphi_2) & \\
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\begin{align*}
\text{pair} \ \varphi_1 \ \varphi_2 & \\
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$\text{pair } \varphi_1 \varphi_2$

$\equiv (\lambda f. \lambda s. \lambda c. ((c f) s) \varphi_1) \varphi_2$

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$\rightarrow \varphi_1$
Encoding Data Structures in the $\lambda$-Calculus

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<tr>
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<th>Syntax</th>
<th>Variables and Substitution</th>
<th>Reductions</th>
<th>Recursion</th>
<th>Nameless Representation</th>
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$$

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\[ \text{pair } \varphi_1 \varphi_2 \]
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\text{pair } \varphi_1 \varphi_2
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\equiv (\lambda f. \lambda s. \lambda c. ((c f) s) \varphi_1) \varphi_2
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\text{snd } (\text{pair } \varphi_1 \varphi_2)
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Basic reduction: $(\lambda x. \ t_1) \ t_2 \rightarrow [x \mapsto t_2]t_1$, where

$[x \mapsto t_2]t_1$ be the term obtained by replacing all “free” occurrences of $x$ in $t_1$ by $t_2$. 
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- A sub-term of \(t\) of the form \((\lambda x. \ t_1) \ t_2\) is called a **redex** of \(t\).
Evaluating Lambda Expressions: An Informal Intro.

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- In general, there may be many redexes in a term.

Example: Let \(id = (\lambda x. \ x)\) in term
\(id \ (id \ (\lambda z. \ id \ z))\)
Reduction Strategies

A reduction strategy is used to **choose** a redex where the basic reduction step will be done.
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- **Full $\beta$-reduction**: Pick a redex non-deterministically.

\[ \lambda x. ((\lambda x. x) (\lambda x. \lambda z. x)) \]
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- **Call-By-Name:** like normal-order, but ignore redexes inside abstractions.
- **Call-By-Value:** choose the right-most, inner-most redex that is not inside an abstraction.
Evaluating Lambda Expressions

The key step in evaluating an application then is: *replace every occurrence of a formal parameter with the actual argument.*

**Example:** $((\lambda x.(\lambda z. x z)) \ y) \rightarrow (\lambda z. y z)$
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\[(\lambda x.t_1)\ t_2 \rightarrow [x \mapsto t_2]t_1\]
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  \[(\lambda x. t_1) t_2 \rightarrow [x \mapsto t_2] t_1\]

- The central problem now is how we define this substitution function.
Substitutions (1\textsuperscript{st} attempt)

\[
\begin{align*}
[x \mapsto s]x &= s \\
[x \mapsto s]y &= y & \text{if } y \neq x \\
[x \mapsto s](\lambda y. \ t) &= \lambda y. [x \mapsto s]t \\
[x \mapsto s](t_1 \ t_2) &= ([x \mapsto s]t_1) ([x \mapsto s]t_2)
\end{align*}
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- Appears to be correct.
  \textbf{Example:} \([x \mapsto y](\lambda z. x z) = (\lambda z. y z)\)
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- But is incorrect!
  
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## Substitutions (2\textsuperscript{nd} attempt)

<table>
<thead>
<tr>
<th>Substitution</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[x \mapsto s]x$</td>
<td>$s$</td>
</tr>
<tr>
<td>$[x \mapsto s]y$</td>
<td>$y$ if $y \neq x$</td>
</tr>
<tr>
<td>$[x \mapsto s](\lambda y. \ t)$</td>
<td>$\begin{cases} \lambda y. \ t &amp; \text{if } x = y \ \lambda y. [x \mapsto s]t &amp; \text{if } x \neq y \end{cases}$</td>
</tr>
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<td>$[x \mapsto s](t_1 \ t_2)$</td>
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- But going by our definition, there is no way to distinguish between the two \(y\)'s!
- **Solution:** We should get \((\lambda w. y w)\) instead (by suitably renaming “local” variables).
Bound and Free Variables: An Informal Intro.

• Variable $x$ in $\lambda$-expression $\lambda x. t$ is said to be **bound**.

Rough meaning: parameters are local to a function definition.

A variable that is not bound is said to be **free**.

Example 2: in $\lambda x. (x y)$, $y$ is free.

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Bound and Binding Occurrences

\( \lambda x. x \)
Bound and Binding Occurrences

\[ \lambda x. x \]

Binding Occurrence (declaration)
Bound and Binding Occurrences

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- **Bound Occurrence (use)**
- **Binding Occurrence (declaration)**
Bound and Binding Occurrences

\( \lambda x. x \)

- Bound Occurrence (use)
- Binding Occurrence (declaration)
Bound and Binding Occurrences

\[(\lambda x. x) (\lambda z. (x z))\]

- Bound Occurrence (use)
- Binding Occurrence (declaration)
Bound and Binding Occurrences

Bound Occurrence (use)

Binding Occurrence (declaration)

\((\lambda \, x. \, x) (\lambda \, z. \, (x \, z))\)
Bound and Binding Occurrences

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Bound and Binding Occurrences

- \( \lambda x. x \)  
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- \(( \lambda x. x )( \lambda z. (x z) )\)
  - Free Occurrence
Bound and Binding Occurrences

- $\lambda x.x$
  - Bound Occurrence (use)
  - Binding Occurrence (declaration)

- $(\lambda x.x)(\lambda z.(x z))$
  - Free Occurrence

- $(\lambda z. (\lambda x. z (x x)) (\lambda x. z (x x)))$
Bound and Binding Occurrences

- \( \lambda x. x \) (Bound Occurrence (use))
- \( \lambda x. \) (Boundary Occurrence (declaration))

- \( (\lambda x. x)(\lambda z.(x z)) \) (Free Occurrence)
- \( (\lambda z.(\lambda x. z(x x)))(\lambda x. z(x x)) \)
Bound and Binding Occurrences

- Bound Occurrence (use)
- Binding Occurrence (declaration)
- Free Occurrence

\[(\lambda x. x)(\lambda z. (x z))\]
Bound and Binding Occurrences

- Bound Occurrence (use)
- Binding Occurrence (declaration)
- Free Occurrence

\[(\lambda x. x)(\lambda z. (x z))\]
Bound Variables

**Formal definition:** $bv(t)$, the set of all bound variables of $t$, is such that:
- $t$ is an abstraction of the form $\lambda x.t'$:
Bound Variables

**Formal definition:** \( bv(t) \), the set of all bound variables of \( t \), is such that:

- \( t \) is an abstraction of the form \( \lambda x.t' \):
  - \( bv(t) = bv(t') \cup \{x\} \)
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- \( t \) is an abstraction of the form \( \lambda x.t' \):
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- \( t \) is an application of the form \( t_1 \ t_2 \):
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  - $bv(t) = bv(t_1) \cup bv(t_2)$
- **Example:**
  - $bv( (\lambda x. x) (\lambda z. (x z)) )$
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**Example:**

$$bv( \ (\lambda x. \ x) \ (\lambda z. \ (x\ z)) )$$

$$= bv(\lambda x. \ x) \cup bv(\lambda z. \ (x\ z))$$
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- \( t \) is an application of the form \( t_1 \ t_2 \):
  - \( \text{bv}(t) = \text{bv}(t_1) \cup \text{bv}(t_2) \)

**Example:**

\[
\text{bv}( (\lambda x. x) \ (\lambda z. (x \ z)) ) = \text{bv}(\lambda x. x) \cup \text{bv}(\lambda z. (x \ z)) = \{x\} \cup \{z\} = \{x, z\}
\]
Free Variables

**Formal definition:** $fv(t)$, the set of all free variables of $t$, is such that:

- $t$ is a variable of the form $x$:
Free Variables

**Formal definition:** $fv(t)$, the set of all free variables of $t$, is such that:

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  - $fv(t) = \{x\}$
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- $t$ is a variable of the form $x$:
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  \[fv( (\lambda x. x) (\lambda z. (x z)) )\]
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**Example:**

