Lecture 15:
Breadth/Depth-First Search (1997)

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The square *of a directed graph* $G = (V, E)$ *is the graph* $G^2 = (V, E^2)$ *such that* $(u, w) \in E^2$ *iff for some* $v \in V$*, both* $(u, v) \in E$ *and* $(v, w) \in E$*; ie. there is a path of exactly two edges.*

Give efficient algorithms for both adjacency lists and matricies.

Given an adjacency matrix, we can check in constant time whether a given edge exists. To discover whether there is an edge $(u, w) \in G^2$, for each possible intermediate vertex v we can check whether (u, v) and (v, w) exist in $O(1)$.

Since there are at most n intermediate vertices to check, and n^2 pairs of vertices to ask about, this takes $O(n^3)$ time.

With adjacency lists, we have a list of all the edges in the graph. For a given edge (u, v), we can run through all the edges from v in $O(n)$ time, and fill the results into an adjacency matrix of G^2, which is initially empty.

It takes $O(mn)$ to construct the edges, and $O(n^2)$ to initialize and read the adjacency matrix, a total of $O((n+m)n)$. Since $n \leq m$ unless the graph is disconnected, this is usually simplified to $O(mn)$, and is faster than the previous algorithm on sparse graphs.

Why is it called the square of a graph? Because the square of the adjacency matrix is the adjacency matrix of the square! This provides a theoretically faster algorithm.

Traversal Orders

The order we explore the vertices depends upon what kind of data structure is used:

- *Queue* – by storing the vertices in a first-in, first out (FIFO) queue, we explore the oldest unexplored vertices first. Thus our explorations radiate out slowly from the starting vertex, defining a so-called *breadth-first search*.
- *Stack* - by storing the vertices in a last-in, first-out (LIFO) stack, we explore the vertices by lurching along a path, constantly visiting a new neighbor if one is available, and backing up only if we are surrounded by previously discovered vertices. Thus our explorations

quickly wander away from our starting point, defining a so-called *depth-first search*.

The three possible colors of each node reflect if it is unvisited (white), visited but unexplored (grey) or completely explored (black).

Breadth-First Search

BFS(G,s)
for each vertex $u \in V[G] - \{s\}$ do
    color[u] = white
    $d[u] = \infty$, ie. the distance from s
    $p[u] = NIL$, ie. the parent in the BFS tree
color[u] = grey
$d[s] = 0$
$p[s] = NIL$
$Q = \{s\}$
while $Q \neq \emptyset$ do
    $u = head[Q]$
    for each $v \in Adj[u]$ do

if $color[v] = white$ then

$color[v] = gray$

$d[v] = d[u] + 1$

$p[v] = u$

enqueue[Q,v]

dequeue[Q]

$color[u] = black$

Depth-First Search

DFS has a neat recursive implementation which eliminates the need to explicitly use a stack.
Discovery and final times are sometimes a convenience to maintain.

DFS(G)
for each vertex $u \in V[G]$ do
    $color[u] = white$
    $parent[u] = nil$
$time = 0$
for each vertex $u \in V[G]$ do
    if $color[u] = white$ then DFS-VISIT[u]

Initialize each vertex in the main routine, then do a search from each connected component. BFS must also start from a vertex in each component to completely visit the graph.

DFS-VISIT[u]

$color[u] = grey$ (*u had been white/undiscovered*)

$discover[u] = time$

$time = time + 1$

for each $v \in Adj[u]$ do

    if $color[v] = white$ then

        $parent[v] = u$

        DFS-VISIT(v)

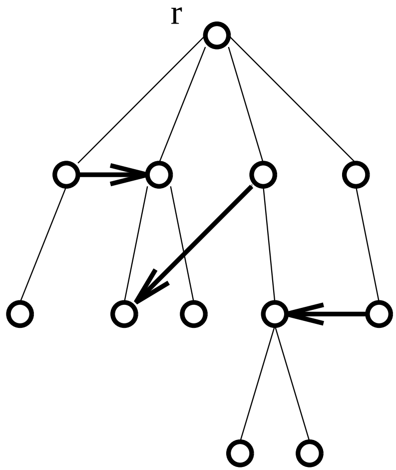
$color[u] = black$ (*now finished with u*)

$finish[u] = time$

$time = time + 1$

BFS Trees

If BFS is performed on a connected, undirected graph, a tree is defined by the edges involved with the discovery of new nodes:



This tree defines a shortest path from the root to every other

node in the tree.
The proof is by induction on the length of the shortest path from the root:

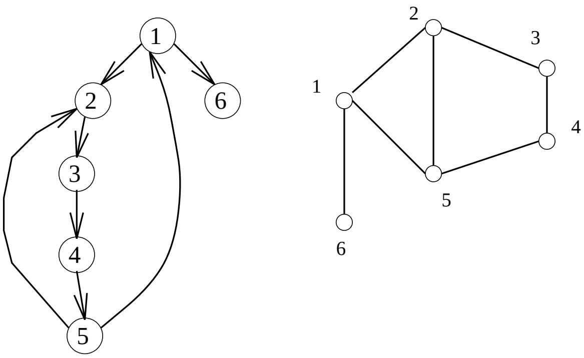
- *Length = 1* First step of BFS explores all neighbors of the root. In an unweighted graph one edge must be the shortest path to any node.

