# *Solid and Shape Modeling 2005*

# *Manifold Splines*

Xianfeng David Gu, Ying He, Hong Qin

`gu,yhe,qin@cs.sunysb.edu`

Center of Visual Computing

State University of New York at Stony Brook

# *Motivation*

- Splines with planar domains have been successfully applied in industrial and academic fields for many years.
- The theoretic foundation of planar splines has been fully developed, which is rigorous and complete.
- Splines with manifold domains have not been well studied or fully understood.
- The theoretic foundation of Manifold Splines hasn't been laid down.

*Goal:*

1. *Establish the theoretic foundation of Manifold Splines;*
2. *Discover the practical method to construct Manifold Splines.*

# *Motivation*

- Most geometric shapes are manifolds with complicated topologies, metrics and embeddings.
- Current Splines are defined on planar domains. Spline patches have to be glued together. The patching procedure is difficult.
  1. The manifold is partitioned to several patches.
  2. Splines with planar domains are constructed for each patch.
  3. The knots and control points along the patch boundaries are adjusted in order to make the patches meet smoothly.
- Most naturally, Splines should be defined on manifolds directly.

# *Problem Statement*

In order to solve the manifold spline problem, we need to answer the following questions.

**Question**: Given a domain manifold $M$,

1. how to define a manifold spline on $M$ precisely.
2. how to verify whether $M$ admits a manifold spline or not.
3. If it doesn't, what is the obstruction, is the obstruction geometric, combinatorial or topological?
4. If $M$ admits a manifold spline, how to design practical algorithms to construct the manifold spline?

# *Extrinsic and Intrinsic Manifold Splines*

There are two major categories of manifold splines, *extrinsic* and *intrinsic*.

**Extrinsic Manifold Spline**

1. Define a spline function from $\mathbb{R}^3$ to $\mathbb{R}$, $f(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$
2. Embed the domain manifold $M$ in $\mathbb{R}^3$, $M \hookrightarrow R^3$.
3. The restriction of $f$ on $M$ is the extrinsic manifold spline,

$$f(x,y,z)|_M : M \longrightarrow \mathbb{R}.$$

**Intrinsic Manifold Spline**

- Define spline functions on each chart of $M$.
- The splines are coherently defined.

# *Extrinsic vs. Intrinsic*

**Extrinsic Manifold Spline**

- $M$ must be realizable in $\mathbb{R}^3$, $M \hookrightarrow \mathbb{R}^3$. The splines depends on the embedding of $M$.
- In nature, the splines are tri-variate.
- If $M$ is a mesh, the differentiation is not well defined.
- Appropriate for special domain manifolds, such as sphere, hyperboloid with closed analytical forms.
- Easy to define, difficult to evaluate.

**Intrinsic Manifold Spline**

- $M$ could be an abstract manifold, it is unnecessary to be realizable in $\mathbb{R}^3$. The evaluation is independent of the embedding of $M$.
- In nature, the splines are bi-variate.
- The differentiation is well defined.
- Appropriate for arbitrary manifolds.
- Difficult to define, easy to evaluate.

# *Fundamental Difficulty for Extrinsic Splines*

**Extrinsic manifold splines** have the fundamental disadvantages.

- Extrinsic manifold splines require embedding of the domain manifolds.
- Domain manifolds are represented as polygonal meshes, which are not smooth (lack differential structure).
- The differentiation of extrinsic manifold splines is not well defined.

**Intrinsic manifold splines** are the major focus of the current work.

# *Intrinsic Manifold Splines*

The following two questions are inspiring for understanding intrinsic manifold splines.

1. Given a point on a conventional subdivision surface ( such as Catmull-Clark or Loop subdivision surface), it has a neighborhood, in this neighborhood, the surface patch can be treated as a conventional $B$-spline patch. The question is:
   - With respect to which parameter domain, the surface patch is a BSpline patch?
   - Are those parameter domains finite or Infinite? Do the domains overlap? How the domains are organized?

   Answer : Affine atlas.

2. Can one use the same domains to generalize triangular $B$-spline (DMS) on the surface?
   - DMS has no requirement for the connectivities, will there still be extraordinary points?