$fv( (\lambda x. x) (\lambda z. (x z)) )$

$= fv(\lambda x. x) \cup fv(\lambda z. (x z))$
Free Variables

**Formal definition:** \(fv(t)\), the set of all free variables of \(t\), is such that:

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  - \(fv(t) = fv(t_1) \cup fv(t_2)\)

Example:

\[
fv( (\lambda x. x) (\lambda z. (x \ z)) ) \\
= fv(\lambda x. x) \cup fv(\lambda z. (x \ z)) \\
= \{ \} \cup \{x\} = \{x\}
\]
\(\alpha\)-Conversion (Renaming)

- **Intuition:** We can rename a bound variable as long as
**α-Conversion (Renaming)**

- **Intuition:** We can rename a bound variable as long as
  - the new name is not also the name of a free variable, and
\[\alpha\text{-Conversion (Renaming)}\]

- **Intuition:** We can rename a bound variable as long as
  - the new name is not also the name of a free variable, and
  - we replace every occurrence of the bound variable
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- **Example 1**: \((\lambda y. x y)\) is equivalent to \((\lambda z. x z)\)
\(\alpha\)-Conversion (Renaming)

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  - the new name is not also the name of a free variable, and
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- Example 1: \((\lambda y. x y)\) is equivalent to \((\lambda z. x z)\)

- Example 2: \((\lambda y. x y)\) is *not* equivalent to \((\lambda x. x x)\) (the name of new variable is same as that of a free variable)
**α-Conversion (Renaming)**

- **Intuition:** We can rename a bound variable as long as
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- Example 1: \((\lambda y. x \ y)\) is equivalent to \((\lambda z. x \ z)\)

- Example 2: \((\lambda y. x \ y)\) is *not* equivalent to \((\lambda x. x \ x)\) (the name of new variable is same as that of a free variable)

- Example 3: \((\lambda y. x \ y)\) is *not* equivalent to \((\lambda y. x \ z)\) (not every occurrence of \(y\) has been replaced).
\(\alpha\)-Conversion (Renaming)

**Intuition:** We can rename a bound variable as long as
- the new name is not also the name of a free variable, and
- we replace every occurrence of the bound variable

**Example 1:** \((\lambda y. x y)\) is equivalent to \((\lambda z. x z)\)

**Example 2:** \((\lambda y. x y)\) is not equivalent to \((\lambda x. x x)\) (the name of new variable is same as that of a free variable)

**Example 3:** \((\lambda y. x y)\) is not equivalent to \((\lambda y. x z)\) (not every occurrence of \(y\) has been replaced).

Two terms \(t\) and \(t'\) are said to be “\(\alpha\)-equivalent” (denoted by \(t \equiv_\alpha t'\)) if they are identical modulo the names of bound variables.
Substitutions (3rd attempt)

\[
\begin{align*}
[x \mapsto s]x &= s \\
[x \mapsto s]y &= y & \text{if } y \neq x \\
[x \mapsto s](\lambda y. \ t) &= \lambda y. [x \mapsto s]t & \text{if } x \neq y \text{ and } y \not\in \text{fv}(s) \\
[x \mapsto s](t_1 \ t_2) &= ([x \mapsto s]t_1) ([x \mapsto s]t_2)
\end{align*}
\]
Substitutions (3\textsuperscript{rd} attempt)

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[x \mapsto s]x &= s \\
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- The definition is now incomplete! e.g. \([x \mapsto y](\lambda y. x \ y) = ?\)
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- This drawback is not serious:
Substitutions (3\textsuperscript{rd} attempt)

\[
[x \mapsto s]x = s
\]
\[
[x \mapsto s]y = y \quad \text{if } y \neq x
\]
\[
[x \mapsto s](\lambda y. \ t) = \lambda y. [x \mapsto s]t \quad \text{if } x \neq y \text{ and } y \not\in \text{fv}(s)
\]
\[
[x \mapsto s](t_1 \ t_2) = ([x \mapsto s]t_1) ([x \mapsto s]t_2)
\]

- The definition is now incomplete! e.g. \([x \mapsto y](\lambda y. x \ y) = ??\)
- This drawback is not serious:
- We can apply a substitution on an \(\alpha\)-equivalent term instead.
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\end{align*}
\]

- The definition is now incomplete! e.g. \([x \mapsto y](\lambda y. x \ y) = ??\]
- This drawback is not serious:
- We can apply a substitution on an $\alpha$-equivalent term instead.
- E.g. \([x \mapsto y](\lambda z. x \ z) = (\lambda z. y \ z)\)
Operational Semantics: Full $\beta$-Reduction

\[
\begin{align*}
\frac{t_1 \rightarrow t'_1}{t_1 \ t_2 \rightarrow t'_1 \ t_2} & \quad \text{E-APP1} \\
\frac{t_2 \rightarrow t'_2}{t_1 \ t_2 \rightarrow t'_1 \ t'_2} & \quad \text{E-APP2} \\
\frac{t \rightarrow t'}{\lambda x. \ t \rightarrow \lambda x. \ t'} & \quad \text{E-ABS} \\
(\lambda x. \ t_1) \ t_2 & \rightarrow [x \mapsto t_2] t_1 & \quad \text{E-APPABS}
\end{align*}
\]
Operational Semantics: Call-By-Value

\[ t ::= \ldots \text{Terms (all } \lambda\text{-terms)} \]
\[ v ::= \lambda x. t \text{ Values} \]

Evaluation:

\[
\frac{t_1 \rightarrow t'_1}{t_1 \ t_2 \rightarrow t'_1 \ t_2} \quad \text{E-APP1}
\]

\[
\frac{t_2 \rightarrow t'_2}{v_1 \ t_2 \rightarrow v_1 \ t'_2} \quad \text{E-APP2}
\]

\[(\lambda x. \ t_1) \ v_2 \rightarrow [x \mapsto v_2]t_1 \quad \text{E-APPABS}\]
Operational Semantics: Call-By-Value

\[
\begin{align*}
  t &::= \ldots \\
  \nu &::= \lambda x. \; t
\end{align*}
\]

Terms (all \( \lambda \)-terms)

Values

Evaluation:

\[
\begin{align*}
  t_1 &\rightarrow t'_1 \\
  t_1 \; t_2 &\rightarrow t'_1 \; t_2 & \text{E-APP1}
\end{align*}
\]

\[
\begin{align*}
  t_2 &\rightarrow t'_2 \\
  \nu_1 \; t_2 &\rightarrow \nu_1 \; t'_2 & \text{E-APP2}
\end{align*}
\]

\[
(\lambda x. \; t_1) \; \nu_2 \rightarrow [x \mapsto \nu_2] \; t_1 & \text{E-APPABS}
\]

- In an application of the form \((t_1 \; t_2)\), if \(t_1\) is a \(\lambda\)-abstraction, then \(t_2\) has to be reduced to a value before the application is done.
Operational Semantics: Call-By-Value

\[ t ::= \ldots \quad \text{Terms (all } \lambda\text{-terms)} \]
\[ v ::= \lambda x. t \quad \text{Values} \]

Evaluation:

\[ \frac{t_1 \rightarrow t'_1}{t_1 \ t_2 \rightarrow t'_1 \ t_2} \quad \text{E-APP1} \]
\[ \frac{t_2 \rightarrow t'_2}{v_1 \ t_2 \rightarrow v_1 \ t'_2} \quad \text{E-APP2} \]
\[ (\lambda x. \ t_1) \ v_2 \rightarrow [x \mapsto v_2]t_1 \quad \text{E-APPABS} \]

