Introduction

Prolog

Principles of Programming Languages

CSE 526
Logic and Programs

“All men are mortal; Socrates is a man; Hence Socrates is mortal”
Logic and Programs

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\[ \forall X. \ man(X) \Rightarrow \ mortal(X) \]
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\[ \forall X. \text{man}(X) \Rightarrow \text{mortal}(X) \]

\[ \text{man(socrates)} \]
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\[ \forall X. \text{man}(X) \Rightarrow \text{mortal}(X) \]

\[ \text{man}(\text{socrates}) \]

\[
\begin{array}{c}
\text{man}(\text{socrates}) & \text{R-MAN} \\
\text{man}(x) & \text{R-MORTAL} \\
\hline
\text{mortal}(x) & \\
\end{array}
\]
Logic and Programs

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\hline \\
\text{mortal}(x)
\end{array} \]

- Predicate logic
Logic and Programs

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  \[ \forall X. \, \text{man}(X) \Rightarrow \text{mortal}(X) \]
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- Predicate logic
  - Predicates (e.g. \text{man}, \text{mortal}) which define sets.
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Predicate logic

- Predicates (e.g. $man$, $mortal$) which define sets.
- Atoms (e.g. $socrates$) which are data values
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“All men are mortal; Socrates is a man; Hence Socrates is mortal”

\[ \forall X. \text{man}(X) \Rightarrow \text{mortal}(X) \]

\[ \text{man}(\text{socrates}) \]

\[ \frac{\text{man}(x)}{\text{mortal}(x)} \]

Predicate logic

- Predicates (e.g. \text{man}, \text{mortal}) which define sets.
- Atoms (e.g. \text{socrates}) which are data values
- Variables (e.g. \(X\)) which range over data values
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“All men are mortal; Socrates is a man; Hence Socrates is mortal”

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\begin{array}{c}
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\text{man(x)} & \text{R-MORTAL} \\
\text{mortal(x)} & \text{R-MORTAL}
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\]

Predicate logic

- Predicates (e.g. `man`, `mortal`) which define sets.
- Atoms (e.g. `socrates`) which are data values
- Variables (e.g. `X`) which range over data values
- Rules (e.g. `\forall X. \text{man}(X) \Rightarrow \text{mortal}(X)`) which define relationships between predicates.
Inference Rules and Logic Programs

\[
\frac{\text{man}(socrates)}{\text{man}(x)} \quad \text{R-MAN} \\
\frac{\text{man}(x)}{\text{mortal}(x)} \quad \text{R-MORTAL}
\]

Logic "Program":

\[
\text{man}(socrates).
\text{mortal}(X) \iff \text{man}(X).
\]

Queries:

?- \text{mortal}(socrates).
yes

?- \text{mortal}(X).
X = socrates; no
Inference Rules and Logic Programs

Logic “Program”:
man(socrates).
mortal(X) :- man(X).

\[
\begin{array}{c}
\text{man(socrates)} & \text{R-MAN} \\
\text{man(x)} & \text{R-MORTAL} \\
\text{mortal(x)}
\end{array}
\]
Inference Rules and Logic Programs

```
\[ \begin{array}{c}
  \text{man}(socrates) \quad \text{R-MAN} \\
  \text{man}(x) \quad \text{R-MORTAL} \\
  \text{mortal}(x)
\end{array} \]
```

Logic “Program”:

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man(socrates).
mortal(X) :- man(X).
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```
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Inference Rules and Logic Programs

Logic “Program”:

\[
\begin{align*}
\text{man}(socrates) & \quad \text{R-MAN} \\
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\text{mortal}(x) & \\
\end{align*}
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\text{Logic “Program”}: \\
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\end{align*}
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Queries:

\[
\begin{align*}
? - \text{mortal}(socrates). \\
\text{yes} \\
? - \text{mortal}(X). \\
X = socrates; \\
\end{align*}
\]
Inference Rules and Logic Programs

Logic “Program”:
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\text{man(socrates).}
\]
\[
\text{mortal(X) :- man(X).}
\]

Queries:
\[
?- \text{mortal(socrates).}
\]
\[
\text{yes}
\]
\[
?- \text{mortal(X).}
\]
\[
X=\text{socrates};
\]
\[
\text{no}
\]
Prolog

Programming in Logic

- Early development: Kowalski & van Emden (Edinburgh); Colmerauer (Marseilles) (early ’70s)
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  - Memoization: Tamaki & Sato (Tokyo); Warren et al (Stony Brook)
Prolog Systems

- SWI Prolog (www.swi-prolog.org)
  - Can be obtained for free and installed on Windows, Linux, Mac.
  - Has a good development environment (command completion, help, graphical debugger, etc.)
  - On compute* (Unix) servers: ~cram/bin/swipl

- XSB Prolog (xsb.sourceforge.net)
  - Can be obtained for free and installed on Windows, Linux, Mac.
  - Supports a powerful extension (memoization) to Prolog
  - Command-line interface (e.g. no graphical debugger)
  - On compute* (Unix) servers: ~cram/bin/xsb
Using Prolog Systems

- Prolog programs are in files with ".pl" extension (".P" for XSB)
- Prolog systems typically support an interactive mode.
- "[filename]." to compile and load a program in filename.pl (filename.P in XSB).
- "halt." to exit the system.
Logic Programs

- Programs are a set of *rules* (also called *clauses*).
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- Example:
  
  \[
  \text{inc}(X,Y) :- Y \text{ is } X+1. 
  \]
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  \[ \text{inc}(X,Y) :\text{-} \ Y \text{ is } X+1. \]

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- Example:
  
  \[
  \text{inc}(X,Y) :- Y \text{ is } X+1.
  \]
  
  - \(X\) and \(Y\) are *variables*.
  - \(\text{inc}\) is a predicate.
Logic Programs

- Programs are a set of *rules* (also called *clauses*).
- *Predicates* in a logic program are analogous to *procedures* in imperative programs.
- One or more rules are used to define a predicate.
- Example:
  
  \[
  \text{inc}(X,Y) :\neg \ Y \text{ is } X+1.
  \]
  
  - \(X\) and \(Y\) are *variables*.
  - \text{inc} is a predicate.
  - The predicate is defined using a single rule.
Logic Programs (contd.)

\[ \text{inc}(X, Y) :- Y \text{ is } X+1. \]

- "\:-" separates the *body* of the rule from its head.
- "\(X\)" and "\(Y\)" are also "parameters" of the predicate. In this case, \(X\) is the input parameter, and \(Y\) is the return parameter (where the return values are stored).
- "\(Y \text{ is } X+1\)" defines \(Y\) in terms of \(X\).
- The period ("\.:") marks the end of a rule.
- The predicate is *called* by giving values to its parameters. e.g.
  \[ \text{inc}(6, B) \text{ returns with } B=7. \]
  \[ \text{inc}(11, B) \text{ returns with } B=12. \]
Syntax of Prolog

- **Variables** are identifiers that begin with an upper case letter or underscore.
  - An underscore, by itself, represents an *anonymous variable*.
- **Predicate** names (and later, data structure symbols) are identifiers that begin with a lower case letter.
- All variables are *local* to the clause in which they occur.
- Different occurrences of the same variable in a clause denote the same data.
- Variables need not be declared, and have no type.
How Prolog Works (An Example)

big(bear).
big(elephant).

brown(bear).

black(cat).

small(cat).

gray(elephant).

dark(Z) :- black(Z).
dark(Z) :- brown(Z).

dangerous(X) :- dark(X), big(X).
Derivations

big(bear).
big(elephant).
small(cat).
brown(bear).
brown(elephant).
black(cat).
black(elephant).
grey(elephant).
dark(Z) :- black(Z).
dark(Z) :- brown(Z).
dark(Z) :- black(Z), big(Z).
dark(Z) :- brown(Z), big(Z).
dangerous(X) :- dark(X), big(X).

