Problem of the Day

The *knapsack problem* is as follows: given a set of integers $S = \{s_1, s_2, \ldots, s_n\}$, and a given target number $T$, find a subset of $S$ which adds up exactly to $T$. For example, within $S = \{1, 2, 5, 9, 10\}$ there is a subset which adds up to $T = 22$ but not $T = 23$.

Find counterexamples to each of the following algorithms for the knapsack problem. That is, give an $S$ and $T$ such that the subset is selected using the algorithm does not leave the knapsack completely full, even though such a solution exists.
Solution

- Put the elements of $S$ in the knapsack in left to right order if they fit, i.e. the first-fit algorithm?
- Put the elements of $S$ in the knapsack from smallest to largest, i.e. the best-fit algorithm?
- Put the elements of $S$ in the knapsack from largest to smallest?
The RAM Model of Computation

Algorithms are an important and durable part of computer science because they can be studied in a machine/language independent way. This is because we use the RAM model of computation for all our analysis.

- Each “simple” operation (+, -, =, if, call) takes 1 step.

- Loops and subroutine calls are not simple operations. They depend upon the size of the data and the contents of a subroutine. “Sort” is not a single step operation.
• Each memory access takes exactly 1 step.

We measure the run time of an algorithm by counting the number of steps, where:
This model is useful and accurate in the same sense as the flat-earth model (which is useful)!
Worst-Case Complexity

The *worst case complexity* of an algorithm is the function defined by the maximum number of steps taken on any instance of size $n$. 
Best-Case and Average-Case Complexity

The *best case complexity* of an algorithm is the function defined by the minimum number of steps taken on any instance of size $n$.

The *average-case complexity* of the algorithm is the function defined by an average number of steps taken on any instance of size $n$.

Each of these complexities defines a numerical function: time vs. size!
Exact Analysis is Hard!

Best, worst, and average are difficult to deal with precisely because the details are very complicated:

It easier to talk about *upper and lower bounds* of the function. Asymptotic notation ($O, \Theta, \Omega$) are as well as we can practically deal with complexity functions.
Names of Bounding Functions

- $g(n) = O(f(n))$ means $C \times f(n)$ is an upper bound on $g(n)$.
- $g(n) = \Omega(f(n))$ means $C \times f(n)$ is a lower bound on $g(n)$.
- $g(n) = \Theta(f(n))$ means $C_1 \times f(n)$ is an upper bound on $g(n)$ and $C_2 \times f(n)$ is a lower bound on $g(n)$.

$C$, $C_1$, and $C_2$ are all constants independent of $n$. 
\( \Theta, \Omega, \text{ and } \Theta \)

The definitions imply a constant \( n_0 \) beyond which they are satisfied. We do not care about small values of \( n \).
Formal Definitions

- \( f(n) = O(g(n)) \) if there are positive constants \( n_0 \) and \( c \) such that to the right of \( n_0 \), the value of \( f(n) \) always lies on or below \( c \cdot g(n) \).

- \( f(n) = \Omega(g(n)) \) if there are positive constants \( n_0 \) and \( c \) such that to the right of \( n_0 \), the value of \( f(n) \) always lies on or above \( c \cdot g(n) \).

- \( f(n) = \Theta(g(n)) \) if there exist positive constants \( n_0, c_1, \) and \( c_2 \) such that to the right of \( n_0 \), the value of \( f(n) \) always lies between \( c_1 \cdot g(n) \) and \( c_2 \cdot g(n) \) inclusive.
Big Oh Examples

\[ 3n^2 - 100n + 6 = O(n^2) \text{ because } 3n^2 > 3n^2 - 100n + 6 \]
\[ 3n^2 - 100n + 6 = O(n^3) \text{ because } .01n^3 > 3n^2 - 100n + 6 \]
\[ 3n^2 - 100n + 6 \neq O(n) \text{ because } c \cdot n < 3n^2 \text{ when } n > c \]

Think of the equality as meaning in the set of functions.
Big Omega Examples

\[ 3n^2 - 100n + 6 = \Omega(n^2) \text{ because } 2.99n^2 < 3n^2 - 100n + 6 \]

\[ 3n^2 - 100n + 6 \neq \Omega(n^3) \text{ because } 3n^2 - 100n + 6 < n^3 \]

\[ 3n^2 - 100n + 6 = \Omega(n) \text{ because } 10^{10^{10}} n < 3n^2 - 100 + 6 \]
Big Theta Examples

\[ 3n^2 - 100n + 6 = \Theta(n^2) \text{ because } O \text{ and } \Omega \]
\[ 3n^2 - 100n + 6 \neq \Theta(n^3) \text{ because } O \text{ only} \]
\[ 3n^2 - 100n + 6 \neq \Theta(n) \text{ because } \Omega \text{ only} \]
Big Oh Addition/Subtraction

Suppose $f(n) = O(n^2)$ and $g(n) = O(n^2)$.

- What do we know about $g'(n) = f(n) + g(n)$? Adding the bounding constants shows $g'(n) = O(n^2)$.
- What do we know about $g''(n) = f(n) - |g(n)|$? Since the bounding constants don’t necessary cancel, $g''(n) = O(n^2)$.

We know nothing about the lower bounds on $g'$ and $g''$ because we know nothing about lower bounds on $f$ and $g$. 
Big Oh Multiplication by Constant

Multiplication by a constant does not change the asymptotics:

\[ O(c \cdot f(n)) \rightarrow O(f(n)) \]
\[ \Omega(c \cdot f(n)) \rightarrow \Omega(f(n)) \]
\[ \Theta(c \cdot f(n)) \rightarrow \Theta(f(n)) \]
Big Oh Multiplication by Function

But when both functions in a product are increasing, both are important:

\[
O(f(n)) \times O(g(n)) \rightarrow O(f(n) \times g(n))
\]

\[
\Omega(f(n)) \times \Omega(g(n)) \rightarrow \Omega(f(n) \times g(n))
\]

\[
\Theta(f(n)) \times \Theta(g(n)) \rightarrow \Theta(f(n) \times g(n))
\]