- In an application of the form \((t_1 \ t_2)\), if \(t_1\) is a \(\lambda\)-abstraction, then \(t_2\) has to be reduced to a value before the application is done.
- This corresponds to Call-By-Value parameter passing: evaluate the actual arguments first before passing them as parameters to a called function.
Operational Semantics: Call-By-Name

\[ t ::= \ldots \] Terms (all \( \lambda \)-terms)

\[ v ::= \lambda x. t \] Values

Evaluation:

\[
\frac{t_1 \rightarrow t_1'}{t_1 \quad t_2 \rightarrow t_1' \quad t_2} \quad \text{E-APP}
\]

\[
(\lambda x. t_1) \quad t_2 \rightarrow [x \mapsto t_2]t_1 \quad \text{E-APPABS}
\]
Operational Semantics: Call-By-Name

\[ t ::= \ldots \quad \text{Terms (all } \lambda\text{-terms)} \]
\[ v ::= \lambda x. t \quad \text{Values} \]

Evaluation:

\[
\frac{t_1 \rightarrow t_1'}{t_1 \  t_2 \rightarrow t_1' \ t_2} \quad \text{E-APP}
\]

\[
(\lambda x. \ t_1) \ t_2 \rightarrow [x \mapsto t_2]t_1 \quad \text{E-APPABS}
\]

- In an application of the form \((t_1 \ t_2)\), if \(t_1\) is a \(\lambda\)-abstraction, then \(t_1\) has to be reduced to a value before the application is done.
Operational Semantics: Call-By-Name

\[ t ::= \ldots \quad \text{Terms (all } \lambda\text{-terms)} \]
\[ v ::= \lambda x. \ t \quad \text{Values} \]

Evaluation:

\[
\frac{t_1 \rightarrow t'_1}{t_1 \ t_2 \rightarrow t'_1 \ t_2} \quad \text{E-APP}
\]

\[
(\lambda x. \ t_1) \ t_2 \rightarrow [x \mapsto t_2] t_1 \quad \text{E-APPABS}
\]

- In an application of the form \((t_1 \ t_2)\), if \(t_1\) is a \(\lambda\)-abstraction, then \(t_1\) has to be reduced to a value before the application is done.
- In terms of familiar languages, the actual arguments are passed \textit{unevaluated} to the called function. They will be evaluated in the called function if needed.
Infinite and Diverging Computations in the $\lambda$-Calculus

\[ \text{omega} : \quad (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \]
Infinite and Diverging Computations in the $\lambda$-Calculus

\[ \text{omega : } (\lambda x. x x) (\lambda x. x x) \]

Evaluation:

\[ \text{omega} \equiv (\lambda x. x x) (\lambda x. x x) \]
Infinite and Diverging Computations in the $\lambda$-Calculus

\[ \omega : \quad (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \]

Evaluation:

\[ \omega \\
\equiv \quad (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \\
\rightarrow \quad (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \\
\equiv \quad \omega \]
Infinite and Diverging Computations in the $\lambda$-Calculus

$\omega : \ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x)$

Evaluation:

$\omega$

$\equiv \ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x)$

$\rightarrow \ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x)$

$\equiv \ \omega$

$\rightarrow \ \omega$
Infinite and Diverging Computations in the $\lambda$-Calculus

**omega**: \((\lambda x. x \ x) (\lambda x. x \ x)\)

Evaluation:

omega

\[\equiv (\lambda x. x \ x) (\lambda x. x \ x)\]

\[\rightarrow (\lambda x. x \ x) (\lambda x. x \ x)\]

\[\equiv omega\]

\[\rightarrow omega\]

\[\vdots\]
Infinite and Diverging Computations in the λ-Calculus

\[ \omega : (\lambda x. x x) (\lambda x. x x) \]
\[ \text{Evaluation:} \]
\[ \omega \equiv (\lambda x. x x) (\lambda x. x x) \]
\[ \rightarrow (\lambda x. x x) (\lambda x. x x) \]
\[ \equiv \omega \]
\[ \rightarrow \omega \]
\[ \therefore \]

\[ \inf : (\lambda x. (x x) x) \]
\[ \text{Evaluation:} \]
\[ (\inf \inf) \equiv (\lambda x. (x x) x) \inf \]
\[ \rightarrow (\lambda x. (x x) x) \inf \]
\[ \rightarrow (\inf \inf) \inf \]
\[ \rightarrow \ldots \ldots \]
\[ \therefore \]
### Infinite and Diverging Computations in the $\lambda$-Calculus

#### Omega

- **Definition:**
  
  \[
  \text{omega} : \ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x)
  \]

- **Evaluation:**
  
  \[
  \begin{align*}
  \text{omega} & \equiv (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \\
  \rightarrow & \ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \\
  & \equiv \text{omega} \\
  & \rightarrow \text{omega} \\
  & \ldots
  \end{align*}
  \]

#### Inf

- **Definition:**
  
  \[
  \text{inf} : \ (\lambda x. \ (x \ x) \ x)
  \]

- **Evaluation:**
  
  \[
  \begin{align*}
  (\text{inf} \ \text{inf}) & \equiv (\lambda x. \ (x \ x) \ x) \ \text{inf} \\
  \rightarrow & \ (\lambda x. \ (x \ x) \ x) \ \text{inf} \\
  \rightarrow & \ldots
  \end{align*}
  \]
Infinite and Diverging Computations in the $\lambda$-Calculus

$\text{omega} : (\lambda x. x x) (\lambda x. x x)$

Evaluation:

\[
\begin{align*}
\text{omega} & \equiv (\lambda x. x x) (\lambda x. x x) \\
\rightarrow & (\lambda x. x x) (\lambda x. x x) \\
\equiv & \text{omega} \\
\rightarrow & \text{omega}
\end{align*}
\]

\[\text{inf} : (\lambda x. (x x) x)\]

Evaluation:

\[
\begin{align*}
(\text{inf inf}) & \equiv (\lambda x. (x x) x) \text{inf} \\
\rightarrow & (\text{inf inf}) \text{inf}
\end{align*}
\]
Infinite and Diverging Computations in the $\lambda$-Calculus

omega : $(\lambda x. x x) \ (\lambda x. x x)$

Evaluation:

\[
\omega \equiv (\lambda x. x x) \ (\lambda x. x x) \\
\longrightarrow (\lambda x. x x) \ (\lambda x. x x) \\
\equiv \omega \\
\longrightarrow \omega
\]

inf : $(\lambda x. (x x) x)$

Evaluation:

\[
inf \inf \equiv (\lambda x. (x x) x) \ inf \\
\longrightarrow (inf \ inf) \ inf \\
\longrightarrow (inf \ inf) \ inf \\
\longrightarrow \ ...
\]
Infinite and Diverging Computations in the $\lambda$-Calculus

$$\text{omega} : \quad (\lambda x. \ x \ x) \ (\lambda x. \ x \ x)$$

Evaluation:

$$\begin{align*}
\text{omega} & \equiv (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \\
\rightarrow & \quad (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \\
\equiv & \quad \text{omega} \\
\rightarrow & \quad \text{omega} \\
\vdots
\end{align*}$$

$$\begin{align*}
\text{inf} & : \quad (\lambda x. \ (x \ x) \ x)
\end{align*}$$

Evaluation:

$$\begin{align*}
(\text{inf} \ \text{inf}) & \equiv (\lambda x. \ (x \ x) \ x) \ \text{inf} \\
\rightarrow & \quad (\text{inf} \ \text{inf}) \ \text{inf} \\
\rightarrow & \quad ( (\text{inf} \ \text{inf}) \ \text{inf}) \ \text{inf} \\
\rightarrow & \quad \ldots \\
\vdots
\end{align*}$$
Consider the function to compute *factorial* of a natural number, written as follows:

\[
\text{fact} \equiv \lambda n. (\text{if} \ (\text{iszero} \ n) \ 1 \ (\text{times} \ n \ (\text{fact} \ (\text{dec} \ n))))
\]

where \(\text{dec}\) is the function that decrements a number by 1.
Consider the function to compute *factorial* of a natural number, written as follows:

\[
\text{fact} \equiv \lambda n. \ (\text{if} \ (\text{iszero} \ n) \ 1 \ (\text{times} \ n \ (\text{fact} \ (\text{dec} \ n))) \ )
\]

where \(\text{dec}\) is the function that decrements a number by 1.

