Basic Maneuvers:
You should see WHY each of these works!

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition:</td>
<td>$\alpha F(z) + \beta G(z) = \sum_{n} (\alpha f_n + \beta g_n) z^n$</td>
</tr>
<tr>
<td>Right shift:</td>
<td>$z^m G(z) = \sum_{n} g_{n-m} z^n$, integer $m \geq 0$</td>
</tr>
<tr>
<td>Left shift:</td>
<td>Delete low order terms</td>
</tr>
<tr>
<td>$G(z) = g_0 - g_1 z - \cdots - g_{m-1} z^{m-1}$</td>
<td>$\sum_{n \geq 0} g_{n+m} z^n$, integer $m \geq 0$</td>
</tr>
<tr>
<td>Variable substitution:</td>
<td>$G(cz) = \sum_{n} c^n g_n z^n$</td>
</tr>
<tr>
<td>Differentiation:</td>
<td>$G'(z) = \sum_{n} (n+1) g_{n+1} z^n$</td>
</tr>
<tr>
<td>Lower degree by 1</td>
<td>$zG'(z) = \sum_{n} n g_n z^n$</td>
</tr>
<tr>
<td>Integration:</td>
<td>$\int_0^z G(t) , dt = \sum_{n \geq 1} \frac{1}{n} g_{n-1} z^n$</td>
</tr>
<tr>
<td>Convolution:</td>
<td>$F(z) G(z) = \sum_{n} \left( \sum_{k} f_k g_{n-k} \right) z^n$</td>
</tr>
<tr>
<td>Special case</td>
<td>$f_i = 1$ gives $\frac{1}{1-z} G(z) = \sum_{n} \left( \sum_{k \leq n} g_k \right) z^n$</td>
</tr>
</tbody>
</table>

Prefix summation
You should understand how to move between each sequence, generating function, and closed form!

### Table 32.1 Simple sequences and their generating functions.

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Generating function</th>
<th>Closed Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle 1, 0, 0, 0, 0, 0, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} [n = 0] z^n$</td>
<td>1</td>
</tr>
<tr>
<td>$\langle 0, \ldots, 0, 1, 0, 0, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} [n = m] z^n$</td>
<td>$z^m$</td>
</tr>
<tr>
<td>$\langle 1, 1, 1, 1, 1, 1, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} z^n$</td>
<td>$\frac{1}{1-z}$</td>
</tr>
<tr>
<td>$\langle 1, -1, 1, -1, 1, -1, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} (-1)^n z^n$</td>
<td>$\frac{1}{1+z}$</td>
</tr>
<tr>
<td>$\langle 1, 1, 1, 1, 1, 1, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} [2 \backslash n] z^n$</td>
<td>$\frac{1}{1-z^2}$</td>
</tr>
<tr>
<td>$\langle 1, 0, 1, 0, 1, 0, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} [m \backslash n] z^n$</td>
<td>$\frac{1}{1-z^m}$</td>
</tr>
<tr>
<td>$\langle 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} (n+1) z^n$</td>
<td>$\frac{1}{(1-z)^2}$</td>
</tr>
<tr>
<td>$\langle 1, 2, 3, 4, 5, 6, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} 2^n z^n$</td>
<td>$\frac{1}{1-2z}$</td>
</tr>
<tr>
<td>$\langle 1, 2, 4, 8, 16, 32, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} \binom{4}{n} z^n$</td>
<td>$(1+z)^4$</td>
</tr>
<tr>
<td>$\langle 1, 4, 6, 4, 1, 0, 0, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} \binom{c}{n} z^n$</td>
<td>$(1+z)^c$</td>
</tr>
<tr>
<td>$\langle 1, c, \binom{c}{2}, \binom{c}{3}, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} \binom{c+n-1}{n} z^n$</td>
<td>$\frac{1}{(1-z)^c}$</td>
</tr>
<tr>
<td>$\langle 1, c, c^2, c^3, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} c^n z^n$</td>
<td>$\frac{1}{1-cz}$</td>
</tr>
<tr>
<td>$\langle 1, \binom{m+1}{m}, \binom{m+2}{m}, \binom{m+3}{m}, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} \binom{m+n}{m} z^n$</td>
<td>$\frac{1}{(1-z)^{m+1}}$</td>
</tr>
<tr>
<td>$\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$</td>
<td>$\sum_{n \geq 1} \frac{1}{n} z^n$</td>
<td>$\ln \frac{1}{1-z}$</td>
</tr>
<tr>
<td>$\langle 0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots \rangle$</td>
<td>$\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} z^n$</td>
<td>$\ln(1+z)$</td>
</tr>
<tr>
<td>$\langle 1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots \rangle$</td>
<td>$\sum_{n \geq 0} \frac{1}{n!} z^n$</td>
<td>$e^z$</td>
</tr>
</tbody>
</table>
What is the Expansion of \( \frac{1}{(1-z)^{n+1}} \)?