dangerous(Q)
Derivations

\[
\begin{align*}
\text{big(bear).} & \quad \text{brown(bear).} & \quad \text{dark(Z) :- black(Z).} \\
\text{big(elephant).} & \quad \text{black(cat).} & \quad \text{dark(Z) :- brown(Z).} \\
\text{small(cat).} & \quad \text{gray(elephant).} & \\
\text{dangerous(X) :- dark(X), big(X).} & \\
\text{dangerous(Q) :- dark(Q), big(Q).}
\end{align*}
\]
Derivations

\[
\text{big(bear). brown(bear). dark(Z) :- black(Z).}
\]
\[
\text{big(elephant). black(cat). dark(Z) :- brown(Z).}
\]
\[
\text{small(cat). gray(elephant).}
\]
\[
\text{dangerous(X) :- dark(X), big(X).}
\]
\[
dangerous(Q) \mid \text{dark}(X), \text{big}(X) \mid \text{dark}(Q), \text{big}(Q)
\]
\[
dark(X) \mid \text{black}(X) \mid \text{black}(Q), \text{big}(Q)
\]
Derivations

\[
\begin{align*}
\text{big(bear).} & & \text{brown(bear).} & & \text{dark(Z) :- black(Z).} \\
\text{big(elephant).} & & \text{black(cat).} & & \text{dark(Z) :- brown(Z).} \\
\text{small(cat).} & & \text{gray(elephant).} & & \\
\text{dangerous(X) :- dark(X), big(X).} & & \\
\text{dangerous(Q)} & & \\
\text{dangerous(X) :- } & & \\
\text{dark(X), big(X) } & & \\
\text{dark(Q), big(Q) } & & \\
\text{dark(X) :- } & & \\
\text{black(X)} & & \\
\text{black(Q), big(Q) } & & \\
\text{black(cat)} & & \\
\text{big(cat)} & & \\
\end{align*}
\]
Derivations

\[
\text{big(bear).} \quad \text{brown(bear).} \quad \text{dark(Z) :- black(Z).}
\]
\[
\text{big(elephant).} \quad \text{black(cat).} \quad \text{dark(Z) :- brown(Z).}
\]
\[
\text{small(cat).} \quad \text{gray(elephant).}
\]
\[
\text{dangerous(X) :- dark(X), big(X).}
\]

```
  dangerous(Q) :-
     dangerous(X), big(X) |
     dark(Q), big(Q)
  dark(X) :-
     black(X) | black(Q), big(Q)
  black(cat) | big(cat)

fail
```

Programming Languages

Logic Programming
big(bear).  brown(bear).  dark(Z) :- black(Z).
big(elephant).  black(cat).  dark(Z) :- brown(Z).
small(cat).  gray(elephant).
dangerous(X) :- dark(X), big(X).

dangerous(Q)
  dangerous(X) :-
    dark(X), big(X)
    dark(Q), big(Q)

dark(X) :-
  black(Q), big(Q)
  brown(Q), big(Q)

black(Q)
  big(Q)

black(cat)

big(cat)

fail
Derivations

\[
\begin{align*}
\text{big(bear).} & \quad \text{brown(bear).} & \quad \text{dark(Z) :- black(Z).} \\
\text{big(elephant).} & \quad \text{black(cat).} & \quad \text{dark(Z) :- brown(Z).} \\
\text{small(cat).} & \quad \text{gray(elephant).} \\
\text{dangerous(X) :- dark(X), big(X).} \\
\end{align*}
\]

\[
\text{dangerous(Q) :}-
\begin{align*}
\text{dangerous(X) :-} \\
\text{dark(X), big(X)} \\
\text{dark(Q), big(Q)} \\
\text{dark(X) :-} \\
\text{black(X)} \\
\text{black(Q), big(Q)} \\
\text{black(cat)} \\
\text{big(Q)} \\
\text{brown(Q), big(Q)} \\
\text{brown(bear)} \\
\text{big(cat)} \\
\text{big(bear)} \\
\text{fail}
\end{align*}
\]
Derivations

big(bear).
brown(bear).
dark(Z) :- black(Z).
big(elephant).
black(cat).
dark(Z) :- brown(Z).
small(cat).
gray(elephant).

dangerous(X) :- dark(X), big(X).

dangerous(X) :-
dark(X), big(X)
  dark(Q), big(Q)

dark(X) :-
  black(X)
  black(Q), big(Q)
  brown(Q), big(Q)

black(cat)
big(cat)

fail

big(bear)
succeed
How Prolog Works (the procedure)

- A *query* is, in general, a conjunction of *goals*
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- To prove $G_1, G_2, \ldots, G_n$: 
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- A *query* is, in general, a conjunction of *goals*
- To prove $G_1, G_2, \ldots, G_n$:
  - Find a clause $H : -B_1, B_2, \ldots, B_k$ such that $G_1$ and $H$ match.
How Prolog Works (the procedure)

- A query is, in general, a conjunction of goals
- To prove $G_1, G_2, \ldots, G_n$:
  - Find a clause $H : \neg B_1, B_2, \ldots, B_k$ such that $G_1$ and $H$ match.
  - Under that substitution for variables, prove $B_1, B_2, \ldots, B_k, G_2, \ldots, G_n$. 
How Prolog Works (the procedure)

- A query is, in general, a conjunction of goals
- To prove $G_1, G_2, \ldots, G_n$:
  - Find a clause $H : -B_1, B_2, \ldots, B_k$ such that $G_1$ and $H$ match.
  - Under that substitution for variables, prove $B_1, B_2, \ldots, B_k, G_2, \ldots, G_n$.
  - If nothing is left to prove then the proof is complete. If there are no more clauses to match, the proof attempt fails.
How Prolog Works (an example)

To prove dangerous(Q):

1. Select dangerous(X) :- dark(X), big(X) and prove dark(Q), big(Q).
2. To prove dark(Q) select the first clause of dark, i.e. dark(Z) :- black(Z), and prove black(Q), big(Q).
3. Now select the fact black(cat) and prove big(cat).
   This proof attempt fails!
4. Go back to step 2, and select the second clause of dark, i.e. dark(Z) :- brown(Z), and prove brown(Q), big(Q).
5. Now select brown(bear) and prove big(bear).
6. Select the fact big(bear).
   There is nothing left to prove, so the proof is complete.
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To prove dangerous(Q):

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3. Now select the fact black(cat) and prove big(cat).
   
   This proof attempt fails!
How Prolog Works (an example)

To prove $\text{dangerous}(Q)$:

1. Select $\text{dangerous}(X) :- \text{dark}(X), \text{big}(X)$ and prove $\text{dark}(Q), \text{big}(Q)$.

2. To prove $\text{dark}(Q)$ select the first clause of $\text{dark}$, i.e. $\text{dark}(Z) :- \text{black}(Z)$, and prove $\text{black}(Q), \text{big}(Q)$.

3. Now select the fact $\text{black}(\text{cat})$ and prove $\text{big}(\text{cat})$.
   
   This proof attempt fails!

4. Go back to step 2, and select the second clause of $\text{dark}$, i.e. $\text{dark}(Z) :- \text{brown}(Z)$, and prove $\text{brown}(Q), \text{big}(Q)$.
How Prolog Works (an example)

To prove dangerous(Q):

1. Select dangerous(X) :- dark(X), big(X) and prove dark(Q), big(Q).

2. To prove dark(Q) select the first clause of dark, i.e. dark(Z) :- black(Z), and prove black(Q), big(Q).

3. Now select the fact black(cat) and prove big(cat).
   
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3. Now select the fact black(cat) and prove big(cat).
   This proof attempt fails!