Note this is not a proper encoding: \(\text{fact}\) is being defined in terms of itself!
Consider the function to compute \textit{factorial} of a natural number, written as follows:

\[
\text{fact} \equiv \lambda n. \ (\text{if} \ (\text{iszero} \ n) \ 1 \ (\text{times} \ n \ (\text{fact} \ (\text{dec} \ n))) )
\]

where \text{dec} is the function that decrements a number by 1.

Note this is not a proper encoding: \text{fact} is being defined in terms of itself!

The solution is to “lift” factorial into a \textit{functional}:

\[
F \equiv \lambda f. \ \lambda n. \ (\text{if} \ (\text{iszero} \ n) \ 1 \ (\text{times} \ n \ (f \ (\text{dec} \ n))) )
\]
Consider the function to compute *factorial* of a natural number, written as follows:

\[ \text{fact} \equiv \lambda n. \left( \text{if} \ (\text{iszero} \ n) \ 1 \ \left( \times \ n \ (\text{fact} \ (\text{dec} \ n)) \right) \right) \]

where *dec* is the function that decrements a number by 1.

Note this is not a proper encoding: *fact* is being defined in terms of itself!

The solution is to “lift” factorial into a *functional*:

\[ F \equiv \lambda f. \lambda n. \left( \text{if} \ (\text{iszero} \ n) \ 1 \ \left( \times \ n \ (f \ (\text{dec} \ n)) \right) \right) \]

Note that *F* is well-defined.
Consider the function to compute *factorial* of a natural number, written as follows:

\[
\text{fact} \equiv \lambda n. \ (\text{if } (\text{iszero } n) \ 1 \ (\text{times } n \ (\text{fact } (\text{dec } n))) ) \)
\]

where \( \text{dec} \) is the function that decrements a number by 1.

Note this is not a proper encoding: \( \text{fact} \) is being defined in terms of itself!

The solution is to “lift” factorial into a *functional*:

\[
F \equiv \lambda f. \ \lambda n. \ (\text{if } (\text{iszero } n) \ 1 \ (\text{times } n \ (f \ (\text{dec } n))) ) \)
\]

Note that \( F \) is well-defined.

\( F \) is a very special function, as we’ll see in the next...
Recursive Functions in the $\lambda$-Calculus

\[ F \equiv \lambda f. \lambda n. (if \ (iszero \ n) \ 1 \ (times \ n \ (f \ (dec \ n))) ) \]

- Consider $fact_0 \equiv F \ \omega$:
  
  \[
  fact_0 \equiv F \ \omega \\
  \equiv (\lambda f. \lambda n. (if \ (iszero \ n) \ 1 \ (times \ n \ (f \ (dec \ n)))) )) \ \omega \\
  \rightarrow \lambda n. (if \ (iszero \ n) \ 1 \ (times \ n \ (\omega \ (dec \ n))))
  \]

When non-strict evaluation is used, $fact_0$ computes the same as $fact_0$ for $0$, but diverges elsewhere.
Recursive Functions in the $\lambda$-Calculus

\[ F \equiv \lambda f. \lambda n. (\text{if } (\text{iszero } n) \ 1 \ (\text{times } n \ (f \ (\text{dec } n))) \ ) \ ) \]

- Consider $fact_0 \equiv F \omega$:
  \[
  fact_0 \equiv F \omega \\
  \equiv (\lambda f. \lambda n. (\text{if } (\text{iszero } n) \ 1 \ (\text{times } n \ (f \ (\text{dec } n))) \ ) \ )) \ \omega \\
  \rightarrow \lambda n. (\text{if } (\text{iszero } n) \ 1 \ (\text{times } n \ (\omega (\text{dec } n))) \ )
  \]

- When non-strict evaluation is used, $fact_0$ computes the same as $fact$ for $0$, but diverges elsewhere.
Recursive Functions in the $\lambda$-Calculus —(3)

\[ F \equiv \lambda f. \lambda n. (\text{if } (\text{iszero } n) 1 (\text{times } n (f (\text{dec } n))) ) \]

Now consider $\text{fact}_1 \equiv F \text{ fact}_0$:

\[
\text{fact}_1 \equiv F \text{ fact}_0 \\
\equiv (\lambda f. \lambda n. (\text{if } (\text{iszero } n) 1 (\text{times } n (f (\text{dec } n))) ) ))) \text{ fact}_0 \\
\rightarrow \lambda n. (\text{if } (\text{iszero } n) 1 (\text{times } n (\text{fact}_0 (\text{dec } n))) )
\]
Recursive Functions in the λ-Calculus —(3)

\[ F \equiv \lambda f. \lambda n. (\text{if} \ (\text{iszero} \ n) \ 1 \ (\text{times} \ n \ (f \ (\text{dec} \ n))) ) \)

- Now consider \( fact_1 \equiv F \ fact_0 \):
  \[
  fact_1 \equiv F \ fact_0 \\
  \equiv (\lambda f. \lambda n. (\text{if} \ (\text{iszero} \ n) \ 1 \ (\text{times} \ n \ (f \ (\text{dec} \ n))) ) ) \) \( fact_0 \\
  \rightarrow \lambda n. (\text{if} \ (\text{iszero} \ n) \ 1 \ (\text{times} \ n \ (fact_0 \ (\text{dec} \ n))) )
  \]

- \textit{When non-strict evaluation is used}, \( fact_1 \) computes the same as \( fact \) for 0 and 1, but diverges elsewhere.
Consider the sequence of functions $fact_0, fact_1, fact_2, \ldots$ such that $fact_0 = \omega$, and $fact_{n+1} = (F \ fact_n)$. 
Consider the sequence of functions $fact_0$, $fact_1$, $fact_2$, ... such that $fact_0 = \omega$, and $fact_{n+1} = (F \, fact_n)$.

None of these functions is the same as $fact$, but as we construct more and more members of this sequence, we get functions that approximate $fact$ closer and closer.

$fact$ is indeed the limit of this sequence of functions!
Consider the sequence of functions \( \text{fact}_0, \text{fact}_1, \text{fact}_2, \ldots \) such that 
\[
\text{fact}_0 = \text{omega}, \quad \text{and} \quad \text{fact}_{n+1} = (F \text{ fact}_n).
\]
None of these functions is same as \( \text{fact} \), but as we construct more and more members of this sequence, we get functions that approximate \( \text{fact} \) closer and closer.

\( \text{fact} \) is indeed the limit of this sequence of functions!
Recursive Functions in the $\lambda$-Calculus —(4)

- Consider the **sequence** of functions $\text{fact}_0, \text{fact}_1, \text{fact}_2, \ldots$ such that $\text{fact}_0 = \text{omega}$, and $\text{fact}_{n+1} = (F \text{ fact}_n)$.
- None of these functions is same as $\text{fact}$, but as we construct more and more members of this sequence, we get functions that approximate $\text{fact}$ closer and closer.
- $\text{fact}$ is indeed the **limit** of this sequence of functions!
- If only we had a way, in the $\lambda$-calculus, to generate such a sequence...
The Y-Combinator

\[ Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x)) \]

Consider \( (Y \ F) \):

\[
\begin{align*}
(Y \ F) & \equiv (\lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))) \ F \\
\Downarrow & (\lambda x. F (x x)) (\lambda x. F (x x)) \\
\Downarrow & F ( (\lambda x. F (x x)) (\lambda x. F (x x)) ) \\
\Rightarrow & F (Y \ F)
\end{align*}
\]
The Y-Combinator

\[ Y = \lambda f. (\lambda x. f (x \ x)) (\lambda x. f (x \ x)) \]

- Consider \((Y \ F)\):
  \[ (Y \ F) \equiv (\lambda f. (\lambda x. f (x \ x)) (\lambda x. f (x \ x))) \ F \]
  \[ \rightarrow (\lambda x. F (x \ x)) (\lambda x. F (x \ x)) \]
  \[ \rightarrow F ( (\lambda x. F (x \ x)) (\lambda x. F (x \ x)) ) \]
  \[ \Rightarrow F (Y \ F) \]

- Recall \(F \equiv \lambda f. \lambda n. (if \ (iszero \ n) \ 1 \ (times \ n \ (f \ (dec \ n)) ))\).
The Y-Combinator

\[ Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x)) \]

- Consider \((Y F)\):
  \[
  (Y F) \equiv (\lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))) F \\
  \rightarrow (\lambda x. F (x x)) (\lambda x. F (x x)) \\
  \rightarrow F ((\lambda x. F (x x)) (\lambda x. F (x x))) \\
  \equiv F (Y F)
  \]

- Recall \(F \equiv \lambda f. \lambda n. (\text{if } (\text{iszero } n) 1 \ (\text{times } n \ (f \ (\text{dec } n))) \ ))\).