The expansion will require binomial coefficients with negative arguments, which is OK since we define \( \binom{n}{k} = \frac{n^k}{k!} \).

Since
\[
\frac{n^k}{k!} = n(n-1) \ldots (n-k+1)
\]
\[
= (-1)^k (n)(1-n) \ldots (k-n-1)
\]
\[
= (-1)^k \frac{k}{k-n-1}
\]

We have the identity
\[
\binom{n}{k} = (-1)^k \frac{k}{k-n-1}
\]
By the binomial theorem:

\[
\frac{1}{(1-z)^{m+1}} = \left(1-\frac{z}{1-z}\right)^{-m-1} = \sum_{n=0}^{\infty} \binom{-(m+1)}{n} (-z)^n \frac{1}{(1-z)^{m-n}}
\]

\[
= \sum_{n=0}^{\infty} \binom{m-1}{n} (-1)^n z^n
\]

\[
= \sum_{n=0}^{\infty} \binom{n+m-1}{n-1} (-1)^n z^n
\]

Since \( \binom{r}{k} = (-1)^n \binom{k-r-1}{k} \) with the substitution \( k = n \) and \( r = m+n \)

\[
= \sum_{n=0}^{\infty} \binom{n+m}{n} z^n
\]

Note that \( \binom{n+m}{n} \) is an \( m \)th degree polynomial in \( n \).
Solving Recurrences

The main advantage of the generating function machinery is providing a systematic way to solve recurrences:

1. Express the recurrence as a single equation
2. Multiply both sides by $z^n$ and sum over all $n$, giving generating function $G(z)$.
3. Find a closed form for $G(z)$
4. Expand $G(z)$ into a power series and read the coefficients.

Ex: Fibonacci Numbers

1. $F_n = F_{n-1} + F_{n-2} + [n=1]$ 
2. $G(z) = \sum F_{n-1} z^n + \sum F_{n-2} z^n + z$ 
   $= z G(z) + z^2 G(z) + z$ 
3. \[ G(z) = \frac{z}{1-z-z^2} = \frac{1}{\sqrt{5}} \left( \phi z - \frac{1}{\phi z} \right) \] 
4. $F_n = \frac{1}{\sqrt{5}} (\phi^n - \frac{1}{\phi^n})$
Expansion Theorem for Rational Generating Functions

The heart of Step 4, expanding the power series, takes a closed form $G(z) = \frac{P(z)}{Q(z)}$ and expresses it as a partial fraction.

This means we take the denominator $Q(z)$ and factor it, and find appropriate numerators so $G(z) = \frac{A}{(1-\beta_1 z)} + \frac{B}{(1-\beta_2 z)}$ ......

where $\frac{1}{\beta_i}$ is a root of $Q(z)$

From this representation, we can expand each term into a geometric series. Thus if constant can be found for $A, B$ ....., it makes sense the solution of any recurrence will largely be powers of the reciprocals of the roots of the denominator of its generating function!

$C_1 \left(\frac{1}{\beta_1}\right)^n + C_2 \left(\frac{1}{\beta_2}\right)^n$ ......
But what does the partial fraction expansion look like with repeated roots?

\[ R(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{\prod_{i=1}^{d_2} (1-P_i z)^{d_i}} \]

\[ = \frac{A_1}{1-P_1 z} + \frac{A_2}{(1-P_1 z)^2} + \cdots + \frac{A_{d_1}}{(1-P_1 z)^{d_1}} + + \]

\[ \frac{B_1}{1-P_2 z} + \frac{B_2}{(1-P_2 z)^2} + \cdots + \frac{B_{d_2}}{(1-P_2 z)^{d_2}} \]

where \( A_i \) and \( B_i \) are all constants if \( Q(z) \) is of higher degree than \( P(z) \).

