Matchings

A subset of the edges of a graph form a matching if no two of the edges are incident upon the same vertex.

As we will see, matchings have applications to a variety of assignment problems.

A maximal matching is one which cannot be made larger by adding edges.

A maximum matching is the largest possible matching for the graph.

A perfect matching contains $N$ edges in a 2n vertex graph.
Not all graphs with an even number of vertices have perfect matchings, but many interesting classes of graphs do: cycles, complete graphs, and hypercubes.

![Graphs](image)

This is just one of the properties which make hypercubes nice. They are also Hamiltonian for \( k \geq 2 \), and the distance between any two nodes is \( \leq k \). Since the diameter is small and the connectivity is high with only \( k \cdot 2^{k-1} \) edges, it is a good topology for parallel computers.
Finding a Maximum Matching

There is a fairly simple condition to test if a matching is maximum, and if not, make it larger.

Given a graph \( G \) and a matching \( M \), an augmenting path in \( G \) is a path of odd length such that the first and last vertices are not in \( M \), and path edges alternate not in \( M \), then in \( M \). The path cannot contain a cycle.
Clearly, a matching is not maximum if it contains an augmenting path, since we can reverse all edges, thus making the matching one edge larger.

Berge's theorem states this necessary condition is also sufficient. A matching is maximum if and only if it contains no augmenting path!

Proof: Let $m'$ be a maximum matching of $G$, and $m$ be non-maximum matching. Thus: $|m'| > |m|.$

Let's consider the graph formed by the symmetric difference of $M + M'$ – the union of the matchings minus the intersection.
Example:

\[ G \]

\[ M', M, H = (M \cup M') - (M \cap M') \]

The connected components of \( H \) must be cycles or paths since the maximum degree of \( H \) is 2. Further, the edges on each cycle or path must alternate red-green, i.e., from different matchings. Since the number of red edges on any cycle = the number of green edges, for \( |M'| > |M| \) there exists at least one red-green-...red alternation path!
Bipartite Matching

We live in a world where we must pair off elements of one set with elements of another:

- People <-> Jobs
- Men <-> Women
- Tasks <-> equipment

If we represent each element by a vertex and draw an edge between elements of different sets which can be matched, we create a bipartite graph:

- Abe
- Bill
- Chris
- Dive

- Amy
- Betty
- Chris
- Dive

Bipartite matching is an important special case of matching in graphs.
Given a set of boys $B$, and a set of girls $G$, is it possible for each boy to marry a girl he knows without bigamy?

In graph theoretic terms, when does a bipartite graph have a matching of size $|B|$, where $V = B \cup G$ and $|B| \leq |G|$?

**Hall's Marriage Theorem**: Such a matching exists iff any subset of boys knows at least as many girls between them.

It is clear that this condition is necessary, since if $|B|$ boys know less than $|B|$ girls, each can't get a girl, without sharing.

<table>
<thead>
<tr>
<th>Abe</th>
<th>Amy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bill</td>
<td>Betty</td>
</tr>
<tr>
<td>Chris</td>
<td>Chris</td>
</tr>
<tr>
<td>Dave</td>
<td>Dique</td>
</tr>
</tbody>
</table>

Since \{Bill, Chris, Dave\} only know \{Betty, Dique\}, no matching exists.
Proof of Hall's Theorem:

By induction on the number of boys $m$.

For $m = 1$, a matching obviously exists if the boy knows at least one girl.

Assume true for up to $m-1$ boys.

Case 1: Suppose every subset of $K < m$ boys knew at least $K+1$ girls.

Then marry off any boy to any girl he knows. This leaves a smaller problem which satisfies Hall's condition, so by induction can be matched.

Case 2: There exists a subset of $K < m$ boys which knows exactly $K$ girls.

By induction, these $K$ boys can be married off, leaving $N-K$ boys. Suppose now there exist a set of $K$ boys who know less than $K$ girls (for otherwise we can complete the matching by induction).

But then these $k$th boys in the original graph knew less than $k$th girls, a contradiction!
The Most Important Algorithm
Problem you will ever solve

Given a set of numbers, find which is the largest one.

15 96 35 42 7 13 84

Ah, but what if the numbers are given to you one at a time, and you cannot go back: N=5

75
14
33
88
92
Finding the Optimal Mate

Most of us seek Mr. or Mrs. Right, the single spouse which will maximize our happiness. But how do we recognize them when we see them?

The only way to determine someone’s absolute desirability is to compare them with everyone else. But we are normally able to date (evaluate) one person at a time, or else they get mad and ditch us.

Abstracting the problem - suppose we can potentially date $n$ people in our lifetime, one at a time, no peeking ahead, no going back. Suppose we are dating the $i^{th}$ person. We must decide whether to marry them before looking at the $(i+1)^{st}$ person.

What strategy can we use to maximize our chances of getting the best?
what I see | what the future holds
\[ w_1, \ w_2, \ w_3, \ w_4, \ w_5 \]
\[ w_1 < w_2 \]
\[ w_3 < w_1, \ < w_2 \]

Should I marry \( w_3 \) or try my luck further?

If any permutation of women is equally likely, and none of them will turn me down (a happy if unlikely assumption), what strategy will maximize my chance of getting #1?

Clearly, the probability that the 1st person I see is best is \( 1/n \), and so it might be the first person I encounter!
The optimal strategy is to look at some fraction of the \( n \) people, reject them all, and pick the next person better than anyone I have seen to date.

Suppose I reject the first \( n/2 \) people. At least \( 1/4 \) of the time I get the best

\[
\omega_1, \; \omega_2, \ldots, \omega_{n/2} \rightarrow \omega_{n/2+1}, \ldots, \omega_n
\]

\[
\Pr(\text{\#2 is in the first } \frac{n}{4} \text{ of } n) = \frac{1}{2}
\]

\[
\Pr(\text{\#1 is in the second } \frac{n}{4} \text{ of } n) = \frac{1}{2}
\]

If these were independent (they almost are), I win this way \( 1/4 \) of the time! Plus there are other ways I win:

\[
\ldots \; 3 \ldots \rightarrow \ldots \; 1 \; 2 \ldots
\]

In fact, the optimal strategy is to wait \( \frac{1}{e^2} \) of your life - so don't be fooled by the very first one...