- Putting it all together:
  \[
  (Y F) \equiv F (Y F) \\
  \equiv (\lambda f. \lambda n. (\text{if } (\text{iszero } n) 1 \ (\text{times } n \ (f \ (\text{dec } n))) \ )) (Y F) \\
  \rightarrow \lambda n. (\text{if } (\text{iszero } n) 1 \ (\text{times } n ((Y F) (\text{dec } n))) \ ))
  \]
The Y-Combinator

\[ Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x)) \]

- Consider \((Y F)\):
  \[
  (Y F) \equiv (\lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))) F
  \]
  \[
  \rightarrow (\lambda x. F (x x)) (\lambda x. F (x x))
  \]
  \[
  \rightarrow F \left( (\lambda x. F (x x)) (\lambda x. F (x x)) \right)
  \]
  \[
  \equiv F \left( Y F \right)
  \]

- Recall \(F \equiv \lambda f. \lambda n. (if \ (iszero\ n)\ 1\ (times\ n\ (f\ (dec\ n)))\ )\).

- Putting it all together:
  \[
  (Y F) \equiv F \left( Y F \right)
  \]
  \[
  \equiv (\lambda f. \lambda n. (if \ (iszero\ n)\ 1\ (times\ n\ (f\ (dec\ n)))\ )\ )\ ) (Y F)
  \]
  \[
  \rightarrow \lambda n. (if \ (iszero\ n)\ 1\ (times\ n\ ((Y F)\ (dec\ n)))\ )
  \]

- \((Y F)\) looks like the mythical function \textit{fact}.  

\textbf{Programming Languages}

\textbf{The Untyped Lambda Calculus}
The Z-Combinator

- \((Y \ F) \cong F \ (Y \ F)\)

With *call-by-name* evaluation strategy, the next steps in reduction will first substitute the formal parameter of \(F\) with \((Y \ F)\).

With *call-by-value* strategy, \(F \ (Y \ F)\) will first reduce \((Y \ F)\), which result in:

\[
\begin{align*}
\rightarrow^* & \quad F \ (F \ (Y \ F)) \\
\rightarrow^* & \quad F \ (F \ (F \ (Y \ F))) \\
\rightarrow^* & \quad \ldots
\end{align*}
\]

For *call-by-value* strategy, we should use the Z combinator instead:

\[
Z = \lambda f. \ (\lambda x. \ f \ (\lambda y. \ x \ x \ y)) \ (\lambda x. \ f \ (\lambda y. \ x \ x \ y))
\]
Recursive Functions in the $\lambda$-Calculus — (5)

$$Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$$
$$F = \lambda f. \lambda n. (if (iszero n) 1 \ (times n (f (dec n))) ) )$$
$$fact = (Y F)$$

Note that the definitions of $Y$, $F$ and $fact$ are all non-recursive.
Recursive Functions in the $\lambda$-Calculus —(5)

\[ Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x)) \]
\[ F = \lambda f. \lambda n. (if \ (iszero \ n) \ 1 \ (times \ n \ (f \ (dec \ n))) \ ) \ ) \]
\[ fact = (Y \ F) \]

- Note that the definitions of $Y$, $F$ and $fact$ are all non-recursive.
- The above recipe can be used for writing any recursive function.
Recursion in the λ-Calculus — (5)

\[ Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x)) \]
\[ F = \lambda f. \lambda n. (\text{if } (\text{iszero } n) 1 (\text{times } n (f (\text{dec } n))) ) ) \]
\[ \text{fact} = (Y F) \]

- Note that the definitions of \( Y \), \( F \) and \( \text{fact} \) are all non-recursive.
- The above recipe can be used for writing any recursive function.
- Say, we have a mythical recursive definition \( f = \lambda x. e \) where \( e \) uses \( f \).
Recursive Functions in the $\lambda$-Calculus

\[
Y = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x)) \\
F = \lambda f. \lambda n. (\text{if } (\text{iszero } n) 1 (\text{times } n (f (\text{dec } n)))) \\
fact = (Y F)
\]

- Note that the definitions of $Y$, $F$ and $fact$ are all non-recursive.
- The above recipe can be used for writing any recursive function.
- Say, we have a mythical recursive definition $f = \lambda x. e$ where $e$ uses $f$.
- We simply rewrite the definition as $f = (Y (\lambda f. \lambda x. e))$. 
Nameless Representation of Terms

- Consider variables in a $\lambda$-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the $\lambda$ that binds them.

Examples:
Nameless Representation of Terms

- Consider variables in a $\lambda$-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the $\lambda$ that binds them.

Examples:

- $\lambda x. \ x$ can be written as

```latex
\begin{align*}
\lambda x \\
\downarrow \\
x
\end{align*}
```
Nameless Representation of Terms

- Consider variables in a $\lambda$-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the $\lambda$ that binds them.

Examples:

- $\lambda x. x$ can be written as

  $\begin{array}{c}
  \lambda x \\
  1 \uparrow \\
  x \\
  0
  \end{array}$
Nameless Representation of Terms

- Consider variables in a \( \lambda \)-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the \( \lambda \) that binds them.

Examples:

- \( \lambda x. x \) can be written as \( \lambda. 0 \)

\[
\begin{array}{c|c}
\lambda x & \lambda \\
\downarrow & \downarrow \\
x & 0
\end{array}
\]
Nameless Representation of Terms

- Consider variables in a $\lambda$-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the $\lambda$ that binds them.

Examples:

- $\lambda x$. $x$ can be written as $\lambda$. 0
- $\lambda x. \lambda y. x$ can be written as...
Nameless Representation of Terms

- Consider variables in a $\lambda$-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the $\lambda$ that binds them.

Examples:

\[
\begin{align*}
\lambda x & \quad \lambda \\
| & \quad | \\
\lambda y & \quad \lambda \\
| & \quad | \\
x & \quad 1
\end{align*}
\]

- $\lambda x. x$ can be written as $\lambda. 0$
- $\lambda x. \lambda y. x$ can be written as
Nameless Representation of Terms

- Consider variables in a $\lambda$-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the $\lambda$ that binds them.

**Examples:**

- $\lambda x$. $x$ can be written as $\lambda$. 0
- $\lambda x. \lambda y. x$ can be written as $\lambda. \lambda. 1$
Nameless Representation of Terms

- Consider variables in a $\lambda$-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the $\lambda$ that binds them.

Examples:

- $\lambda x$. $x$ can be written as $\lambda. 0$
- $\lambda x. \lambda y. x$ can be written as $\lambda. \lambda. 1$
- $\lambda x. \lambda y. x (y x)$ can be written as
Nameless Representation of Terms

- Consider variables in a \( \lambda \)-term as named “holes” to be filled in.
- Instead of using symbolic names for variables, one can name the holes w.r.t. the \( \lambda \) that binds them.