   Answer: Topological Obstruction

# *Intrinsic Manifold Splines*

The manifold splines should have the following desired properties,

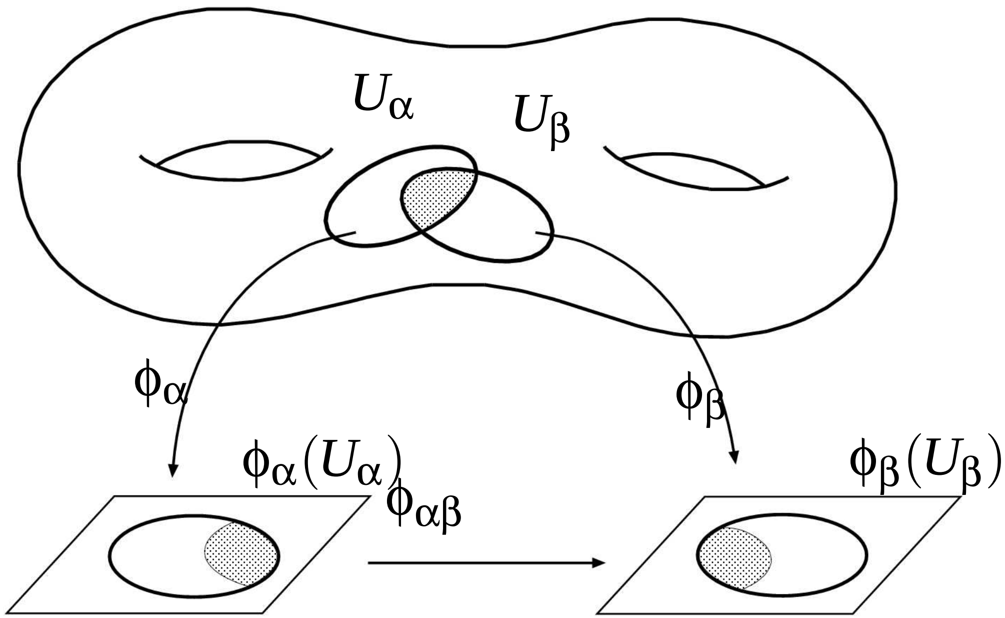
- Locally, in a neighborhood, it is just like a spline defined on a planar domain.
- Globally, the evaluation of the spline is independent of the choice of the neighborhoods.
- It preserves all the properties of planar splines
  - Local support,
  - Convex hull,
  - Polynomial reproduction,
  - Affine invariance,
  - Smoothness.

# *Definition of Manifold*

A *manifold* of dimension $n$ is a connected Hausdorfff space $M$ for which every point has a neighborhood $U$ that is homeomorphic to an open subset $V$ of $\mathbb{R}^n$. Such a homeomorphism

$$\phi : U \rightarrow V$$

is called a coordinate chart. An *atlas* is a family of charts $\{U_\alpha, \phi_\alpha\}$ for which $U_\alpha$ constitute an open covering of $M$.

