# Lecture 11:
# Introduction to Dynamic Programming (1997)

**Steven Skiena**

Department of Computer Science
State University of New York
Stony Brook, NY 11794–4400

http://www.cs.sunysb.edu/∼skiena

# The Partition Problem

Suppose the job scanning through a shelf of books is to be split between $k$ workers. To avoid the need to rearrange the books or separate them into piles, we can divide the shelf into $k$ regions and assign each region to one worker.

What is the fairest way to divide the shelf up?

If each book is the same length, partition the books into equal-sized regions,

$$100\ 100\ 100 \mid 100\ 100\ 100 \mid 100\ 100\ 100$$

But what if the books are not the same length? This partition would yield

$$100\ 200\ 300 \mid 400\ 500\ 600 \mid 700\ 800\ 900$$

Which part of the job would you volunteer to do?
How can we find the fairest possible partition, i.e.

$$100\ 200\ 300\ 400\ 500 \mid 600\ 700 \mid 800\ 900$$

# The Linear Partition Problem

Input: A given arrangement $S$ of nonnegative numbers $\{s_1, \ldots, s_n\}$ and an integer $k$.

Problem: Partition $S$ into $k$ ranges, so as to minimize the maximum sum over all the ranges.

Try to find an algorithm which always gives the optimal solution.

Does fixed partition positions always work?

Does taking the average value of a part $\Sigma_{i=1}^{n} s_i/k$ from the left always work?

How about the right?

Any other ideas?

# Recursive Idea

Consider a recursive, exhaustive search approach. Notice that the $k$th partition starts right after we placed the $(k-1)$st divider.

Where can we place this last divider? Between the $i$th and $(i+1)$st elements for some $i$, where $1 \leq i \leq n$.

What is the cost of this? The total cost will be the larger of two quantities, (1) the cost of the last partition $\Sigma_{j=i+1}^{n} s_j$ and (2) the cost of the largest partition cost formed to the left of $i$.

What is the size of this left partition? To partition the elements $\{s_1, \ldots, s_i\}$ as equally as possible. *But this is a smaller instance of the same problem!*

# Dynamic Programming Recurrence

Define $M[n,k]$ to be the minimum possible cost over all partitionings of $\{s_1,\ldots,s_n\}$ into $k$ ranges, where the cost of a partition is the largest sum of elements in one of its parts. Thus defined, this function can be evaluated:

$$M[n,k] = \min_{i=1}^{n} \max(M[i,k-1], \sum_{j=i+1}^{n} s_j)$$

with the natural basis cases of

$$M[1,k] = s_1, \text{ for all } k>0 \text{ and,}$$

$$M[n,1] = \sum_{i=1}^{n} s_i$$

What is the running time?

It is the number of cells times the running time per cell.
A total of $k \cdot n$ cells exist in the table.
Each cell depends on $n$ others, and can be computed in linear time, for a total of $O(kn^2)$.

# Implementation

To evaluate this efficiently, we must make sure we do the smaller cases before the larger cases that depend upon them.

Partition$[S, k]$

(* compute prefix sums: $p[k] = \Sigma_{i=1}^{k} s_i$ *)
$p[0] = 0$
for $i = 1$ to $n$ do $p[i] = p[i-1] + s_i$

(* initialize boundary conditions *)
for $i = 1$ to $n$ do $M[i, 1] = p[i]$
for $i = 1$ to $k$ do $M[1, j] = s_1$

(* evaluate main recurrence *)

```
for $i = 2$ to $n$ do
    for $j = 2$ to $k$ do
        $M[i, j] = \infty$
        for $x = 1$ to $i - 1$ do
            $s = \max(M[x, j - 1], p[i] - p[x])$
            if $(M[i, j] > s)$ then
                $M[i, j] = s$
                $D[i, j] = x$
```

# DP Matrices

For the input $\{1,2,3,4,5,6,7,8,9\}$

| $M$ | $k$ | | | | $D$ | $k$ | | |
|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | | $n$ | 1 | 2 | 3 |
| 1 | 1 | 1 | 1 | | 1 | – | – | – |
| 2 | 3 | 2 | 2 | | 2 | – | 1 | 1 |
| 3 | 6 | 3 | 3 | | 3 | – | 2 | 2 |
| 4 | 10 | 6 | 4 | | 4 | – | 3 | 3 |
| 5 | 15 | 9 | 6 | | 5 | – | 3 | 4 |
| 6 | 21 | 11 | 9 | | 6 | – | 4 | 5 |
| 7 | 28 | 15 | 11 | | 7 | – | 5 | 6 |
| 8 | 36 | 21 | 15 | | 8 | – | 5 | 6 |
| 9 | 45 | 24 | 17 | | 9 | – | 6 | 7 |