Examples:

- \( \lambda x \). \( x \) can be written as \( \lambda. \ 0 \)
- \( \lambda x. \ \lambda y. \ x \) can be written as \( \lambda. \ \lambda. \ 1 \)
- \( \lambda x. \ \lambda y. \ x \ (y \ x) \) can be written as
Consider variables in a λ-term as named “holes” to be filled in.
Instead of using symbolic names for variables, one can name the holes w.r.t. the λ that binds them.

Examples:

\(\lambda x\)
\(\lambda y\)
apply
\(x\)
apply
\(y\)
\(x\)

\(\lambda\)
\(\lambda\)
apply
\(1\)
apply
\(0\)
\(1\)

\(\lambda x. x\) can be written as \(\lambda. 0\)
\(\lambda x. \lambda y. x\) can be written as \(\lambda. \lambda. 1\)
\(\lambda x. \lambda y. x\) \((y\ x)\) can be written as \(\lambda. \lambda. 1\) \((0\ 1)\)
De Bruijn terms are defined by a family of sets (each set being a set of terms) \( \{ \mathcal{T}_0, \mathcal{T}_1, \ldots \} \) such that \( \mathcal{T}_n \) represents \( \lambda \)-terms with \( n \) or fewer free variables.

Formally, \( \mathcal{T} \) is the smallest family of sets \( \{ \mathcal{T}_0, \mathcal{T}_1, \ldots \} \) such that

- \( k \in \mathcal{T}_n \) whenever \( 0 \leq k < n \)
- if \( t_1 \in \mathcal{T}_n \) then \( \lambda \cdot t_1 \in \mathcal{T}_{n-1} \)
- if \( t_1, t_2 \in \mathcal{T}_n \) then \( (t_1 \ t_2) \in \mathcal{T}_n \)

\( \alpha \)-equivalent closed \( \lambda \)-terms will have the same de Bruijn representation.
Naming Context

- When a $\lambda$-term has free variables, we need information on their relative positions.
- E.g. given $\{ v \mapsto 2, w \mapsto 1, x \mapsto 0 \}$:
Naming Context

- When a $\lambda$-term has free variables, we need information on their relative positions.
- E.g. given $\{v \mapsto 2, w \mapsto 1, x \mapsto 0\}$:
Naming Context

- When a $\lambda$-term has free variables, we need information on their relative positions.
- E.g. given $\{v \mapsto 2, w \mapsto 1, x \mapsto 0\}$:

\[
\lambda v \quad \lambda w \quad \lambda x \\
\downarrow \quad \downarrow \quad \downarrow \\
apply \\
\downarrow \quad \downarrow \\
v \quad apply \\
\downarrow \quad \downarrow \\
w \quad x
\]

- $v \ (w \ x)$ can be written as $\lambda y. \lambda c. \lambda v. \lambda.$ $\lambda y. \lambda c. v$. $\lambda.$ $\lambda.$ $\lambda$ $\lambda$ $\lambda$.

Programming Languages

The Untyped Lambda Calculus

CSE 526
When a $\lambda$-term has free variables, we need information on their relative positions.

E.g. given $\{v \mapsto 2, w \mapsto 1, x \mapsto 0\}$:

$v (w \: x)$ can be written as $2 \: (1 \: 0)$
Naming Context

- When a λ-term has free variables, we need information on their relative positions.
- E.g. given \( \{ v \mapsto 2, w \mapsto 1, x \mapsto 0 \} \):

  - \( \lambda v \lambda w x \) can be written as \( 2 (1 0) \)
  - \( \lambda y. w y \) can be written as

```
λy (w x)
```

Programming Languages
Naming Context

- When a $\lambda$-term has free variables, we need information on their relative positions.
- E.g. given $\{v \mapsto 2, w \mapsto 1, x \mapsto 0\}$:

  $\lambda v \lambda w w y (\lambda x \lambda y v (w x))$ can be written as $2 (1 0)$
  $\lambda y. w y$ can be written as $\lambda. 2 0$
Naming Context

- When a λ-term has free variables, we need information on their relative positions.
- E.g. given \( \{ v \mapsto 2, w \mapsto 1, x \mapsto 0 \} \):

  - \( \lambda y. \lambda c. v \) can be written as \( \lambda. \lambda. 4 \)
  - \( v \ (w \ x) \) can be written as 2 (1 0)
  - \( \lambda y. w \ y \) can be written as \( \lambda. 2 \ 0 \)
  - \( \lambda y. \lambda c. v \) can be written as...
Naming Context

- When a \( \lambda \)-term has free variables, we need information on their relative positions.
- E.g. given \( \{ v \mapsto 2, w \mapsto 1, x \mapsto 0 \} \):

\[
\lambda v \lambda w \lambda x \lambda y \lambda y \lambda c \lambda v
\]

- \( v \ (w \ x) \) can be written as \( 2 \ (1 \ 0) \)
- \( \lambda y. \ w \ y \) can be written as \( \lambda. \ 2 \ 0 \)
- \( \lambda y. \lambda c. \ v \) can be written as \( \lambda. \ \lambda. \ 4 \)
Naming Context

- When a \( \lambda \)-term has free variables, we need information on their relative positions.
- E.g. given \( \{ v \mapsto 2, w \mapsto 1, x \mapsto 0 \} \):

  \[
  \lambda c \lambda v \lambda w v \lambda x \lambda y v (w x) \text{ can be written as } 2 (1 0)
  \]
  \[
  \lambda y v \lambda w \lambda x \lambda y. w y \text{ can be written as } \lambda. 2 0
  \]
  \[
  \lambda y. \lambda c. v \text{ can be written as } \lambda. \lambda. 4
  \]

- Naming contexts are often written as a sequence, where \( x_n, x_{n-1}, \ldots, x_1, x_0 \), represents a context where each \( x_i \) has de Bruijn index \( i \).
Substitution

Term \( (\lambda y. \lambda z. (x \ y) \ (w \ z)) \) under naming context \( v, w, x \) has the following de Bruijn representation:

\[ \lambda. \lambda. (2 \ 1) \ (3 \ 0) \]
Substitution

- Term $(\lambda y. \lambda z. (x \; y) \; (w \; z))$ under naming context $v, w, x$ has the following de Bruijn representation:

  $\lambda. \lambda. (2 \; 1) \; (3 \; 0)$

- Term $(v \; w)$ under naming context $v, w, x$ has the following de Bruijn representation:

  $(2 \; 1)$
Substitution

- Term \((\lambda y. \lambda z. (x \ y) (w \ z))\) under naming context \(v, w, x\) has the following de Bruijn representation:

\[
\lambda. \lambda. (2 \ 1) (3 \ 0)
\]

- Term \((v \ w)\) under naming context \(v, w, x\) has the following de Bruijn representation:

\[(2 \ 1)\]

- Substitution \([x \mapsto (v \ w)](\lambda y. \lambda z. (x \ y) (w \ z))\) will yield the term

\[
\lambda y. \lambda z. ((v \ w) \ y) (w \ z)
\]
Substitution

- Term \((\lambda y. \lambda z. (x \ y) \ (w \ z))\) under naming context \(v, w, x\) has the following de Bruijn representation:

  \[
  \lambda. \lambda. \(2 \ 1\) \(3 \ 0\)
  \]

- Term \((v \ w)\) under naming context \(v, w, x\) has the following de Bruijn representation:

  \[
  (2 \ 1)
  \]

- Substitution \([x \mapsto (v \ w)](\lambda y. \lambda z. (x \ y) \ (w \ z))\) will yield the term

  \[
  \lambda y. \lambda z. ((v \ w) \ y) \ (w \ z)
  \]

- Assuming the naming context is \(v, w, x\), the above term has the following de Bruijn representation:

  \[
  (\lambda. \lambda. ((4 \ 3) \ 1) \ (3 \ 0))
  \]
Substitution

- Term \((\lambda y. \lambda z. (x \ y) (w \ z))\) under naming context \(v, w, x\) has the following de Bruijn representation:
  \[
  \lambda. \lambda. (2 \ 1) (3 \ 0)
  \]

- Term \((v \ w)\) under naming context \(v, w, x\) has the following de Bruijn representation:
  \[(2 \ 1)\]

- Substitution \([x \mapsto (v \ w)](\lambda y. \lambda z. (x \ y) (w \ z))\) will yield the term
  \[
  \lambda y. \lambda z. ((v \ w) \ y) (w \ z)
  \]

- Assuming the naming context is \(v, w, x\), the above term has the following de Bruijn representation: \((\lambda. \lambda. ((4 \ 3) \ 1) (3 \ 0))\)

- Hence, when carrying out substitution, we need to renumber the indices of free variables in the replacement term, and retain the indices of bound variables.