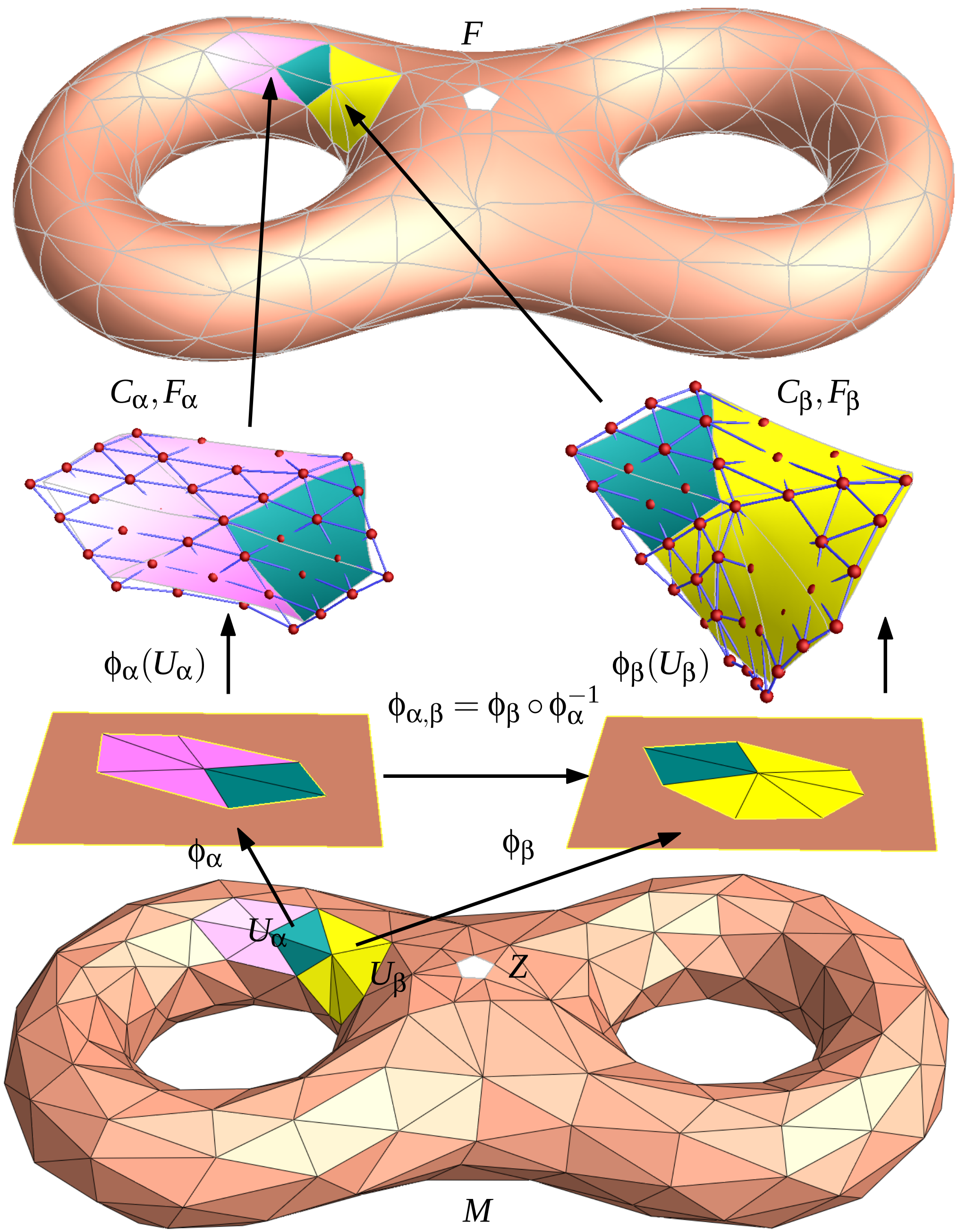


# *Transition Functions*

Transition function plays an vital role in the manifold spline problem.

- *Transition function*: Suppose $\{U_\alpha, \phi_\alpha\}$ and $\{U_\beta, \phi_\beta\}$ are two overlapping charts on a manifold $M$, $U_\alpha \cap U_\beta \neq \emptyset$, the chart transition is
$$\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$$
- Atlas can be classified by transition functions.

# *Definition of Manifold Splines*



# *Formulation of Manifold Splines*

Suppose $M$ is the domain manifold with an atlas $\{(U_\alpha, \phi_\alpha)\}$, $\phi_\alpha : U_\alpha \to \mathbb{R}^2$, chart transition functions are

$$\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\beta \cap U_\alpha),$$

$$f : M \to \mathbb{R}^3, f_\alpha = f \circ \phi_\alpha,$$

- *Manifold*: The function is defined on the manifold, namely, the evaluation is independent of the choice of the chart:

$$\begin{tikzcd} \phi_\alpha(U_\alpha \cap U_\beta) \arrow[r, "\phi_{\alpha\beta}"] \arrow[d, "f_\alpha"'] & \phi_\beta(U_\alpha \cap U_\beta) \arrow[d, "f_\beta"] \\ \mathbb{R}^3 \arrow[r, "id"'] & \mathbb{R}^3 \end{tikzcd} \qquad f_\alpha(p) \equiv f_\beta \circ \phi_{\alpha\beta}(p).$$

- *Spline*: On each chart, the function is a conventional Spline with planar domain, namely $f_\alpha, f_\beta$ are conventional splines and can be represented as polar forms. Namely, the above diagram is commutable.

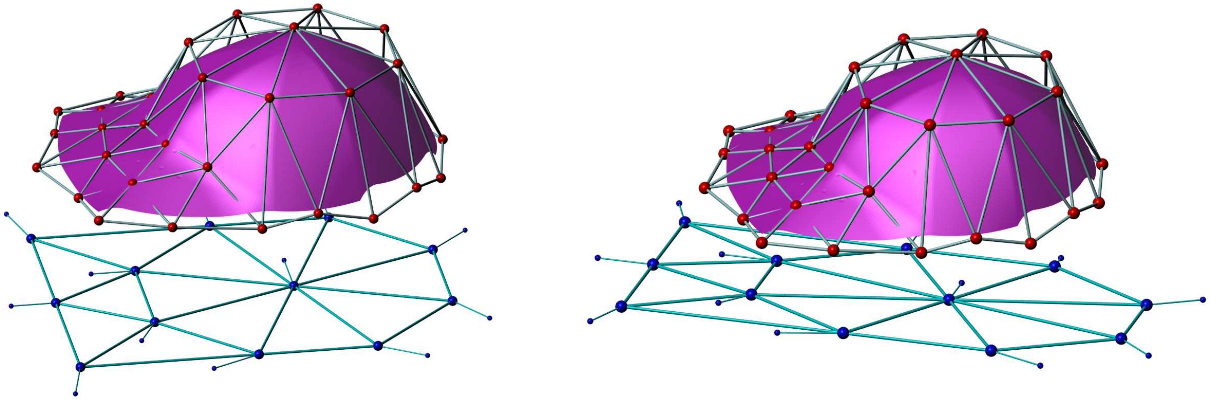
# *Fundamental Theorem of Manifold Splines*