For the input $\{1, 1, 1, 1, 1, 1, 1, 1, 1\}$

| $M$ | $k$ | | | | $D$ | $k$ | | |
|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | | $n$ | 1 | 2 | 3 |
| 1 | 1 | 1 | 1 | | 1 | – | – | – |
| 1 | 2 | 1 | 1 | | 1 | – | 1 | 1 |
| 1 | 3 | 2 | 1 | | 1 | – | 1 | 2 |
| 1 | 4 | 2 | 2 | | 1 | – | 2 | 2 |
| 1 | 5 | 3 | 2 | | 1 | – | 2 | 3 |
| 1 | 6 | 3 | 2 | | 1 | – | 3 | 4 |
| 1 | 7 | 4 | 3 | | 1 | – | 3 | 4 |
| 1 | 8 | 4 | 3 | | 1 | – | 4 | 5 |
| 1 | 9 | 5 | 3 | | 1 | – | 4 | 6 |

# Parallel Bubblesort

In order for me to give back your midterms, please form a line and sort yourselves in alphabetical order, from A to Z.

There is traditionally a strong correlation between the midterm grades and the number of daily problems attempted:

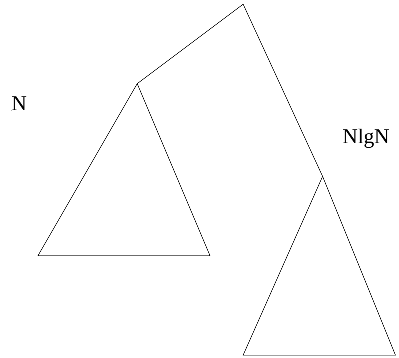
daily: 0, sum: 134, count: 3, avg: 44.67
daily: 1, sum: 0, count: 2, avg: XXXXX
daily: 2, sum: 63, count: 1, avg: 63.00
daily: 3, sum: 194, count: 3, avg: 64.67
daily: 4, sum: 335, count: 5, avg: 67.00
daily: 5, sum: 489, count: 8, avg: 61.12
daily: 6, sum: 381, count: 6, avg: 63.50
daily: 7, sum: 432, count: 6, avg: 72.00

daily: 8, sum: 217, count: 3, avg: 72.33
daily: 9, sum: 293, count: 4, avg: 73.25

*Show that there is no sorting algorithm which sorts at least* $(1/2^n) \times n!$ *instances in* $O(n)$ *time.*

---

Think of the decision tree which can do this. What is the shortest tree with $(1/2^n) \times n!$ leaves?



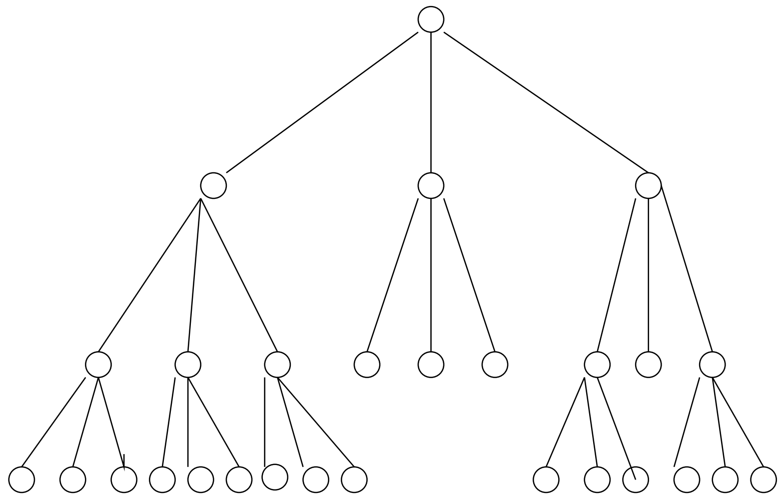

$$h > \lg(n!/2^n) \;=\; \lg(n!) - \lg(2^n)$$

$$
\begin{aligned}
&= \Theta(n \lg n) - n \\
&= \Theta(n \lg n)
\end{aligned}
$$

Moral: there cannot be too many good cases for any sorting algorithm!

*Show that the $\Omega(n \lg n)$ lower bound for sorting still holds with ternary comparisons.*

---



The maximum number of leaves in a tree of height h is $3^h$,

$$\lg_3(n!) = \Theta(n \lg n)$$

So it goes for any constant base.

# Optimization Problems

In the algorithms we have studied so far, correctness tended to be easier than efficiency. In optimization problems, we are interested in finding a *thing* which maximizes or minimizes some function.

In designing algorithms for optimization problem - we must prove that the algorithm in fact gives the best possible solution.

*Greedy* algorithms, which makes the best local decision at each step, occasionally produce a global optimum - but you need a proof!