This will be done using the shifting operation, defined next.
Shifting

For substitution, we need to

- renumber the indices of free variables (say, by $d$), and
- retain the indices of bound variables (say, those numbered below $c$).

This is done using the *shifting* operation, defined as follows:

$$\uparrow^d_c (k) = \begin{cases} k & \text{if } k < c \\ k + d & \text{if } k \geq c \end{cases}$$

$$\uparrow^d_c (\lambda. t_1) = \lambda. \uparrow^d_{c+1} (t_1)$$

$$\uparrow^d_c (t_1 t_2) = (\uparrow^d_c t_1 \uparrow^d_c t_2)$$

$$\uparrow^d (t) = \uparrow^d_0 (t)$$

Examples

$$\uparrow^2 (\lambda. \lambda. 1 (0 2)) = \lambda. \lambda. 1 (0 4)$$

$$\uparrow^2 (\lambda. 0 1 (\lambda. 0 1 2)) = \lambda. 0 3 (\lambda. 0 1 4)$$
Shifting

For substitution, we need to
- renumber the indices of free variables (say, by \(d\)), and
- retain the indices of bound variables (say, those numbered below \(c\)).

This is done using the *shifting* operation, defined as follows:

\[
\uparrow_c^d (k) = \begin{cases} 
  k & \text{if } k < c \\
  k + d & \text{if } k \geq c
\end{cases}
\]

\[
\uparrow_c^d (\lambda. \ t_1) = \lambda. \uparrow_{c+1}^d (t_1)
\]

\[
\uparrow_c^d (t_1 \ t_2) = (\uparrow_c^d t_1 \uparrow_c^d t_2)
\]

\[
\uparrow^d (t) = \uparrow_0^d (t)
\]

Examples
- \(\uparrow^2 (\lambda. \lambda. 1 (0 \ 2)) = \)
For substitution, we need to
- renumber the indices of free variables (say, by $d$), and
- retain the indices of bound variables (say, those numbered below $c$).

This is done using the *shifting* operation, defined as follows:

\[
\uparrow^d_c (k) = \begin{cases} 
  k & \text{if } k < c \\
  k + d & \text{if } k \geq c 
\end{cases}
\]

\[
\uparrow^d_c (\lambda. t_1) = \lambda. \uparrow^{d+1}_c (t_1)
\]

\[
\uparrow^d_c (t_1 t_2) = (\uparrow^d_c t_1 \uparrow^d_c t_2)
\]

\[
\uparrow^d (t) = \uparrow^0 (t)
\]

Examples
- $\uparrow^2 (\lambda. \lambda. 1 (0 2)) = \lambda. \lambda. 1 (0 4)$
For substitution, we need to
- renumber the indices of free variables (say, by $d$), and
- retain the indices of bound variables (say, those numbered below $c$).

This is done using the *shifting* operation, defined as follows:

$$\uparrow^d_c (k) = \begin{cases} 
  k & \text{if } k < c \\
  k + d & \text{if } k \geq c
\end{cases}$$

$$\uparrow^d_c (\lambda. t_1) = \lambda. \uparrow^d_{c+1} (t_1)$$

$$\uparrow^d_c (t_1 t_2) = (\uparrow^d_c t_1 \uparrow^d_c t_2)$$

$$\uparrow^d (t) = \uparrow^d_0 (t)$$

Examples
- $\uparrow^2 (\lambda. \lambda. 1 (0 2)) = \lambda. \lambda. 1 (0 4)$
- $\uparrow^2 (\lambda. 0 1 (\lambda. 0 1 2)) =$
For substitution, we need to
- renumber the indices of free variables (say, by $d$), and
- retain the indices of bound variables (say, those numbered below $c$).

This is done using the *shifting* operation, defined as follows:

$$\uparrow^d_c (k) = \begin{cases} 
  k & \text{if } k < c \\
  k + d & \text{if } k \geq c
\end{cases}$$

$$\uparrow^d_c (\lambda. t_1) = \lambda. \uparrow^{d+c+1}_c (t_1)$$

$$\uparrow^d_c (t_1 \ t_2) = (\uparrow^d_c t_1 \ \uparrow^d_c t_2)$$

$$\uparrow^d (t) = \uparrow^d_0 (t)$$

**Examples**

- $\uparrow^2 (\lambda. \lambda. 1 (0 \ 2)) = \lambda. \lambda. 1 (0 \ 4)$
- $\uparrow^2 (\lambda. 0 \ 1 (\lambda. 0 \ 1 \ 2)) = \lambda.0 \ 3 (\lambda. 0 \ 1 \ 4)$
Substitution using Shifting

\[
[j \mapsto s]k = \begin{cases} 
s & \text{if } k = j \\ 
k & \text{otherwise} \end{cases}
\]

\[
[j \mapsto s](\lambda. t_1) = \lambda. [j + 1 \mapsto s](s)\]

\[
[j \mapsto s](t_1 t_2) = ([j \mapsto s] t_1 [j \mapsto s] t_2)
\]

Examples:

\[
[0 \mapsto 1](0 (\lambda. \lambda. 2)) =
\]
Substitution using Shifting

\[ [j \mapsto s]k = \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases} \]

\[ [j \mapsto s](\lambda \cdot t_1) = \lambda \cdot [j + 1 \mapsto s^1 (s)]t_1 \]

\[ [j \mapsto s](t_1 \ t_2) = ([j \mapsto s]t_1 [j \mapsto s]t_2) \]

Examples:

- \[ [0 \mapsto 1](0 (\lambda \cdot \lambda \cdot 2)) = 1 (\lambda \cdot \lambda \cdot 3) \]
Substitution using Shifting

\[ [j \mapsto s]k = \begin{cases} 
  s & \text{if } k = j \\
  k & \text{otherwise}
\end{cases} \]

\[ [j \mapsto s](\lambda. t_1) = \lambda. [j + 1 \mapsto t_1] \]

\[ [j \mapsto s](t_1 t_2) = ([j \mapsto s]t_1 [j \mapsto s]t_2) \]

Examples:

- \([0 \mapsto 1](0 (\lambda. \lambda. 2)) = 1 (\lambda. \lambda. 3)\)
- \([0 \mapsto (1 (\lambda. 2))] (0 (\lambda. 1)) = \)
Substitution using Shifting

\[ [j \mapsto s]k = \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases} \]

\[ [j \mapsto s](\lambda. t_1) = \lambda. [j + 1 \mapsto \uparrow^1 (s)]t_1 \]

\[ [j \mapsto s](t_1 t_2) = ([j \mapsto s]t_1 [j \mapsto s]t_2) \]

Examples:

- \([0 \mapsto 1](0 (\lambda. \lambda. 2)) = 1 (\lambda. \lambda. 3)\)
- \([0 \mapsto (1 (\lambda. 2))](0 (\lambda. 1)) = (1 (\lambda. 2)) (\lambda(2 (\lambda. 3)))\)
Substitution using Shifting

\[ [j \mapsto s]k = \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases} \]

\[ [j \mapsto s](\lambda. t_1) = \lambda. [j + 1 \mapsto \uparrow^1 (s)]t_1 \]

\[ [j \mapsto s](t_1 t_2) = ([j \mapsto s]t_1 [j \mapsto s]t_2) \]