4. Go back to step 2, and select the second clause of dark, i.e. dark(Z) :- brown(Z), and prove brown(Q), big(Q).

5. Now select brown(bear) and prove big(bear).

6. Select the fact big(bear).
   There is nothing left to prove, so the proof is complete.
Data Representation in Prolog

- Prolog has no notion of data types
- All data is represented as *terms*, which can be:
  - Variables
  - Non-variable Terms
    - Atomic data (Integers, floating point numbers, constants, ...)
    - Compound Terms (Structures)
Atomic Data

- **Numeric constants**: Integers, floating point numbers (e.g. 1024, -42, 3.1415, 6.023e23 ...)

- **Atoms**:
  - Strings of characters enclosed in single quotes (e.g. ’cram’, ’Stony Brook’)
  - Identifiers: sequence of letters, digits, underscore, beginning with a letter (e.g. cram, r2d2, x_24).
Data Structures

- If \( f \) is an identifier and \( t_1, t_2, \ldots t_n \) are terms, then \( f(t_1, t_2, \ldots t_n) \) is a term.

- In the above, \( f \) is called a function symbol (or functor) and \( t_i \) is an argument.

- Structures are used to group related data items together (in some ways similar to struct in C and objects in Java).

- Structures are used to construct trees (and, as a special case, lists).
Trees

- Example: expression trees:
  \[
  \text{plus}(\text{minus}(\text{num}(3), \text{num}(1)), \text{star}(\text{num}(4), \text{num}(2)))
  \]

- Data structures may have variables. And the same variable may occur multiple times in a data structure.
(We’ll extend this to *unification* later)

- $t_1 = t_2$: find substitutions for variables in $t_1$ and $t_2$ that make the two terms identical.

$\begin{array}{c}
\text{plus} \\
\text{star} \\
\text{num} \\
3 \\
\text{minus} \\
\text{num} \\
X
\end{array} \quad \begin{array}{c}
\text{plus} \\
\text{star} \\
\text{num} \\
? \\
\text{num} \\
Y \\
2
\end{array} = \begin{array}{c}
\text{plus} \\
\text{star} \\
\text{num} \\
3 \\
\text{num} \\
1 \\
4 \\
2
\end{array}$
Matching

(We’ll extend this to \textit{unification} later)

- \( t_1 = t_2 \): find substitutions for variables in \( t_1 \) and \( t_2 \) that make the two terms identical.

\[
\begin{array}{c}
\text{plus} \\
\text{star} \\
\text{minus} \\
\text{num} \\
3 \\
\text{num} \\
X \\
\text{num} \\
Y \\
\text{num} \\
2
\end{array}
\quad \text{=} \quad
\begin{array}{c}
\text{plus} \\
\text{star} \\
\text{minus} \\
\text{num} \\
3 \\
\text{num} \\
1 \\
\text{num} \\
4 \\
\text{num} \\
2
\end{array}
\]

Yes, with \( X = 1, \ Y = 4 \).
Matching (contd.)

\[
\begin{array}{c}
\text{\texttt{plus}} \\
\text{\texttt{minus}} \\
\text{\texttt{star}} \\
\text{\texttt{num}} \\
3 \\
\end{array}
\quad = 

\begin{array}{c}
\text{\texttt{plus}} \\
\text{\texttt{minus}} \\
\text{\texttt{star}} \\
\text{\texttt{num}} \\
X \\
\end{array}
\quad \begin{array}{c}
\text{\texttt{num}} \\
Y \\
2 \\
\end{array}
\]

Yes, with \(X = 1, Y = 4\).
Matching (contd.)

Yes, with $X = 1$, $Y = 4$. 
Matching (contd.)

```
plus
  minus num
  star num

3 1 4 2
```

```
plus
  minus num
  star num

X X X 2
```
No! $X$ cannot be 1 and 4 at the same time.
Accessing arguments of a structure

- Matching is the predominant means for accessing a structures arguments.
- Let \( \text{date('Sep', 1, 2005)} \) be a structure used to represent dates, with the month, day and year as the three arguments (in that order).
- Then \( \text{date(M, D, Y)} = \text{date('Sep', 1, 2005)} \) makes \( M = \text{'Sep'} \), \( D = 1 \), \( Y = 2005 \).
- If we want to get only the day, we can write \( \text{date(\_\_, D, \_\_\_)} = \text{date('Sep', 1, 2005)} \). Then we get \( D = 1 \).
Lists

Prolog uses a special syntax to represent and manipulate lists.

- \([1,2,3,4]\): represents a list with 1, 2, 3 and 4, respectively.
- This can also be written as \([1 \mid [2,3,4]]\): a list with 1 as the head (its first element) and \([2,3,4]\) as its tail (the list of remaining elements).
- If \(X = 1\) and \(Y = [2,3,4]\) then \([X\mid Y]\) is same as \([1,2,3,4]\).
- The empty list is represented by \([\ ]\).
- The symbol “\(\mid\)” (called cons) and is used to separate the beginning elements of a list from its tail.
  For example: \([1,2,3,4] = [1 \mid [2,3,4]]\)
  \(= [1 \mid [2 \mid [3,4]]]\)
  \(= [1,2 \mid [3,4]]\).
Lists are special cases of trees. For instance, the list \([1,2,3,4]\) is represented by the following structure:

![Tree Diagram](image)

- The function symbol `.\(2\)` is the list constructor.
- \([1,2,3,4]\) is same as `.\((1, .\((2, .\((3, .\((4, [])))))\))`
First example: `member/2`, to find if a given element occurs in a list:
First example: member/2, to find if a given element occurs in a list:

**The program:**

```
member(X, [X|_]).
member(X, [_|Ys]) :- member(X, Ys).
```
Programming with Lists — I

First example: \texttt{member/2}, to find if a given element occurs in a list:

\textbf{The program:}

\begin{verbatim}
member(X, [X|_]).
member(X, [_|Ys]) :- member(X, Ys).
\end{verbatim}

\textbf{Example queries:}

\begin{verbatim}
member(s, [l,i,s,t])
member(X, [l,i,s,t])
member(f(X), [f(1), g(2), f(3), h(4), f(5)])
\end{verbatim}
append/3: concatenate two lists to form the third list.
Programming with Lists — II

append/3: concatenate two lists to form the third list.

The program:

append([], L, L).
append([X|Xs], Ys, [X|Zs]) :- append(Xs, Ys, Zs).

Example queries:
append([f,i,r], [s,t], L)
append(X, Y, [s,e,c,o,n,d])
append(X, [t,h], [f,o,u,r,t,h])
Programming with Lists — II

append/3: concatenate two lists to form the third list.

The program:

```
append([], L, L).
append([X|Xs], Ys, [X|Zs]) :- append(Xs, Ys, Zs).
```

Example queries:

```
append([f,i,r], [s,t], L)
append(X, Y, [s,e,c,o,n,d])
append(X, [t,h], [f,o,u,r,t,h])
```
Define a predicate, `len/2` that finds the length of a list (first argument).
Define a predicate, \texttt{len/2} that finds the length of a list (first argument).

**The program:**

\begin{verbatim}
len([], 0).
len([_|Xs], N+1) :- len(Xs, N).
\end{verbatim}
Define a predicate, `len/2` that finds the length of a list (first argument).

