Supplement to Computing Bisimulation Functions using SOS Optimization and \( \delta \)-Decidability over the Reals

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1 Proofs of Theorems 1 and 2

**Theorem 1.** Let \( S \) be a BF, with parameters \( \lambda \) and \( \gamma \), between dynamical systems \( \Sigma_i, i = 1, 2 \), and let \( x_1(t) \) and \( x_2(t) \) be two trajectories of the systems. For all \( t \geq 0 \),

\[
\| g_1(x_1(t)) - g_2(x_2(t)) \| \leq S(x_1(t), x_2(t)) \\
\leq e^{-\lambda t} S(x_1(0), x_2(0)) + \frac{\gamma}{\lambda} \| u_1(t) - u_2(t) \| \infty
\]

where \( \| u_1 - u_2 \|_{\infty} = sup_{t \geq 0} \| u_1(t) - u_2(t) \| \) denotes the maximum difference in the input signals being fed to the two systems.

**Proof.** From Eq. (4), we have the first inequality. From Eq. (5), we have

\[
\frac{dS(x_1(t), x_2(t))}{dt} \leq -\lambda S(x_1(t), x_2(t)) + \gamma \| u_1(t) - u_2(t) \| \infty \\
\leq -\lambda S(x_1(t), x_2(t)) + \gamma \| u_1 - u_2 \|_{\infty}
\]

Let \( \eta(t) = e^{-\lambda t} S(x_1(0), x_2(0)) + \frac{\lambda}{\gamma} \| u_1 - u_2 \|_{\infty} \). It is a solution of the differential equation \( \dot{\eta}(t) = -\lambda \eta(t) + \gamma \| u_1 - u_2 \|_{\infty} \). Moreover, \( S(x_1(0), x_2(0)) \leq \eta(0) \); then from the funnel theorem [2], it follows that \( \forall t \geq 0, S(x_1(t), x_2(t)) \leq \eta(t) \). \( \square \)

**Theorem 2.** Let \( \Sigma_i = (X_i, A_i^0, U_i, f_i, O_i, g_i), i = 1, 2, A, B \), be dynamical systems such that \( U_1 = O_A, U_A = O_1, U_2 = O_B \) and \( U_B = O_2 \). Let \( S_{12} \), parameterized by \( \lambda_{12} \) and \( \gamma_{12} \), be a BF between \( \Sigma_1 \) and \( \Sigma_2 \). Let \( S_{AB} \), parameterized by \( \lambda_{AB} \) and \( \gamma_{AB} \), be a BF between \( \Sigma_A \) and \( \Sigma_B \).

Let \( \Sigma_{A1} \) and \( \Sigma_{B2} \) be composite systems representing feedback-composition\(^3\) of \( \Sigma_A \) with \( \Sigma_1 \) and \( \Sigma_B \) with \( \Sigma_2 \), respectively. If the small gain condition (SGC) \( \frac{\gamma_{12}}{\lambda_{AB}\lambda_{12}} < 1 \) is met, then a BF \( S \) can be constructed between \( \Sigma_{A1} \) and \( \Sigma_{B2} \) by composing \( S_{AB} \) and \( S_{12} \) as

\[
S(x_{A1}, x_{B2}) = \alpha_1 S_{AB}(x_A, x_B) + \alpha_2 S_{12}(x_1, x_2)
\]

\(^3\) In \( \Sigma_{A1} \), the output of \( \Sigma_A \) is fed as input to \( \Sigma_1 \) and vice versa. Similarly in \( \Sigma_{B2} \), the output of \( \Sigma_B \) is fed as input to \( \Sigma_2 \) and vice versa.
where $x_{A1} = [x_A, x_1]^T$ and $x_{B2} = [x_B, x_2]^T$ and the constants $\alpha_1$ and $\alpha_2$ are as follows.

\[
\begin{cases}
\frac{\gamma_{12}}{\lambda_{12}} < \alpha_1 < \frac{\lambda_{12}}{\gamma_{12}} \quad \text{and} \quad \alpha_2 = 1 \quad \text{if} \quad \lambda_{AB} \leq \gamma_{12} \\
\alpha_1 = 1 \quad \text{and} \quad \frac{\lambda_{AB}}{\lambda_{12}} < \alpha_2 < \frac{\lambda_{AB}}{\gamma_{12}} \quad \text{if} \quad \lambda_{12} \leq \lambda_{AB} \\
\alpha_1 = 1 \quad \text{and} \quad \alpha_2 = 1 \quad \text{in other cases}
\end{cases}
\]

**Proof.** Consider $S$ be the function as per the theorem. We will find conditions on $\alpha_1$ and $\alpha_2$ such that $S$ is a BF between $\Sigma_{A1}$ and $\Sigma_{B2}$. If $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$, then

\[
S(x_{A1}, x_{B2}) \geq S_{AB}(x_A, x_B) + S_{12}(x_1, x_2)
\]

\[
\geq \| g_A(x_A) - g_B(x_B) \| + \| g_1(x_1) - g_2(x_2) \|
\]

because $S_{AB}$ and $S_{12}$ satisfy Eq. (4). The observational difference of $\Sigma_{A1}$ and $\Sigma_{B2}$:

\[
\| g_A(x_A) - g_B(x_B) \| =
\sqrt{\| g_A(x_A) - g_B(x_B) \|^2 + \| g_1(x_1) - g_2(x_2) \|^2}
\]

\[
\leq \| g_A(x_A) - g_B(x_B) \| + \| g_1(x_1) - g_2(x_2) \|
\]

Therefore, it shows that $S$ satisfies Eq. (4). Applying similar steps as in [1], we can write the following:

\[
\frac{\partial S}{\partial x_{A1}} f_{A1}(x_{A1}, u_{A1}) + \frac{\partial S}{\partial x_{B2}} f_{B2}(x_{B2}, u_{B2}) \leq -(\alpha_1 \lambda_{AB} - \alpha_2 \gamma_{12}) S_{AB}(x_A, x_B) - (\alpha_2 \lambda_{12} - \alpha_1 \gamma_{AB}) S_{12}(x_1, x_2)
\]

If $(\alpha_1 \lambda_{AB} - \alpha_2 \gamma_{12}) > 0$ and $(\alpha_2 \lambda_{12} - \alpha_1 \gamma_{AB}) > 0$ and

\[
\lambda = \min\left(\frac{\alpha_1 \lambda_{AB} - \alpha_2 \gamma_{12}}{\alpha_1}, \frac{\alpha_2 \lambda_{12} - \alpha_1 \gamma_{AB}}{\alpha_2}\right)
\]

then

\[
\frac{\partial S}{\partial x_{A1}} f_{A1}(x_{A1}, u_{A1}) + \frac{\partial S}{\partial x_{B2}} f_{B2}(x_{B2}, u_{B2}) \leq -\lambda S(x_{A1}, x_{B2}).
\]

Therefore, $S$ will be a BF if $\alpha_1 \geq 1$, $\alpha_2 \geq 1$, $(\alpha_1 \lambda_{AB} - \alpha_2 \gamma_{12}) > 0$ and $(\alpha_2 \lambda_{12} - \alpha_1 \gamma_{AB}) > 0$. As shown in [1], these four conditions can be expressed as $\frac{\lambda_{12} \lambda_{AB}}{\gamma_{AB} \gamma_{12}} < 1$. $\square$

**References**