**Definition 0.1** (Affine Atlas)**.** *A manifold $M$ has an atlas $\{(U_\alpha, \phi_\alpha)\}$, if all transition functions $\phi_{\alpha\beta}$ are affine, then the atlas is an affine atlas.*

**Definition 0.2** (Affine Structure)**.** *Two affine atlas are equivalent, if their union is still an affine atlas. Each affine atlas equivalence class is an affine structure.*

**Theorem 1** (Fundamental Theorem)**.** *A manifold $M$ admits a manifold spline scheme, if and only if $M$ admits an affine structure.*

# *Fundamental Theorem of Manifold Splines*



$$\begin{tikzcd}
\mathbf{u}, V^I_\beta \arrow[r, "\phi"] \arrow[d] & \phi(\mathbf{u}), \phi(V^I_\beta) \arrow[d] \\
F \arrow[r, "\phi"'] & F \circ \phi
\end{tikzcd}$$

- If $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine map, then both $\mathbf{u} \to F(\mathbf{u})$ and $\phi(\mathbf{u}) \to F \circ \phi(\mathbf{u})$ are triangular $B$-splines (piecewise polar forms).
- If both $\mathbf{u} \to F(\mathbf{u})$ and $\phi(\mathbf{u}) \to F \circ \phi(\mathbf{u})$ are triangular $B$-splines (piecewise polar forms), then $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ must be an affine map.

# *Existence*

**Theorem 2** (Benzécri)**.** *Let $S$ be a closed two dimensional affine manifold, then* $\chi(S) = 0$.

John Milnor generalizes the theorem to arbitrary dimensions using Characteristic Class theory of vector bundles.

The topological obstruction for affine structure is the Euler class. Therefore, any closed manifold admits manifold splines if and only if its Euler class is zero.

# *Existence*

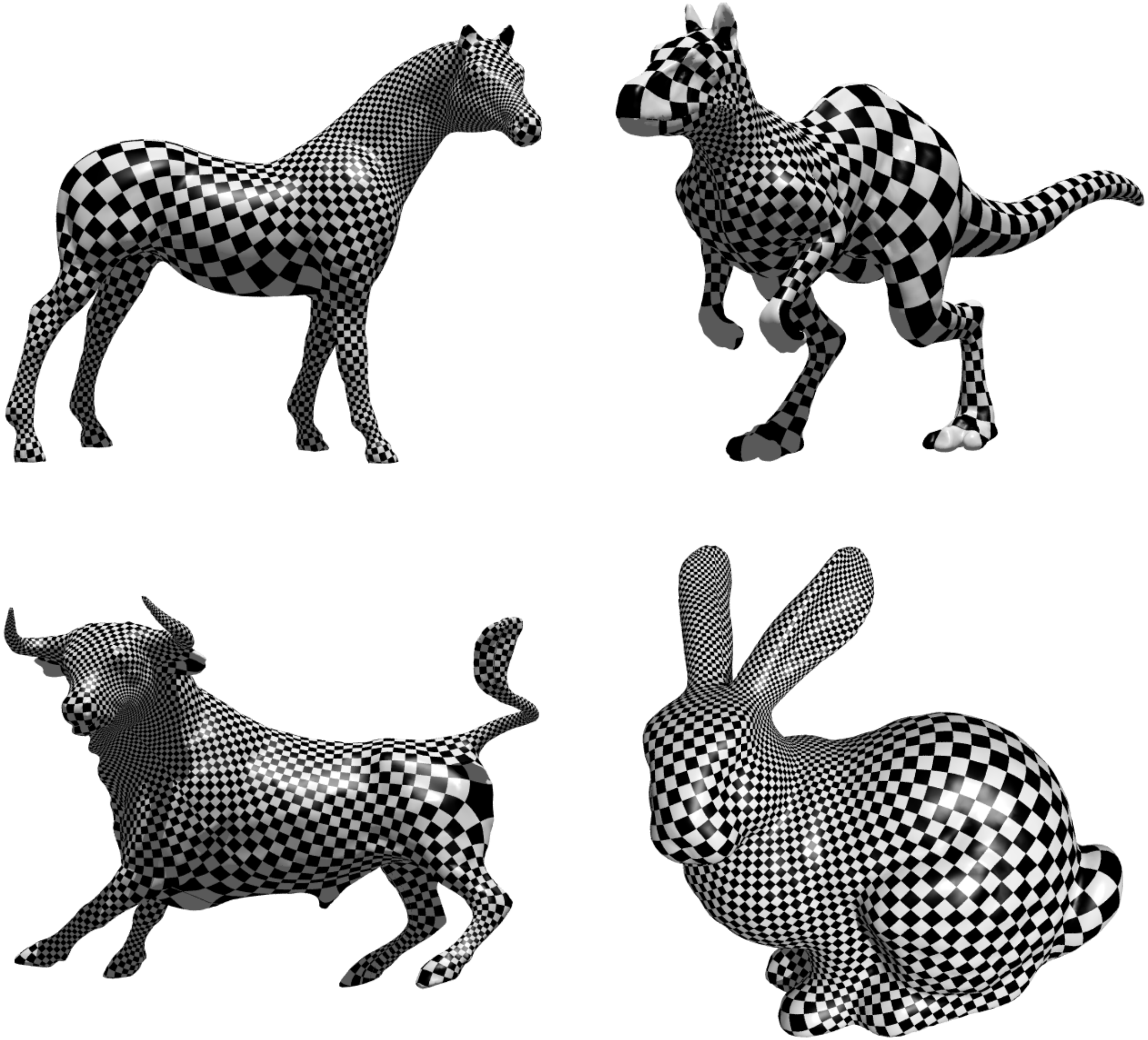
**Theorem 3.** *(Open Surface)Any oriented open surface admits an affine structure.*