**The program:**

```prolog
len([], 0).
len([_|Xs], N+1) :- len(Xs, N).
```

**Example queries:**

```prolog
len([], X)
len([l,i,s,t], 4)
len([l,i,s,t], X)
```
Arithmetic

| ?- 1+2 = 3.

no
Arithmetic

| ?- 1+2 = 3. 

no

- In *Predicate logic*, the basis for Prolog, the only symbols that have a meaning are the predicates themselves.
Arithmetic

| ?- 1+2 = 3. |

no

- In *Predicate logic*, the basis for Prolog, the only symbols that have a meaning are the predicates themselves.
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?- 1+2 = 3.

no

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- Meaning for arithmetic expressions is given by the *built-in* predicate “is”: 
Introduction

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Unification

Arithmetic

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- In *Predicate logic*, the basis for Prolog, the only symbols that have a meaning are the predicates themselves.
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- Meaning for arithmetic expressions is given by the *built-in* predicate “is”:
  - X is 1 + 2 succeeds, binding X to 3.
Arithmetic

\[ ?- 1+2 = 3. \]

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- In \textit{Predicate logic}, the basis for Prolog, the only symbols that have a meaning are the predicates themselves.
- In particular, function symbols are uninterpreted: have no special meaning and can only be used to construct data structures.
- Meaning for arithmetic expressions is given by the \textit{built-in} predicate “\textit{is}”:
  - X \textit{is} 1 + 2 succeeds, binding X to 3.
  - 3 \textit{is} 1 + 2 succeeds.
Arithmetic

| ?- 1+2 = 3.  

no

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- In particular, function symbols are uninterpreted: have no special meaning and can only be used to construct data structures.
- Meaning for arithmetic expressions is given by the *built-in* predicate “is”:
  - X is 1 + 2 succeeds, binding X to 3.
  - 3 is 1 + 2 succeeds.
  - General form: R is E where E is an expression to be evaluated and R is matched with the expression’s value.
| ?- 1+2 = 3. 

no

- In *Predicate logic*, the basis for Prolog, the only symbols that have a meaning are the predicates themselves.
- In particular, function symbols are uninterpreted: have no special meaning and can only be used to construct data structures.
- Meaning for arithmetic expressions is given by the *built-in* predicate “is”:
  - X is 1 + 2 succeeds, binding X to 3.
  - 3 is 1 + 2 succeeds.
  - General form: \( R \ is \ E \) where \( E \) is an expression to be evaluated and \( R \) is matched with the expression’s value.
  - Y is X + 1 will give an error if X does not (yet) have a value.
The list length example revisited

Define a predicate, `length/2` that finds the length of a list (first argument).

**The program:**

```
length([], 0).
length([_|Xs], M) :- length(Xs, N), M is N+1.
```
The list length example revisited

Define a predicate, `length/2` that finds the length of a list (first argument).

**The program:**

```
length([], 0).
length([_|Xs], M) :- length(Xs, N), M is N+1.
```

**Example queries:**

```
length([], X)  
length([l,i,s,t], 4)  
length([l,i,s,t], X)  
length(List, 4)
```
Conditional Evaluation

Consider the computation of $n!$, i.e. the factorial of $n$.

factorial(N, F) :- ...

- $N$ is the input parameter; and $F$ is the output parameter.
Conditional Evaluation

Consider the computation of $n!$, i.e. the factorial of $n$.

```
factorial(N, F) :- ... 
```

- $N$ is the input parameter; and $F$ is the output parameter.
- The body of the rule specifies how the output is related to the input.
Conditional Evaluation

Consider the computation of $n!$, i.e. the factorial of $n$.

```
factorial(N, F) :- ...
```

- $N$ is the input parameter; and $F$ is the output parameter.
- The body of the rule specifies how the output is related to the input.
- For factorial, there are two cases: $N \leq 0$ and $N > 0$. 
Conditional Evaluation

Consider the computation of $n!$, i.e. the factorial of $n$.

\[ \text{factorial}(N, F) :- \ldots \]

- $N$ is the input parameter; and $F$ is the output parameter.
- The body of the rule specifies how the output is related to the input.
- For factorial, there are two cases: $N \leq 0$ and $N > 0$.
  - $N \leq 0$: $F = 1$
Conditional Evaluation

Consider the computation of $n!$, i.e. the factorial of $n$.

factorial(N, F) :- ...

- $N$ is the input parameter; and $F$ is the output parameter.
- The body of the rule specifies how the output is related to the input.
- For factorial, there are two cases: $N \leq 0$ and $N > 0$.
  - $N \leq 0$: $F = 1$
  - $N > 0$: $F = N \times (N - 1)!$
Consider the computation of $n!$, i.e. the factorial of $n$.

factorial(N, F) :- ...

- $N$ is the input parameter; and $F$ is the output parameter.
- The body of the rule specifies how the output is related to the input.
- For factorial, there are two cases: $N \leq 0$ and $N > 0$.
  - $N \leq 0$: $F = 1$
  - $N > 0$: $F = N \cdot (N - 1)!$

factorial(N, F) :-
  (N > 0
   -> N1 is N-1, factorial(N1, F1), F is N*F1
   ; F = 1
  ).
Assignments with arithmetic expressions is done using the keyword “is”.

If-then-else is written as `(cond -> then-part; else-part)`.
Assignments with arithmetic expressions is done using the keyword “is”.

If-then-else is written as (cond -> then-part ; else-part)
Assignments with arithmetic expressions is done using the keyword “is”.

If-then-else is written as (cond -> then-part ; else-part)

If more than one action needs to be performed in a rule, they are written one after another, separated by a comma.
Assignments with arithmetic expressions is done using the keyword “is”.

If-then-else is written as ( cond -> then-part ; else-part )

If more than one action needs to be performed in a rule, they are written one after another, separated by a comma.

Arithmetic expressions are not directly used as arguments when calling a predicate; they are first evaluated, and then passed to the called predicate.
Arithmetic Operators

- Integer/Floating Point operators: +, -, *, /
- Integer operators: mod, // (div)
- Int ↔ Float operators: floor, ceiling
- Comparison operators: <, >, =<, >=, :==, =\=
Sequences, revisited

append/3: concatenate two lists to form the third list (sometimes called conc/3).
Sequences, revisited

append/3: concatenate two lists to form the third list (sometimes called conc/3).

The program:
Sequences, revisited

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The program:
append([], L, L).
Sequences, revisited

append/3: concatenate two lists to form the third list (sometimes called conc/3).

The program:

```prolog
append([], L, L).
append([X|Xs], Ys, [X|Zs]) :- append(Xs, Ys, Zs).
```

Example queries:

- `append([f,i,r], [s,t], L)`
- `append(X, [s,e,c,o,n,d])`
- `append(X, [t,h], [f,o,u,r,t,h])`
Sequences, revisited

append/3: concatenate two lists to form the third list (sometimes called conc/3).

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```
append([], L, L).
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```

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Example queries:
- append([f,i,r], [s,t], L)
- append(X, Y, [s,e,c,o,n,d])
- append(X, [t,h], [f,o,u,r,t,h])
Mystery Program

\[ m(X, X). \]
\[ m(X1, X5) :- a(X1, X2), m(X2, X3), b(X3, X4), m(X4, X5). \]
\[ a([0|Y], Y). \]
\[ b([1|Y], Y). \]
Mystery Program

\[ m(X, X). \]
\[ m(X_1, X_5) :\neg a(X_1, X_2), m(X_2, X_3), b(X_3, X_4), m(X_4, X_5). \]

\[ a([0|Y], Y). \]
\[ b([1|Y], Y). \]

?- \( m([0,1,0,0,1,1], L). \)
Mystery Program

\[
\begin{align*}
m(X, X). \\
m(X_1, X_5) & : - a(X_1, X_2), m(X_2, X_3), b(X_3, X_4), m(X_4, X_5). \\
a([0|Y], Y). \\
b([1|Y], Y).
\end{align*}
\]

?- \ m([0,1,0,0,1,1], L).

\[
L = [0,1,0,0,1,1]
\]
Mystery Program

\[
m(X, X).
m(X1, X5) :\neg\ a(X1, X2), m(X2, X3), b(X3, X4), m(X4, X5).
\]

\[
a([0|Y], Y).
b([1|Y], Y).
\]

?- m([0,1,0,0,1,1], L).

\[
L=[0,1,0,0,1,1]
L=[0,0,1,1]
L=[0,0,1,1]
\]
Mystery Program

\[
m(X, X).
m(X_1, X_5) :- a(X_1, X_2), m(X_2, X_3), b(X_3, X_4), m(X_4, X_5).
\]

\[
a([0|Y], Y).
b([1|Y], Y).
\]

?- m([0,1,0,0,1,1], L).

\[
L=[0,1,0,0,1,1] \\
L=[0,0,1,1] \\
L=[]
\]
Mystery Program

m(X, X).
m(X1, X5) :- a(X1, X2), m(X2, X3), b(X3, X4), m(X4, X5).

a([0|Y], Y).
b([1|Y], Y).