# Dynamic Programming

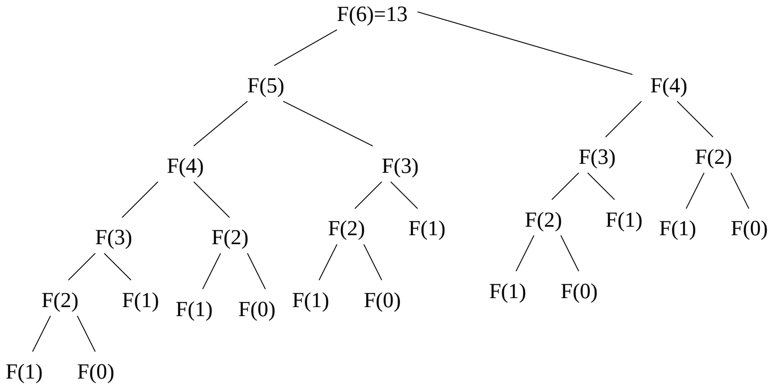
Dynamic Programming is a technique for computing recurrence relations efficiently by sorting partial results.

# Computing Fibonacci Numbers

$$F_n = F_{n-1} + F_{n-2}$$

$$F_0 = 0, F_1 = 1$$

Implementing it as a recursive procedure is easy but slow!
We keep calculating the same value over and over!



# How slow is slow?

$F_{n+1}/F_n \approx \phi = (1+\sqrt{5})/2 \approx 1.61803$
Thus $F_n \approx 1.6^n$, and since our recursion tree has $0$ and $1$ as leaves, means we have $\approx 1.6^n$ calls!

## What about Dynamic Programming?

We can calculate $F_n$ in linear time by storing small values:

$F_0 = 0$
$F_1 = 1$
For $i = 1$ to $n$

$$F_i = F_{i-1} + F_{i-2}$$

*Moral: we traded space for time.*

Dynamic programming is a technique for efficiently computing recurrences by storing partial results.

Once you understand dynamic programming, it is usually easier to reinvent certain algorithms than try to look them up!

Dynamic programming is best understood by looking at a bunch of different examples.

I have found dynamic programming to be one of the most useful algorithmic techniques in practice:

- Morphing in Computer Graphics
- Data Compression for High Density Bar Codes
- Utilizing Grammatical Constraints for Telephone Keypads

# Multiplying a Sequence of Matrices

Suppose we want to multiply a long sequence of matrices $A \times B \times C \times D \ldots$.

Multiplying an $X \times Y$ matrix by a $Y \times Z$ matrix (using the common algorithm) takes $X \times Y \times Z$ multiplications.

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 13 & 18 & 23 \\ 18 & 25 & 32 \\ 23 & 32 & 41 \end{bmatrix}$$

We would like to avoid big intermediate matrices, and since matrix multiplication is *associative*, we can parenthesise however we want.

Matrix multiplication is *not communitive*, so we cannot permute the order of the matrices without changing the result.

## Example

Consider $A \times B \times C \times D$, where $A$ is $30 \times 1$, $B$ is $1 \times 40$, $C$ is $40 \times 10$, and $D$ is $10 \times 25$.
There are three possible parenthesizations:

$$((AB)C)D = 30\times 1\times 40 + 30\times 40\times 10 + 30\times 10\times 25 = 20,700$$

$$(AB)(CD) = 30\times 1\times 40 + 40\times 10\times 25 + 30\times 40\times 25 = 41,200$$

$$A((BC)D) = 1\times 40\times 10 + 1\times 10\times 25 + 30\times 1\times 25 = 1400$$

The order makes a big difference in real computation. How do we find the best order?
Let $M(i,j)$ be the *minimum* number of multiplications necessary to compute $\Pi_{k=i}^{j} A_k$.

The key observations are

- The outermost parentheses partition the chain of matricies $(i, j)$ at some $k$.
- The optimal parenthesization order has optimal ordering on either side of $k$.

A recurrence for this is:

$$
\begin{aligned}
M(i,j) &= Min_{i \le k \le j-1}[M(i,k) + M(k+1,j) + d_{i-1}d_k d_j] \\
M(i,i) &= 0
\end{aligned}
$$

If there are $n$ matrices, there are $n+1$ dimensions.
A direct recursive implementation of this will be exponential, since there is a lot of duplicated work as in the Fibonacci recurrence.
Divide-and-conquer is seems efficient because there is no overlap, but . . .
There are only $\binom{n}{2}$ substrings between $1$ and $n$. Thus it requires only $\Theta(n^2)$ space to store the optimal cost for each of them.
We can represent all the possibilities in a triangle matrix. We can also store the value of $k$ in another triangle matrix to

reconstruct to order of the optimal parenthesisation.