Therefore, the minimum number of extraordinary points is one for all closed surfaces.

# *Conformal Structure*

- A surface $M$ with an atlas $\{(U_\alpha, \phi_\alpha)\}$, $\phi_\alpha : U_\alpha \to C$, if all transition functions $\phi_{\alpha\beta}$ are holomorphic, then the atlas is a *conformal atlas*.
- Two conformal atlas are equivalent, if their union is still a conformal atlas. Each equivalence class of conformal atlas is called a *conformal structure*.
- A surface with a conformal structure is called a *Riemann surface*. All oriented metric surfaces are Riemann surfaces.

# Conformal Structure



# *Holomorphic One-Forms*

Suppose $M$ is a Riemann surface, with conformal structure $\{(U_\alpha, \phi_\alpha)\}$, a complex differential form $\omega$ is called a holomorphic one-form, if it has local representation

$$\omega = f_\alpha(z_\alpha) dz_\alpha,$$

where $z_\alpha = u_\alpha + \sqrt{-1} v_\alpha$ is the local coordinates of $\phi_\alpha(U_\alpha)$, $f_\alpha$ is a holomorphic function.

- Because the coordinate transition functions $\phi_{\alpha\beta} = z_\beta(z_\alpha)$are holomorphic,
  $$f_\beta(z_\beta) = f_\alpha \frac{dz_\beta}{dz_\alpha},$$
  is also holomorphic. Therefore, $\omega$ is globally well defined.
- There are Euler number of zero points of $\omega$, which are independent of the choice of the local coordinates.