?- m([0,1,0,0,1,1], L).
   L=[0,1,0,0,1,1]
   L=[0,0,1,1]
   L=[]

?- m([0,0,1,1,1,0], L).
Mystery Program

\[ m(X, X). \]
\[ m(X_1, X_5) :- a(X_1, X_2), m(X_2, X_3), b(X_3, X_4), m(X_4, X_5). \]

\[ a([0|Y], Y). \]
\[ b([1|Y], Y). \]

?- \ m([0,1,0,0,1,1], L).
  \ \ L=[0,1,0,0,1,1]
  \ \ L=[0,0,1,1]
  \ \ L=[]

?- \ m([0,0,1,1,1,0], L).
  \ \ L=[0,1,0,0,1,1]
Mystery Program

\[
m(X, X).
\]
\[
m(X_1, X_5) :- a(X_1, X_2), m(X_2, X_3), b(X_3, X_4), m(X_4, X_5).
\]
\[
a([0|Y], Y).
\]
\[
b([1|Y], Y).
\]

?- \( m([0,1,0,0,1,1], L). \)

\[
L=[0,1,0,0,1,1]
L=[0,0,1,1]
L=[]
\]

?- \( m([0,0,1,1,1,0], L). \)

\[
L=[0,1,0,0,1,1]
L=[1,0]
\]
Definite Clause Grammars

m --> [].
m --> a, m, b, m.

a --> [0].
b --> [1].
Definite Clause Grammars

m --> [].
m --> a, m, b, m.

a --> [0].
b --> [1].

?- m([0,1,0,0,1,1], L).
Definite Clause Grammars

\[
m \to \text{[]}. \\
m \to a, m, b, m. \\
a \to \text{[0]}.
\]

\[
b \to \text{[1]}. \\
\]

?- m([0,1,0,0,1,1], L).
\[
L=[0,1,0,0,1,1], ... 
\]
Definite Clause Grammars

\[ m \rightarrow \[\]. \]
\[ m \rightarrow a, m, b, m. \]

\[ a \rightarrow [0]. \]
\[ b \rightarrow [1]. \]

?- m([0,1,0,0,1,1], L).

\[ L=[0,1,0,0,1,1], \ldots \]

?- phrase(m, [0,1,0,0,1,1])
Definite Clause Grammars

\[ m \rightarrow \[] \]
\[ m \rightarrow a, m, b, m \]

\[ a \rightarrow [0] \]
\[ b \rightarrow [1] \]

?- \( m([0,1,0,0,1,1], L) \).
\[ L=[0,1,0,0,1,1], \ldots \]

?- phrase(m, [0,1,0,0,1,1])
Definite Clause Grammars

m --> [].
m --> a, m, b, m.

a --> [0].
b --> [1].

?- m([0,1,0,0,1,1], L).
    L=[0,1,0,0,1,1], ...

?- phrase(m, [0,1,0,0,1,1]) \equiv m([0,1,0,0,1,1], [])
    yes
Definite Clause Grammars

\( m \rightarrow \text{[]} \).
\( m \rightarrow a, m, b, m. \)

\( a \rightarrow [0]. \)
\( b \rightarrow [1]. \)

?- \( m([0,1,0,0,1,1], L). \)
    \( L=[0,1,0,0,1,1], \ldots \)

?- phrase(m, [0,1,0,0,1,1]) \equiv m([0,1,0,0,1,1], [])
    yes

?- phrase(m, L). \)
Definite Clause Grammars

\[ m \rightarrow \[] \].
\[ m \rightarrow a, m, b, m. \]

\[ a \rightarrow [0] \].
\[ b \rightarrow [1] \].

?- \ m([0,1,0,0,1,1], L).
    L=[0,1,0,0,1,1], ...

?- phrase(m, [0,1,0,0,1,1]) \equiv m([0,1,0,0,1,1], [])
    yes

?- phrase(m, L).
    L=[]
Definite Clause Grammars

\[
\begin{align*}
m & \rightarrow \square. \\
m & \rightarrow a, m, b, m. \\
a & \rightarrow [0]. \\
b & \rightarrow [1].
\end{align*}
\]

\[\text{?- } m([0,1,0,0,1,1], L). \quad L=[0,1,0,0,1,1], \ldots\]

\[\text{?- phrase}(m, [0,1,0,0,1,1]) \equiv m([0,1,0,0,1,1], []) \quad \text{yes}\]

\[\text{?- phrase}(m, L). \quad L=[] \quad L=[0,1]\]
Definite Clause Grammars

\[ m \rightarrow []. \]
\[ m \rightarrow a, m, b, m. \]

\[ a \rightarrow [0]. \]
\[ b \rightarrow [1]. \]

?- \( m([0,1,0,0,1,1], L). \)
   \( \quad L=[0,1,0,0,1,1], \ldots \)

?- phrase(m, [0,1,0,0,1,1]) \equiv m([0,1,0,0,1,1], [])
   \( \quad \text{yes} \)

?- phrase(m, L).
   \( \quad L=[] \)
   \( \quad L=[0,1] \)
   \( \quad L=[0,1,0,1] \)
Definite Clause Grammars

m --> [].
m --> a, m, b, m.

a --> [0].
b --> [1].

?- m([0,1,0,0,1,1], L).
   L=[0,1,0,0,1,1], ...

?- phrase(m, [0,1,0,0,1,1]) ≡ m([0,1,0,0,1,1], [])
   yes

?- phrase(m, L).
   L=[]
   L=[0,1]
   L=[0,1,0,1]
   ...
   ...

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Definite Clause Grammars (Magic?)

\[ r([]) \rightarrow []. \]
\[ r([X|Xs]) \rightarrow r(Xs), [X]. \]
Definite Clause Grammars (Magic?)

\[
\begin{align*}
  &r([]) \rightarrow []. \\
  &r([X|Xs]) \rightarrow r(Xs), [X].
\end{align*}
\]

?- phrase(r([1,2,3,4]), L).
Definite Clause Grammars (Magic?)

\[
\begin{align*}
    r([]) & \rightarrow []. \\
    r([X|Xs]) & \rightarrow r(Xs), [X].
\end{align*}
\]

?- phrase(r([1,2,3,4]), L).
   L=[4,3,2,1]
Definite Clause Grammars (Magic?)

\[
\begin{align*}
    & r([], \text{--} \rightarrow \text{} []). \\
    & r([X|Xs], \text{--} \rightarrow r(Xs), [X]). \\
\end{align*}
\]

?- phrase(r([1,2,3,4]), L).
    \quad L=[4,3,2,1]

?- phrase(r(Q), [1,2,3,4]).
Definite Clause Grammars (Magic?)

\[
\begin{align*}
\text{r}([],) & \rightarrow \ []. \\
\text{r}([X|Xs]) & \rightarrow \text{r}(Xs), \ [X].
\end{align*}
\]

?- phrase(r([1,2,3,4]), L).
    L=[4,3,2,1]

?- phrase(r(Q), [1,2,3,4]).
    Q=[4,3,2,1]
Definite Clause Grammars (Trick exposed!)

\[
\begin{align*}
    r([]) & \rightarrow []. \\
    r([X\mid Xs]) & \rightarrow r(Xs), [X].
\end{align*}
\]
Definite Clause Grammars (Trick exposed!)

\[
\begin{align*}
\text{r}([\,]) & \rightarrow [\,]. \\
\text{r}([X|Xs]) & \rightarrow \text{r}(Xs), [X]. \\
\end{align*}
\]