Thus we can partition any rational generating function in \( d \) functions:

\[ \frac{C}{(1-P z)^d} \]

where \( d \) is the degree of \( Q(z) \),

**EACH OF WHICH WE KNOW HOW TO EXPAND!**
Further, what does the expansion look like? If the \( i \)th root \( \rho_i \) has multiplicity \( d_i \),

\[
\frac{A_1}{(1-\rho_i z)} + \frac{A_2}{(1-\rho_i z)^2} + \ldots + \frac{A_{d_i}}{(1-\rho_i z)^{d_i}}
\]

has as the \( k \)th term in its expansion,

\[
A_1 \rho_i^k z^k + A_2 \binom{k+1}{k} \rho_i^k z^k + \ldots + A_{d_i} \binom{k+d_i-1}{k} \rho_i^k z^k
\]

which has the form

\[
\left( C_{d_i} z^k + \ldots + C_i \rho_i^{k+C_0} \right) (\rho_i z)^k
\]

So we now the solution to any recurrence whose generating function is rational, except for the constants \( C_i \), and these can be found by solving a system of linear equations from the first \( d \) values of the recurrence.
Ex: \( g_0 = 9, = 1 \)

\( g_n = g_{n-1} + 2g_{n-2} + (-1)^n \)

Solve using the Four Step Method

1. Write it as a single recurrence

\( g_n = g_{n-1} + 2g_{n-2} + (-1)^n [n \geq 0] + [n = 1] \)

The assumption with generating functions is that the coefficients are 0 for \( z^{-n}, n > 0 \)

2. Turn it into a generating function

\[
G(z) = \sum_{n=0}^{\infty} g_n z^n = \sum_{n=0}^{\infty} g_{n-1} z^n + 2 \sum_{n=0}^{\infty} g_{n-2} z^n
\]

\[
= \frac{1}{1+z} + z
\]

\[
= z \cdot G(z) + 2z^2 G(z) + \frac{1+z(1+z)}{1+z}
\]

\( \text{Simple shifts are handled by multiplying by } z \)

3. Solve for \( G(z) \)

\[
G(z) = \frac{1+z+z^2}{(1+z)(1-z-2z^2)}
\]
4. Expand into power series

\[ G(z) = \frac{1+z+z^2}{(1+z)^2(1-zz)} \]

The roots to the denominator are \( z = -1, -1, \frac{1}{2} \)

\[ g_n = (-1)^n(c_1 + c_3n) + 2^n c_3 \]

\[ g_0 = 1 = c_1 + c_3 \]
\[ g_1 = 1 = 2c_3 - c_1 - c_2 \]
\[ g_2 = 1 + 2 + 1 = 4 = c_1 + 2c_2 + 4c_3 \]

we could have gotten the actual constants directly from the Expansion Theorem, but I find it easy to solve equations once I know the form.

One caution. This method only works when the degree of the numerator is less than the denominator. If not, the problem often can be resolved e.g.:

\[ \frac{100}{Z} \]
\[ \frac{Z}{Z^5 + Z^3 + Z + 1} \]

right shift 100 places the solution to \( \frac{10}{Z^5 + Z^3 + Z + 1} \)
Ex: 3 x N Domino Tilings

The total number of ways to tile a 3 x N region is governed by the recurrence

\[ U_0 = 1 \]
\[ U_1 = 0 \]
\[ U_N = U_{N-2} + 2V_{N-1} \]

where

\[ V_0 = 0 \]
\[ V_1 = 1 \]
\[ V_N = U_{N-1} + U_{N-2} \]

Thus:

\[ U_N = U_{N-2} + 2V_{N-1} + [N = 0] \]
\[ V_N = U_{N-1} + V_{N-1} \]

So,

\[ U(z) = z^2 U(z) + z^4 V(z) + 1 \]
\[ V(z) = z U(z) + z^4 V(z) \]

These are two interrelated generating functions.

With two equations and two unknowns, we can solve for \( U(z) + V(z) \):
\[ U(z) = \frac{1 - z^2}{1 - 4z^2 + z^4} \quad V(z) = \frac{z}{1 - 4z^2 + z^4} \]

Since \( U(z) \) is what we are interested in, and it is the proper ratio of polynomials, we can solve it as before, but it involves finding four roots.

The slick solution observes all exponents in the denominator are even, so if:

\[ w(z) = \frac{1}{1 - 4z^2 + z^4}, \quad \text{then} \]

\[ U(z) = (1 - z^2) w(z^2) \quad V(z) = 2 \cdot w(z^2) \]

Since \( w_n = \frac{3 + 2\sqrt{3}}{6} (z + \sqrt{3})^n + \frac{3 - 2\sqrt{3}}{6} (z - \sqrt{3})^n \)

\[ U_{an} = W_n - W_{n-1} = \frac{(z + \sqrt{3})^n}{3 + \sqrt{3}} + \frac{(z - \sqrt{3})^n}{3 - \sqrt{3}} \]

The moral, understand the combinatorial significance of GF (can save much calculation).