# *Holomorphic One-Forms*

Intuitively, a holomorphic 1-form is a pair of vector fields

$$\omega = \omega_1 + \sqrt{-1}\omega_2,$$

such that

1. Conjugate

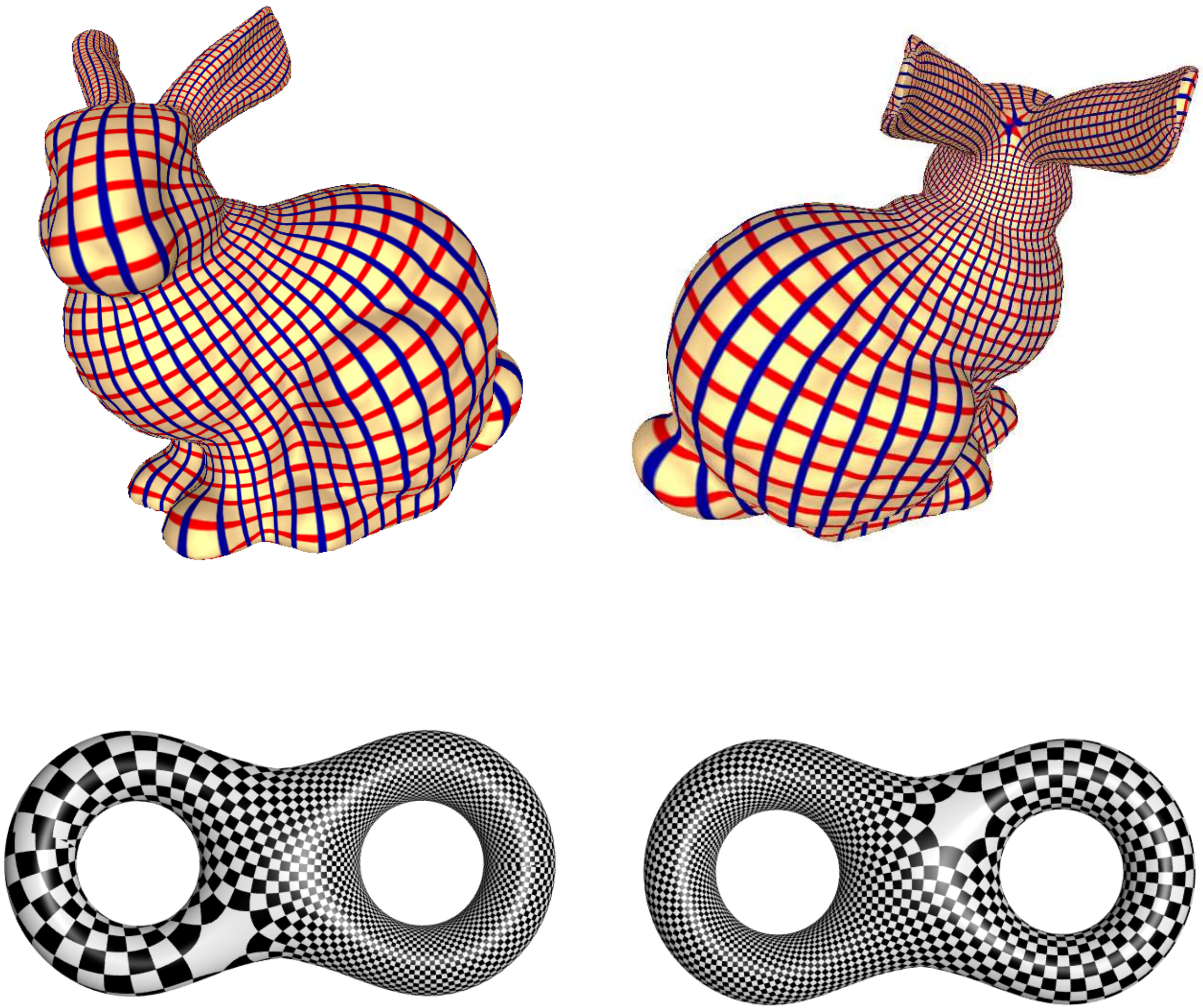
$$\omega_2 = \vec{n} \times \omega_1,$$

   where $\vec{n}$ is the normal field of the surface.

2. Curl free

$$\nabla \times \omega_1 = 0, \nabla \times \omega_2 = 0,$$

# *Holomorphic 1-form*



# *Affine structure induced by conformal structure*

**Theorem 0.3.** *(Conformal Structure vs. Affine structure) Suppose $M$ is a Riemann surface with a conformal structure, $\omega$ is a holomorphic 1-form, then $\omega$ induces an affine structure of $M/Z$, where $Z$ is the zero set of $\omega$. $|Z|$ equals to the Euler number of $M$.*

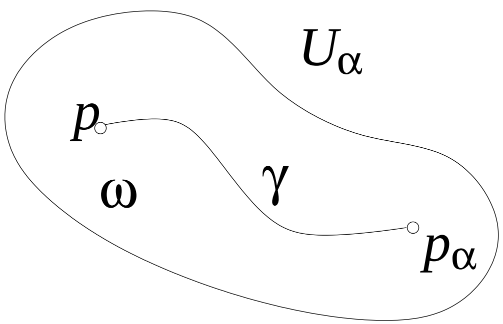
# *Affine structure induced by conformal structure*

A Riemann surface $M$ with a holomorphic 1-form $\omega$, suppose $U_\alpha$ is an open set, choose a base point $p_\alpha \in U_\alpha$, define local coordinates by

$$z_\alpha(p) = \int_\gamma \omega,$$

where $\gamma$ is an arbitrary path connecting $p_\alpha$ to $p$, and contained in $U_\alpha$. If $U_\alpha \cap U_\beta \neq \varnothing$, then the coordinate transition function $\phi_{\alpha\beta}$ is a planar transition, suppose $p \in U_\alpha \cap U_\beta$, then