Translated to:

\[
\begin{align*}
\text{r}([\,], X, X). \\
\text{r}([X|Xs], Z1, Z3) & : \ - \text{r}(Xs, Z1, Z2), Z2 = [X|Z3]. \\
\end{align*}
\]
Definite Clause Grammars (Trick exposed!)

\[
r([]) \rightarrow []. \\
r([X|Xs]) \rightarrow r(Xs), [X].
\]

Translated to:

\[
r([], X, X). \\
r([X|Xs], Z1, Z3) :- r(Xs, Z1, Z2), Z2 = [X|Z3].
\]

Equivalent to:

\[
r([], X, X). \\
r([X|Xs], Z1, Z3) :- r(Xs, Z1, [X|Z3]).
\]

?- phrase(r([1,2,3,4]), L).
Definite Clause Grammars (Trick exposed!)

\[ r([]) \rightarrow []. \]
\[ r([X|Xs]) \rightarrow r(Xs), [X]. \]

Translated to:

\[ r([], X, X). \]
\[ r([X|Xs], Z1, Z3) \leftarrow r(Xs, Z1, Z2), Z2 = [X|Z3]. \]

Equivalent to:

\[ r([], X, X). \]
\[ r([X|Xs], Z1, Z3) \leftarrow r(Xs, Z1, [X|Z3]). \]

\[ ?- \text{phrase}(r([1,2,3,4]), L). \]
\[ \equiv r([1,2,3,4], L, []) \]
Definite Clause Grammars (Trick exposed!)

\[
\begin{align*}
  r([],[]) & \rightarrow \ [] . \\
  r([X|Xs]) & \rightarrow r(Xs), [X] . \\
\end{align*}
\]

Translated to:

\[
\begin{align*}
  r([], X, X) . \\
  r([X|Xs], Z1, Z3) & :\sim r(Xs, Z1, Z2), Z2 = [X|Z3] . \\
\end{align*}
\]

Equivalent to:

\[
\begin{align*}
  r([], X, X) . \\
  r([X|Xs], Z1, Z3) & :\sim r(Xs, Z1, [X|Z3]) . \\
\end{align*}
\]

?- phrase(r([1,2,3,4]), L).
\[
\begin{align*}
  L=\{4,3,2,1\}
\end{align*}
\]
Definite Clause Grammars (Trick exposed!)

\[
\begin{align*}
r([]) & \rightarrow []. \\
r([X|Xs]) & \rightarrow r(Xs), [X].
\end{align*}
\]

Translated to:

\[
\begin{align*}
r([], X, X). \\
r([X|Xs], Z1, Z3) & :\neg r(Xs, Z1, Z2), Z2 = [X|Z3].
\end{align*}
\]

Equivalent to:

\[
\begin{align*}
r([], X, X). \\
r([X|Xs], Z1, Z3) & :\neg r(Xs, Z1, [X|Z3]).
\end{align*}
\]

?- phrase(r([1,2,3,4]), L).

\[
\begin{align*}
\equiv r([1,2,3,4], L, []) \\
L=[4,3,2,1]
\end{align*}
\]

- A way to reverse a list in **polynomial time**!
Unification

- Operation done to “match” the goal atom with the head of a clause in the program.
- Forms the basis for the matching operation we used for Prolog evaluation.
  - \( f(a, Y) \) and \( f(X, b) \) unify when \( X=a \) and \( Y=b \).
  - \( f(a, X) \) and \( f(X, b) \) do not unify.
  - \( X \) and \( f(X) \) do not unify
    (but they “match” in Prolog!)
A substitution is a mapping between variables and values (terms).

- Denoted by \( \{ X_1 \mapsto t_1, X_2 \mapsto t_2, \ldots, X_n \mapsto t_n \} \) such that
  - \( X_i \neq t_i \), and
  - \( X_i \) and \( X_j \) are distinct variables when \( i \neq j \).

- Empty substitution is denoted by \( \epsilon \).

- A substitution is said to be a **renaming** if it is of the form \( \{ X_1 \mapsto Y_1, \ldots, X_n \mapsto Y_n \} \) and \( Y_1, \ldots, Y_n \) is a permutation of \( X_1, \ldots, X_n \).

- Example: \( \{ X \mapsto Y, Y \mapsto X \} \) is a renaming substitution.
Substitutions and Terms

- Application of a substitution:
  - $X\theta = t$ if $X \mapsto t \in \theta$.
  - $X\theta = X$ if $X \mapsto t \notin \theta$ for any term $t$.

- Application of a substitution $\{X_1 \mapsto t_1, \ldots, X_n \mapsto t_n\}$ to a term $s$:
  - is a term obtained by *simultaneously* replacing every occurrence of $X_i$ in $s$ by $t_i$.
  - Denoted by $s\theta$
    and $s\theta$ is said to be an *instance* of $s$.

- Example:

  \[
  p(f(X, Z), f(Y, a)) \{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\} = p(f(g(Y), a), f(Z, a))
  \]
Composition of Substitutions

- Composition of substitutions $\theta = \{X_1 \mapsto s_1, \ldots, X_m \mapsto s_m\}$ and $\sigma = \{Y_1 \mapsto t_1, \ldots, Y_n \mapsto t_n\}$:
  - First form the set $\{X_1 \mapsto s_1\sigma, \ldots, X_m \mapsto s_m\sigma, Y_1 \mapsto t_1, \ldots, Y_n \mapsto t_n\}$
  - Remove from the set $X_i \mapsto s_i\sigma$ if $s_i\sigma = X_i$
  - Remove from the set $Y_j \mapsto t_j$ if $Y_j$ is identical to some variable $X_i$
- Example: Let $\theta = \sigma = \{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\}$. Then $\theta\sigma = \{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\}$
  - More examples: Let $\theta = \{X \mapsto f(Y)\}$ and $\sigma = \{Y \mapsto a\}$
    - $\theta\sigma = \{X \mapsto f(a), Y \mapsto a\}$
    - $\theta\sigma = \{X \mapsto f(Y), Y \mapsto a\}$
- Composition is not commutative but is associative: $\theta(\sigma\gamma) = (\theta\sigma)\gamma$
- Also, $E(\theta\sigma) = (E\theta)\sigma$
Idempotence

- A substitution $\theta$ is **idempotent** iff $\theta \theta = \theta$.
- Examples:
  - $\{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\}$ is not idempotent since
    $$\{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\}\{X \mapsto g(Y), Y \mapsto Z, Z \mapsto a\} = \{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\}$$
  - $\{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\}$ is not idempotent either since
    $$\{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\}\{X \mapsto g(Z), Y \mapsto a, Z \mapsto a\} = \{X \mapsto g(a), Y \mapsto a, Z \mapsto a\}$$
  - $\{X \mapsto g(a), Y \mapsto a, Z \mapsto a\}$ is idempotent

For a substitution $\theta = \{X_1 \mapsto t_1, \ldots, X_n \mapsto t_n\}$,
- $\text{Dom}(\theta) = \{X_1, X_2, \ldots, X_n\}$
- $\text{Range}(\theta) = \text{set of all variables in } t_1, \ldots, t_n$

A substitution $\theta$ is idempotent iff $\text{Dom}(\theta) \cap \text{Range}(\theta) = \emptyset$
Unifiers

- A substitution $\theta$ is a **unifier** of two terms $s$ and $t$ if $s\theta$ is identical to $t\theta$.
- $\theta$ is a unifier of a set of equations $\{s_1 = t_1, \ldots, s_n = t_n\}$, if for all $i$, $s_i\theta = t_i\theta$.
- A substitution $\theta$ is more general than $\sigma$ (written as $\theta \geq \sigma$) if there is a substitution $\omega$ such that $\sigma = \theta\omega$.
- A substitution $\theta$ is a **most general unifier** (mgu) of two terms (or a set of equations) if for every unifier $\sigma$ of the two terms (or equations) $\theta \geq \sigma$.
- Example: Consider two terms $f(g(X), Y, a, b)$ and $f(Z, W, X, b)$.
  - $\theta_1 = \{X \mapsto a, Y \mapsto b, Z \mapsto g(a), W \mapsto b\}$ is a unifier
  - $\theta_2 = \{X \mapsto a, Y \mapsto W, Z \mapsto g(a)\}$ is also a unifier
  - $\theta_2$ is a most general unifier
Equations and Unifiers

- A set of equations $\mathcal{E}$ is in **solved form** if it is of the form
  \[ \{ X_1 \equiv t_1, \ldots, X_n \equiv t_n \} \]  iff
  - all $X_i$’s are distinct, and
  - no $X_i$ appears in any $t_j$.