Examples:

- \([0 \mapsto 1](0 (\lambda. \lambda. 2)) = 1 (\lambda. \lambda. 3)\)
- \([0 \mapsto (1 (\lambda. 2))](0 (\lambda. 1)) = (1 (\lambda. 2)) (\lambda(2 (\lambda. 3)))\)
- \([0 \mapsto 1](\lambda. (0 2)) = \lambda. (0 2)\)
Evaluation

In the calculus with symbolic term representation:

\((\lambda x. t_1) t_2 \rightarrow [x \mapsto t_2]t_1\) \hspace{1cm} E-APPABS
Evaluation

In the calculus with symbolic term representation:

\[(\lambda x. \ t_1) \ t_2 \rightarrow [x \mapsto t_2]t_1 \quad \text{E-AppAbs}\]

In the calculus with de Bruijn representation:

\[(\lambda. \ t_1) \ t_2 \rightarrow_{-1} ([0 \mapsto^{1} (t_2)]t_1) \quad \text{E-AppAbs}\]
Evaluation

In the calculus with symbolic term representation:

$$(\lambda x. \, t_1) \, t_2 \rightarrow [x \mapsto t_2]t_1 \quad \text{E-AppAbs}$$

In the calculus with de Bruijn representation:

$$(\lambda. \, t_1) \, t_2 \rightarrow \uparrow^{-1} ([0 \mapsto \uparrow^1 (t_2)]t_1) \quad \text{E-AppAbs}$$

- The outer $\lambda$ is removed after application, so the indices have to shift \textit{down} by 1.
Evaluation

In the calculus with symbolic term representation:

$$(\lambda x. \ t_1) \ t_2 \rightarrow [x \mapsto t_2]t_1$$ \hspace{1cm} \text{E-AppAbs}

In the calculus with de Bruijn representation:

$$(\lambda. \ t_1) \ t_2 \rightarrow \uparrow^{-1} ([0 \mapsto \uparrow^{1} (t_2)]t_1)$$ \hspace{1cm} \text{E-AppAbs}

- The outer $\lambda$ is removed after application, so the indices have to shift \textit{down} by 1.
- Indices in argument ($t_2$) should \textit{not} be changed in the end, so we shifting them \textit{up} by 1 first.
Evaluation

In the calculus with symbolic term representation:

\[(\lambda x. \; t_1) \; t_2 \rightarrow [x \mapsto t_2]t_1 \quad \text{E-AppAbs}\]

In the calculus with de Bruijn representation:

\[(\lambda. \; t_1) \; t_2 \rightarrow \uparrow^{-1} ([0 \mapsto \uparrow^1 (t_2)]t_1) \quad \text{E-AppAbs}\]

- The outer \(\lambda\) is removed after application, so the indices have to shift \textit{down} by 1.
- Indices in argument \((t_2)\) should \textit{not} be changed in the end, so we shifting them \textit{up} by 1 first.
  - Consider \((\lambda x. \; w \; x \; v) \; (\lambda y. \; (w \; y))\), whose de Bruijn representation is \((\lambda. \; 1 \; 0 \; 2) \; (\lambda. \; 1 \; 0)\) (assuming naming context \(v, w\)).
Evaluation

In the calculus with symbolic term representation:

\[ (\lambda x. \ t_1) \ t_2 \rightarrow [x \mapsto t_2]t_1 \quad \text{E-AppAbs} \]

In the calculus with de Bruijn representation:

\[ (\lambda. \ t_1) \ t_2 \rightarrow \uparrow^{-1} ([0 \mapsto \uparrow^1 (t_2)]t_1) \quad \text{E-AppAbs} \]

- The outer \( \lambda \) is removed after application, so the indices have to shift down by 1.
- Indices in argument \( t_2 \) should not be changed in the end, so we shifting them up by 1 first.
  - Consider \((\lambda x. \ w \times v) \ (\lambda y. \ (w \ y))\), whose de Bruijn representation is \((\lambda. \ 1 \ 0 \ 2) \ (\lambda. \ 1 \ 0)\) (assuming naming context \( v, w \)).
  - The result of the application is \( w \ (\lambda y. \ w \ y) \ v \).
Evaluation

In the calculus with symbolic term representation:

\[(\lambda x. t_1) \; t_2 \rightarrow [x \mapsto t_2]t_1 \quad \text{E-AppAbs}\]

In the calculus with de Bruijn representation:

\[(\lambda. t_1) \; t_2 \rightarrow \uparrow^{-1} ([0 \mapsto \uparrow^1 (t_2)]t_1) \quad \text{E-AppAbs}\]

- The outer \(\lambda\) is removed after application, so the indices have to shift down by 1.
- Indices in argument \((t_2)\) should not be changed in the end, so we shifting them up by 1 first.
  - Consider \((\lambda x. w \; x \; v) \; (\lambda y. (w \; y))\), whose de Bruijn representation is \((\lambda. 1 \; 0 \; 2) \; (\lambda. 1 \; 0)\) (assuming naming context \(v, w\)).
  - The result of the application is \(w \; (\lambda y. w \; y) \; v\).
  - \(\uparrow^1 (\lambda. 1 \; 0) = \lambda. 2 \; 0\)
Evaluation

In the calculus with symbolic term representation:

\[(\lambda x. \ t_1) \ t_2 \to [x \mapsto t_2]t_1 \quad \text{E-AppAbs}\]

In the calculus with de Bruijn representation:

\[(\lambda. \ t_1) \ t_2 \to \uparrow^{-1} ([0 \mapsto \uparrow^1 (t_2)]t_1) \quad \text{E-AppAbs}\]

- The outer \(\lambda\) is removed after application, so the indices have to shift \textit{down} by 1.
- Indices in argument \((t_2)\) should \textit{not} be changed in the end, so we shifting them \textit{up} by 1 first.
  - Consider \((\lambda x. \ w \ x \ v) (\lambda y. \ (w \ y))\), whose de Bruijn representation is \((\lambda. \ 1 \ 0 \ 2) (\lambda. \ 1 \ 0)\) (assuming naming context \(v, w\)).
  - The result of the application is \(w (\lambda y. \ w \ y) \ v\).
  - \(\uparrow^1 (\lambda. \ 1 \ 0) = \lambda. \ 2 \ 0\)
  - \([0 \mapsto (\lambda. \ 2 \ 0)](1 \ 0 \ 2) = 1 (\lambda. \ 2 \ 0) \ 2\)
Evaluation

In the calculus with symbolic term representation:

\[(\lambda x. \ t_1) \ t_2 \rightarrow [x \mapsto t_2]t_1 \quad \text{E-AppAbs}\]

In the calculus with de Bruijn representation:

\[(\lambda. \ t_1) \ t_2 \rightarrow^{\uparrow^{-1}} ([0 \mapsto^{\uparrow^1} (t_2)]t_1) \quad \text{E-AppAbs}\]

- The outer \(\lambda\) is removed after application, so the indices have to shift down by 1.
- Indices in argument \((t_2)\) should not be changed in the end, so we shifting them up by 1 first.
  - Consider \((\lambda x. \ w \ x \ v) \ (\lambda y. \ (w \ y))\), whose de Bruijn representation is \((\lambda. \ 1 \ 0 \ 2) \ (\lambda. \ 1 \ 0)\) (assuming naming context \(v, w\)).
  - The result of the application is \(w \ (\lambda y. \ w \ y) \ v\).
  - \(\uparrow^1 (\lambda. \ 1 \ 0) = \lambda. \ 2 \ 0\)
  - \([0 \mapsto (\lambda. \ 2 \ 0)](1 \ 0 \ 2) = 1 \ (\lambda. \ 2 \ 0) \ 2\)
  - \(\uparrow^{-1} (1 \ (\lambda. \ 2 \ 0) \ 2) = 0 \ (\lambda. \ 1 \ 0) \ 1\)