$$z_\alpha(p) - z_\beta(p) \equiv const$$

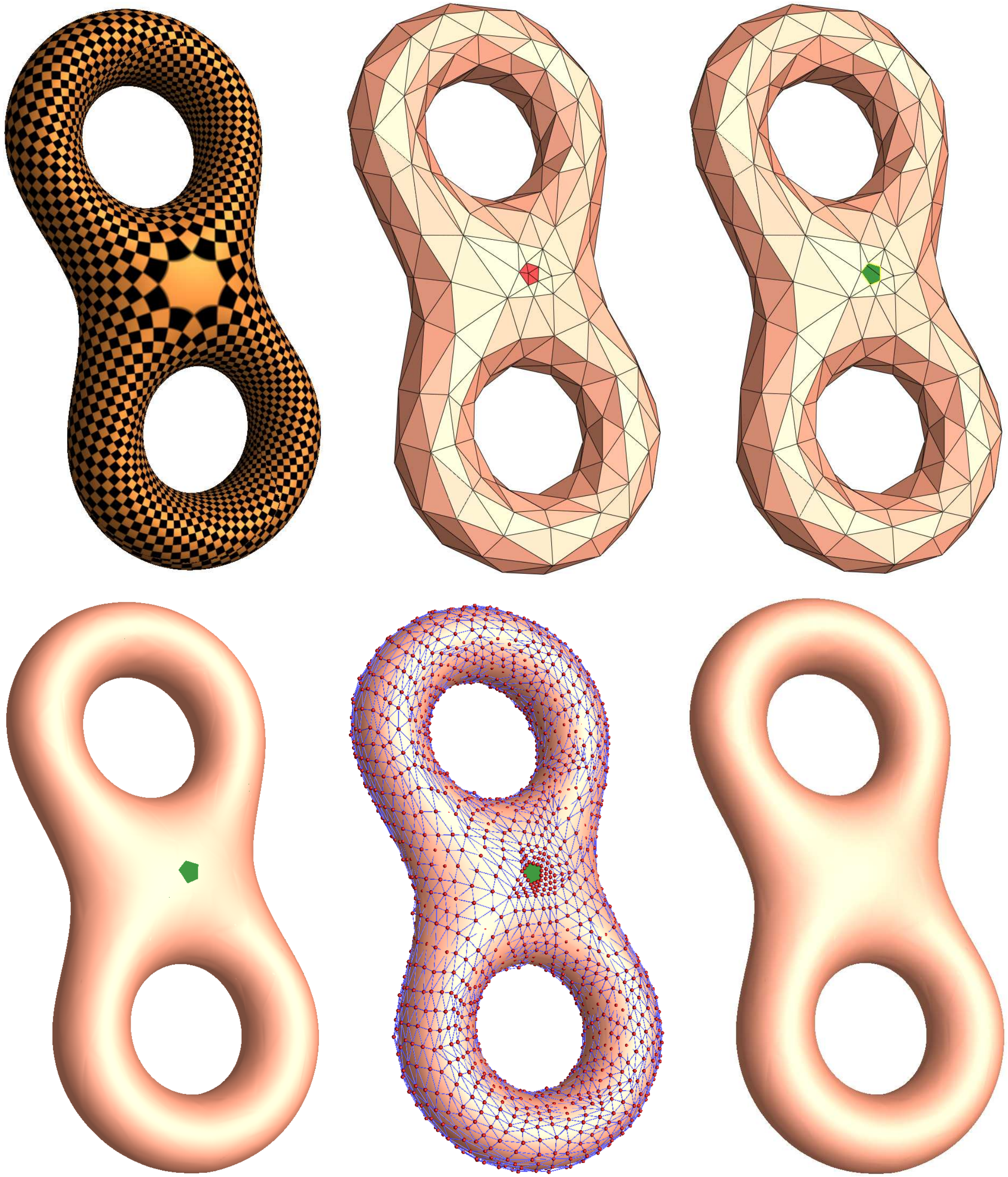


# *Algorithms for Computing Manifold Spline*

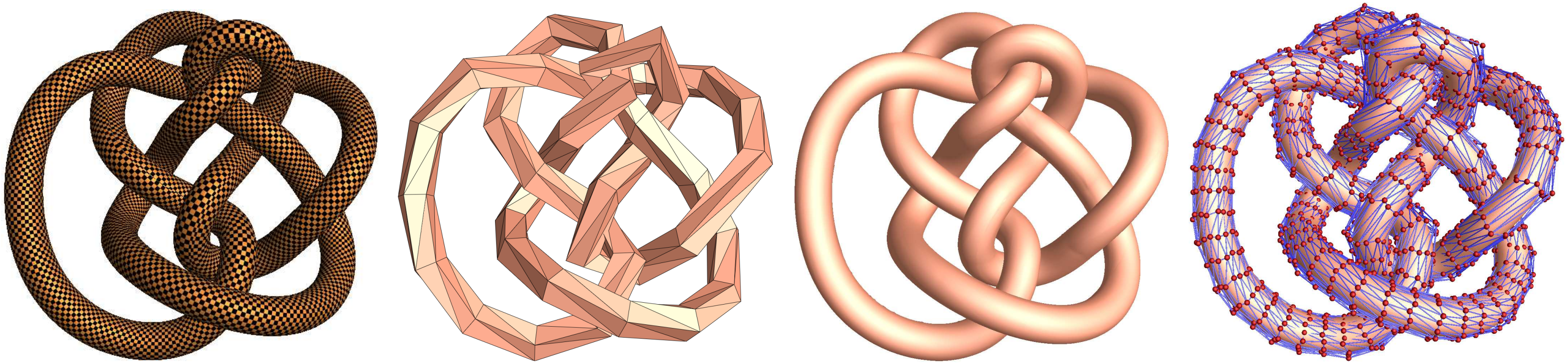
We demonstrate the manifold spline algorithm using DMS spline, for its versatility.
Given a domain mesh $M$, a control net $C$,