- Given a set of equations in solved form $\mathcal{E} = \{ X_1 \equiv t_1, \ldots, X_n \equiv t_n \}$ the substitution $\{ X_1/t_1, \ldots X_n/t_n \}$ is an idempotent mgu of $\mathcal{E}$.

- Two sets of equations $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be **equivalent** iff they have the same set of unifiers.

- To find the mgu of two terms $s$ and $t$, find a set of equations in solved form that is equivalent to $\{ s \equiv t \}$. If there is no equivalent solved form, there is no mgu.
A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

  $\{f(X, g(Y)) \models f(g(Z), Z)\} \Rightarrow$
A Simple Unification Algorithm (via Examples)

Example 1: Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)

\[
\{ f(X, g(Y)) \doteq f(g(Z), Z) \} \quad \Rightarrow \quad \{ X \doteq g(Z), g(Y) \doteq Z \}
\]
A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

\[
\{ f(X, g(Y)) \doteq f(g(Z), Z) \} \Rightarrow \{ X \doteq g(Z), g(Y) \doteq Z \} \\
\Rightarrow \{ X \doteq g(Z), Z \doteq g(Y) \} 
\]
A Simple Unification Algorithm (via Examples)

- Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$
\{f(X, g(Y)) \equiv f(g(Z), Z)\} \Rightarrow \{X \equiv g(Z), g(Y) \equiv Z\}
$$

$$
\Rightarrow \{X \equiv g(Z), Z \equiv g(Y)\}
$$

$$
\Rightarrow \{X \equiv g(g(Y)), Z \equiv g(Y)\}
$$
A Simple Unification Algorithm (via Examples)

- **Example 1**: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

  \[
  \{f(X, g(Y)) \doteq f(g(Z), Z)\} \Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\} \\
  \Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\} \\
  \Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}
  \]

- **Example 2**: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

  \[
  \{f(X, g(X), b) \doteq f(a, g(Z), Z)\} \Rightarrow 
  \]
A Simple Unification Algorithm (via Examples)

Example 1: Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)

\[
\{ f(X, g(Y)) \doteq f(g(Z), Z) \} \quad \Rightarrow \quad \{ X \doteq g(Z), g(Y) \doteq Z \}
\]

\[
\Rightarrow \quad \{ X \doteq g(Z), Z \doteq g(Y) \}
\]

\[
\Rightarrow \quad \{ X \doteq g(g(Y)), Z \doteq g(Y) \}
\]

Example 2: Find the mgu of \( f(X, g(X), b) \) and \( f(a, g(Z), Z) \)

\[
\{ f(X, g(X), b) \doteq f(a, g(Z), Z) \} \quad \Rightarrow \quad \{ X \doteq a, g(X) \doteq g(Z), b \doteq Z \}
\]


A Simple Unification Algorithm (via Examples)

- **Example 1:** Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

  \[
  \{f(X, g(Y)) \doteq f(g(Z), Z)\} \Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\} \\
  \Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\} \\
  \Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}
  \]

- **Example 2:** Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

  \[
  \{f(X, g(X), b) \doteq f(a, g(Z), Z)\} \Rightarrow \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\} \\
  \Rightarrow \{X \doteq a, g(a) \doteq g(Z), b \doteq Z\}
  \]
A Simple Unification Algorithm (via Examples)

- **Example 1:** Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

  \[
  \{f(X, g(Y)) \doteq f(g(Z), Z)\} \Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\} \\
  \Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\} \\
  \Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}
  \]

- **Example 2:** Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

  \[
  \{f(X, g(X), b) \doteq f(a, g(Z), Z)\} \Rightarrow \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\} \\
  \Rightarrow \{X \doteq a, g(a) \doteq g(Z), b \doteq Z\} \\
  \Rightarrow \{X \doteq a, a \doteq Z, b \doteq Z\} \\
  \Rightarrow \text{fail}
  \]
A Simple Unification Algorithm (via Examples)

- **Example 1:** Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)

\[
\{f(X, g(Y)) \vdash f(g(Z), Z)\} \Rightarrow \{X \vdash g(Z), g(Y) \vdash Z\}
\]

\[
\Rightarrow \{X \vdash g(Z), Z \vdash g(Y)\}
\]

\[
\Rightarrow \{X \vdash g(g(Y)), Z \vdash g(Y)\}
\]

- **Example 2:** Find the mgu of \( f(X, g(X), b) \) and \( f(a, g(Z), Z) \)

\[
\{f(X, g(X), b) \vdash f(a, g(Z), Z)\} \Rightarrow \{X \vdash a, g(X) \vdash g(Z), b \vdash Z\}
\]

\[
\Rightarrow \{X \vdash a, g(a) \vdash g(Z), b \vdash Z\}
\]

\[
\Rightarrow \{X \vdash a, a \vdash Z, b \vdash Z\}
\]

\[
\Rightarrow \{X \vdash a, Z \vdash a, b \vdash Z\}
\]
A Simple Unification Algorithm (via Examples)

- **Example 1**: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

  \[
  \{f(X, g(Y)) \doteq f(g(Z), Z)\} \Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\} \\
  \Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\} \\
  \Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}
  \]

- **Example 2**: Find the mgu of $f(X, g(X), b)$ and $f(a, g(Z), Z)$

  \[
  \{f(X, g(X), b) \doteq f(a, g(Z), Z)\} \Rightarrow \{X \doteq a, g(X) \doteq g(Z), b \doteq Z\} \\
  \Rightarrow \{X \doteq a, g(a) \doteq g(Z), b \doteq Z\} \\
  \Rightarrow \{X \doteq a, a \doteq Z, b \doteq Z\} \\
  \Rightarrow \{X \doteq a, Z \doteq a, b \doteq Z\} \\
  \Rightarrow \{X \doteq a, Z \doteq a, b \doteq a\}
  \]
A Simple Unification Algorithm (via Examples)

- **Example 1:** Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)

\[
\{ f(X, g(Y)) \cong f(g(Z), Z) \} \Rightarrow \{ X \cong g(Z), g(Y) \cong Z \}
\]

\[
\Rightarrow \{ X \cong g(Z), Z \cong g(Y) \}
\]

\[
\Rightarrow \{ X \cong g(g(Y)), Z \cong g(Y) \}
\]