- *Length = s* Assume the BFS tree has the shortest paths up to length $s - 1$. Any node at a distance of s will first be discovered by expanding a distance $s - 1$ node.

The *key* idea about DFS

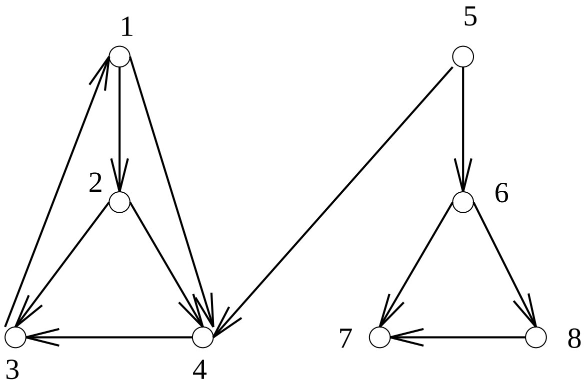
A depth-first search of a graph organizes the edges of the graph in a precise way.

In a DFS of an undirected graph, we assign a direction to each edge, from the vertex which discover it:



In a DFS of a directed graph, every edge is either a tree edge or a black edge.

In a DFS of a directed graph, no cross edge goes to a higher numbered or rightward vertex. Thus, no edge from 4 to 5 is possible:

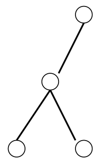


Edge Classification for DFS

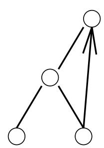
What about the other edges in the graph? Where can they go on a search?

Every edge is either:

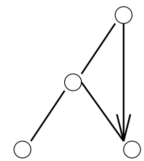
1. A Tree Edge



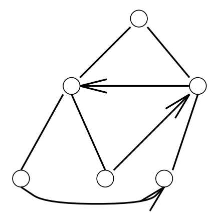
2. A Back Edge
   to an ancestor



3. A Forward Edge
   to a decendant



4. A Cross Edge
   to a different node



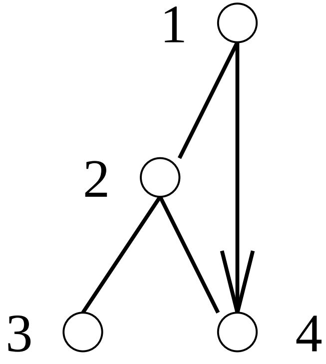
On any particular DFS or BFS of a directed or undirected graph, each edge gets classified as one of the above.

DFS Trees

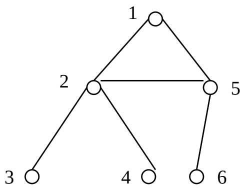
The reason DFS is so important is that it defines a very nice ordering to the edges of the graph.

In a DFS of an undirected graph, every edge is either a tree edge or a back edge.

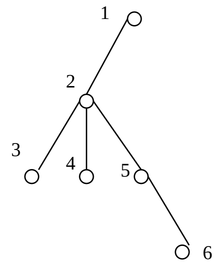
Why? Suppose we have a forward edge. We would have encountered $(4, 1)$ when expanding 4, so this is a back edge.



Suppose we have a cross-edge

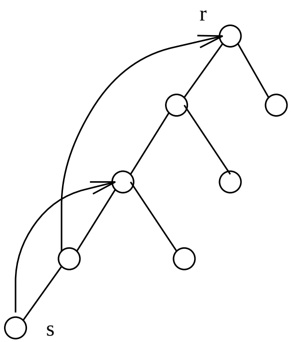


When expanding 2, we would discover 5, so the tree would look like:



Paths in search trees

Where is the shortest path in a DFS?



It could use multiple back and tree edges, where BFS only used tree edges.

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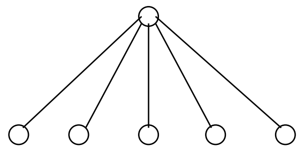
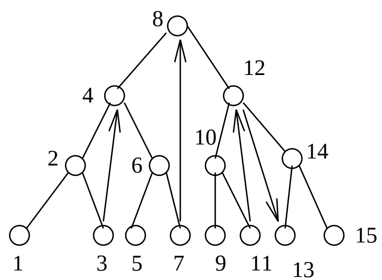
DFS gives a better approximation of the longest path than BFS.



The BFS tree can have height 1, independant of the length of the longest path.



The DFS must always have height >= log P, where P is the length of the longest path.