The diagonal moves up to the right as the computation progresses. On each element of the $k$th diagonal $|j-i|=k$.

For the previous example:

Procedure MatrixOrder

for $i=1$ to $n$ do $M[i,j]=0$

for $diagonal=1$ to $n-1$

  for $i=1$ to $n-diagonal$ do

    $j=i+diagonal$

    $M[i,j]=\min_{i=k}^{j-1}[M[i,k]+M[k+1,j]+d_{i-1}d_kd_j$

    faster$(i,j)=k$

return $[m(1,n)]$

Procedure ShowOrder$(i, j)$
if $(i = j)$ write ($A_i$)
else

$k =$factor$(i, j)$
write "("
ShowOrder$(i, k)$
write "*"
ShowOrder $(k + 1, j)$
write ")"

## A dynamic programming solution has three components:

1. Formulate the answer as a recurrence relation or recursive algorithm.
2. Show that the number of different instances of your recurrence is bounded by a polynomial.
3. Specify an order of evaluation for the recurrence so you always have what you need.

# Approximate String Matching

A common task in text editing is string matching - finding all occurrences of a word in a text.

Unfortunately, many words are mispelled. How can we search for the string closest to the pattern?

Let $p$ be a pattern string and $T$ a text string over the same alphabet.

A $k$-approximate match between $P$ and $T$ is a substring of $T$ with at most $k$ differences.

Differences may be:

1. the corresponding characters may differ: KAT$\longrightarrow$ CAT
2. $P$ is missing a character from $T$: CAAT$\longrightarrow$ CAT

3. $T$ is missing a character from $P$: CT$\longrightarrow$ CAT

Approximate Matching is important in genetics as well as spell checking.

# A 3-Approximate Match

A match with one of each of three edit operations is:

$$P = \text{unescessaraly}$$
$$T = \text{unnecessarily}$$

Finding such a matching seems like a hard problem because we must figure out where you add *blanks*, but we can solve it with dynamic programming.

$D[i, j]$ = the minimum number of differences between $P_1, P_2, \ldots, P_i$ and the segment of $T$ ending at $j$.

$D[i, j]$ is the *minimum* of the three possible ways to extend smaller strings:

1. If $P_i = t_i$ then $D[i-1, j-1]$ else $D[i-1, j-1]+1$ (corresponding characters do or do not match)

2. $D[i-1, j]+1$ (extra character in text – we do not advance the pattern pointer).

3. $D[i, j-1]+1$ (character in pattern which is not in text).

Once you accept the recurrence it is easy.
To fill each cell, we need only consider three other cells, not $O(n)$ as in other examples. This means we need only store two rows of the table. The total time is $O(mn)$.

# Boundary conditions for string matching

What should the value of $D[0, i]$ be, corresponding to the cost of matching the first $i$ characters of the text with none of the pattern?

It depends. Are we doing string matching in the text or substring matching?

- If you want to match all of the pattern against all of the text, this meant that would have to delete the first $i$ characters of the pattern, so $D[0, i] = i$ to pay the cost of the deletions.
- if we want to find the place in the text where the pattern occurs? We do not want to pay more of a cost if the

pattern occurs far into the text than near the front, so it is important that starting cost be equal for all positions. In this case, $D[0, i] = 0$, since we pay no cost for deleting the first $i$ characters of the text.

In both cases, $D[i, 0] = i$, since we cannot excuse deleting the first $i$ characters of the pattern without cost.

# What do we return?

If we want the *cost* of comparing all of the pattern against all of the text, such as comparing the spelling of two words, all we are interested in is $D[n, m]$.

But what if we want the cheapest match between the pattern anywhere in the text? Assuming the initialization for substring matching, we seek the cheapest matching of the full pattern ending anywhere in the text. This means the cost equals $\min_{1 \leq i \leq m} D[n, i]$.

This only gives the cost of the optimal matching. The actual alignment – what got matched, substituted, and deleted – can be reconstructed from the pattern/text and table without an auxiliary storage, once we have identified the cell with the lowest cost.

## How much space do we need?

Do we need to keep all $O(mn)$ cells, since if we evaluate the recurrence filling in the columns of the matrix from left to right, we will never need more than two columns of cells to do what we need. Thus $O(m)$ space is sufficient to evaluate the recurrence without changing the time complexity at all.

Unfortunately, because we won't have the full matrix we cannot reconstruct the alignment, as above.

Saving space in dynamic programming is very important. Since memory on any computer is limited, $O(nm)$ space is more of a bottleneck than $O(nm)$ time.

Fortunately, there is a clever divide-and-conquer algorithm which computes the actual alignment in $O(nm)$ time and

$O(m)$ space.