- **Example 2:** Find the mgu of \( f(X, g(X), b) \) and \( f(a, g(Z), Z) \)

\[
\{ f(X, g(X), b) \cong f(a, g(Z), Z) \} \Rightarrow \{ X \cong a, g(X) \cong g(Z), b \cong Z \}
\]

\[
\Rightarrow \{ X \cong a, g(a) \cong g(Z), b \cong Z \}
\]

\[
\Rightarrow \{ X \cong a, a \cong Z, b \cong Z \}
\]

\[
\Rightarrow \{ X \cong a, Z \cong a, b \cong Z \}
\]

\[
\Rightarrow \{ X \cong a, Z \cong a, b \cong a \}
\]

\[
\Rightarrow \text{fail}
\]
A Simple Unification Algorithm

Given a set of equations $\mathcal{E}$:

repeat
  select $s \equiv t \in \mathcal{E}$;
  case $s \equiv t$ of
    1. $f(s_1, \ldots, s_n) \equiv f(t_1, \ldots, t_n)$:
       replace the equation by $s_i \equiv t_i$ for all $i$
    2. $f(s_1, \ldots, s_n) \equiv g(t_1, \ldots, t_m)$, $f \neq g$ or $n \neq m$:
       halt with failure
    3. $X \equiv X$ : remove the equation
    4. $t \equiv X$ : where $t$ is not a variable
       replace equation by $X \equiv t$
    5. $X \equiv t$ : where $X \neq t$ and $X$ occurs more than once in $\mathcal{E}$:
       if $X$ is a proper subterm of $t$
          then halt with failure (5a)
       else replace all other $X$ in $\mathcal{E}$ by $t$ (5b)
until no action is possible for any equation in $\mathcal{E}$
return $\mathcal{E}$
A Simple Unification Algorithm (More Examples)

Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\{f(X, g(Y)) \models f(g(Z), Z)\} \Rightarrow$$
A Simple Unification Algorithm (More Examples)

Example 1: Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)

\[
\{ f(X, g(Y)) \models f(g(Z), Z) \} \quad \Rightarrow \quad \{ X \models g(Z), g(Y) \models Z \}
\]

\text{case 1}
Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\{f(X, g(Y)) \doteq f(g(Z), Z)\} \Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\}$$

$$\Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\}$$
A Simple Unification Algorithm (More Examples)

Example 1: Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$

$$\{f(X, g(Y)) \vdash f(g(Z), Z)\} \Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\}$$  \hspace{1cm} \text{case 1}

$$\Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\}$$ \hspace{1cm} \text{case 4}

$$\Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}$$ \hspace{1cm} \text{case 5b}
A Simple Unification Algorithm (More Examples)

- **Example 1:** Find the mgu of \(f(X, g(Y))\) and \(f(g(Z), Z)\)
  \[
  \{f(X, g(Y)) \vdash f(g(Z), Z)\} \Rightarrow \{X \overset{\text{case 1}}{=} g(Z), g(Y) \overset{\text{case 4}}{=} Z\}
  \]
  \[
  \Rightarrow \{X \overset{\text{case 4}}{=} g(Z), Z \overset{\text{case 5b}}{=} g(Y)\}
  \]

- **Example 3:** Find the mgu of \(f(X, g(X))\) and \(f(Z, Z)\)
  \[
  \{f(X, g(X)) \vdash f(Z, Z)\} \Rightarrow \{X \overset{\text{case 1}}{=} Z, g(X) \overset{\text{case 5b}}{=} g(Y)\}\]
A Simple Unification Algorithm (More Examples)

Example 1: Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)

\[
\{ f(X, g(Y)) \doteq f(g(Z), Z) \} \Rightarrow \{ X \doteq g(Z), g(Y) \doteq Z \} \\
\Rightarrow \{ X \doteq g(Z), Z \doteq g(Y) \} \\
\Rightarrow \{ X \doteq g(g(Y)), Z \doteq g(Y) \}
\]

Example 3: Find the mgu of \( f(X, g(X)) \) and \( f(Z, Z) \)

\[
\{ f(X, g(X)) \doteq f(Z, Z) \} \Rightarrow \{ X \doteq Z, g(X) \doteq Z \}
\]

case 1

case 4

case 5b
A Simple Unification Algorithm (More Examples)

- Example 1: Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)
  \[
  \{ f(X, g(Y)) \models f(g(Z), Z) \} \quad \Rightarrow \quad \{ X \models g(Z), g(Y) \models Z \} \quad \text{case 1}
  \]
  \[
  \Rightarrow \quad \{ X \models g(Z), Z \models g(Y) \} \quad \text{case 4}
  \]
  \[
  \Rightarrow \quad \{ X \models g(g(Y)), Z \models g(Y) \} \quad \text{case 5b}
  \]

- Example 3: Find the mgu of \( f(X, g(X)) \) and \( f(Z, Z) \)
  \[
  \{ f(X, g(X)) \models f(Z, Z) \} \quad \Rightarrow \quad \{ X \models Z, g(X) \models Z \} \quad \text{case 1}
  \]
  \[
  \Rightarrow \quad \{ X \models Z, g(Z) \models Z \} \quad \text{case 5b}
  \]
A Simple Unification Algorithm (More Examples)

- **Example 1:** Find the mgu of $f(X, g(Y))$ and $f(g(Z), Z)$
  
  $\{f(X, g(Y)) \doteq f(g(Z), Z)\} \Rightarrow \{X \doteq g(Z), g(Y) \doteq Z\}$
  
  $\Rightarrow \{X \doteq g(Z), Z \doteq g(Y)\}$
  
  $\Rightarrow \{X \doteq g(g(Y)), Z \doteq g(Y)\}$

  - case 1
  - case 4
  - case 5b

- **Example 3:** Find the mgu of $f(X, g(X))$ and $f(Z, Z)$
  
  $\{f(X, g(X)) \doteq f(Z, Z)\} \Rightarrow \{X \doteq Z, g(X) \doteq Z\}$
  
  $\Rightarrow \{X \doteq Z, g(Z) \doteq Z\}$
  
  $\Rightarrow \{X \doteq Z, Z \doteq g(Z)\}$

  - case 1
  - case 5b
  - case 4
A Simple Unification Algorithm (More Examples)

- **Example 1:** Find the mgu of \( f(X, g(Y)) \) and \( f(g(Z), Z) \)
  \[
  \{ f(X, g(Y)) \doteq f(g(Z), Z) \} \quad \Rightarrow \quad \{ X \doteq g(Z), g(Y) \doteq Z \} \\
  \Rightarrow \quad \{ X \doteq g(Z), Z \doteq g(Y) \} \\
  \Rightarrow \quad \{ X \doteq g(g(Y)), Z \doteq g(Y) \}
  \]

- **Example 3:** Find the mgu of \( f(X, g(X)) \) and \( f(Z, Z) \)
  \[
  \{ f(X, g(X)) \doteq f(Z, Z) \} \quad \Rightarrow \quad \{ X \doteq Z, g(X) \doteq Z \} \\
  \Rightarrow \quad \{ X \doteq Z, g(Z) \doteq Z \} \\
  \Rightarrow \quad \{ X \doteq Z, Z \doteq g(Z) \} \\
  \Rightarrow \quad \text{fail}
  \]
Complexity of the unification algorithm

Consider

\[ \mathcal{E} = \{ g(X_1, \ldots, X_n) \equiv g(f(X_0, X_0), f(X_1, X_1), \ldots, f(X_{n-1}, X_{n-1}) \}. \]

- By applying **case 1** of the algorithm, we get
  \[ \{ X_1 = f(X_0, X_0), X_2 = f(X_1, X_1), \ldots, X_n = f(X_{n-1}, X_{n-1}) \}. \]

- If terms are kept as trees, the final value for \( X_n \) is a tree of size \( O(2^n) \).
- Recall that for **case 5** we need to first check if a variable appears in a term, and this could now take \( O(2^n) \) time.
- There are **linear-time** unification algorithms that share structures (terms as DAGs).
- \( X = t \) is the most common case for unification in Prolog. The fastest algorithms are linear in \( t \).
- Prolog cuts corners by omitting **case 5a** (the occur check), thereby doing \( X = t \) in constant time.
Most General Unifiers

- Note that mgu stands for a most general unifier.
- There may be more than one mgu. E.g. $f(X) \equiv f(Y)$ has two mgus:
  - $\{X \mapsto Y\}$
  - $\{Y \mapsto X\}$
- If $\theta$ is an mgu of $s$ and $t$, and $\omega$ is a renaming, then $\theta\omega$ is an mgu of $s$ and $t$.
- If $\theta$ and $\sigma$ are mgus of $s$ and $t$, then there is a renaming $\omega$ such that $\theta = \sigma\omega$. 