1. Compute the conformal structure of $M$ using Gu-Yau method.
2. Select an optimal holomorphic 1-form $\omega$, which maximizes the uniformity of the conformal factor (area distortion factor).
3. Locate the zero points of $\omega$. Remove their 1-ring neighbors.
4. Each 1-ring neighbor of a vertex form an open set $U_\alpha$, compute local coordinates by integrating $\omega$, namely $\phi_\alpha$.
5. For each control point, define its knots on one chart $(U_\alpha, \phi_\alpha)$, then transform the knots to other overlapping charts $(U_\beta, \phi_\beta)$ by using transition functions $\phi_{\alpha\beta}$.
6. Evaluate the Spline using polar form on any chart.

# *Algorithm Pipline*

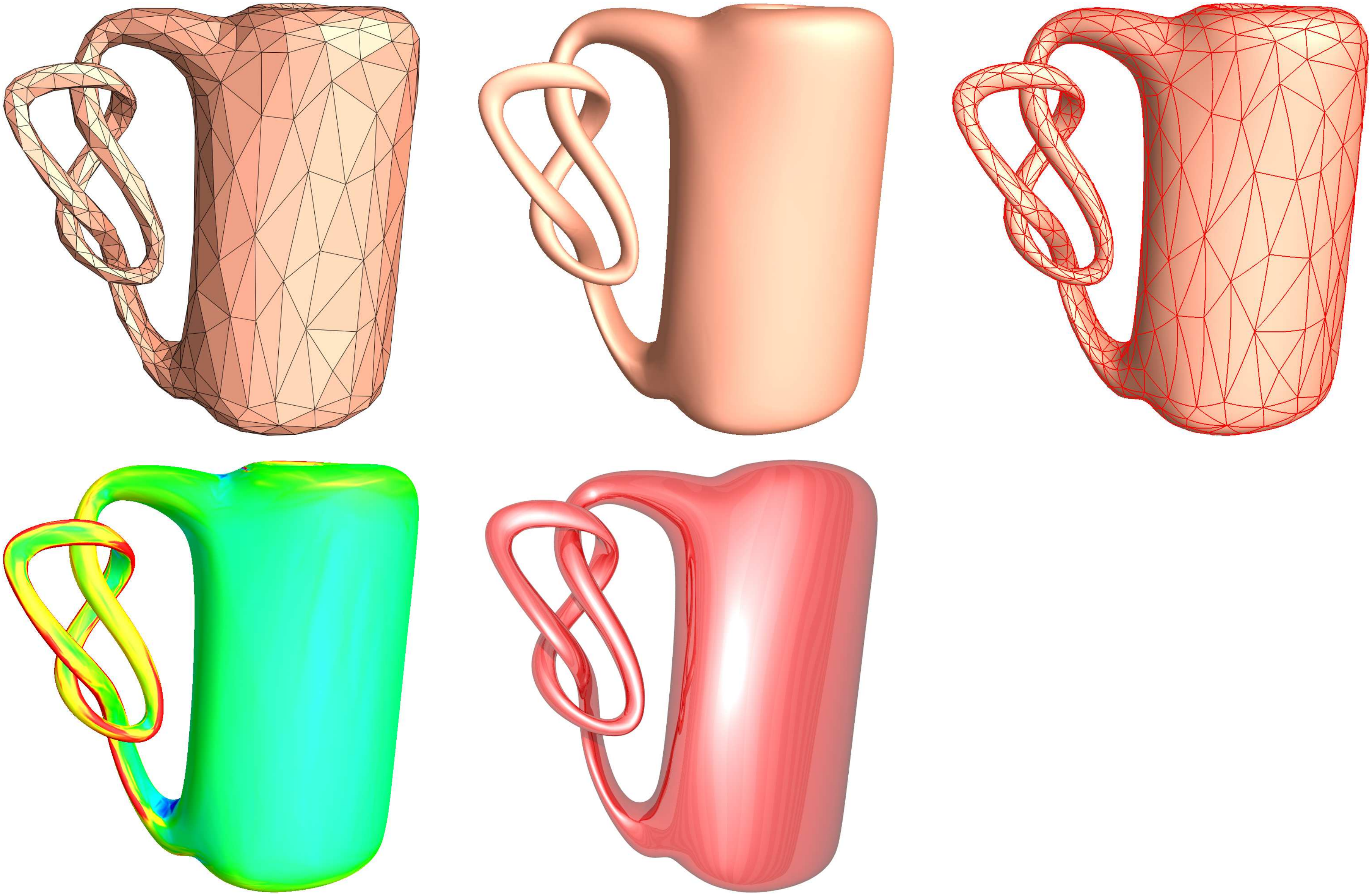


# *Genus 1 manifold DMS Spline -*

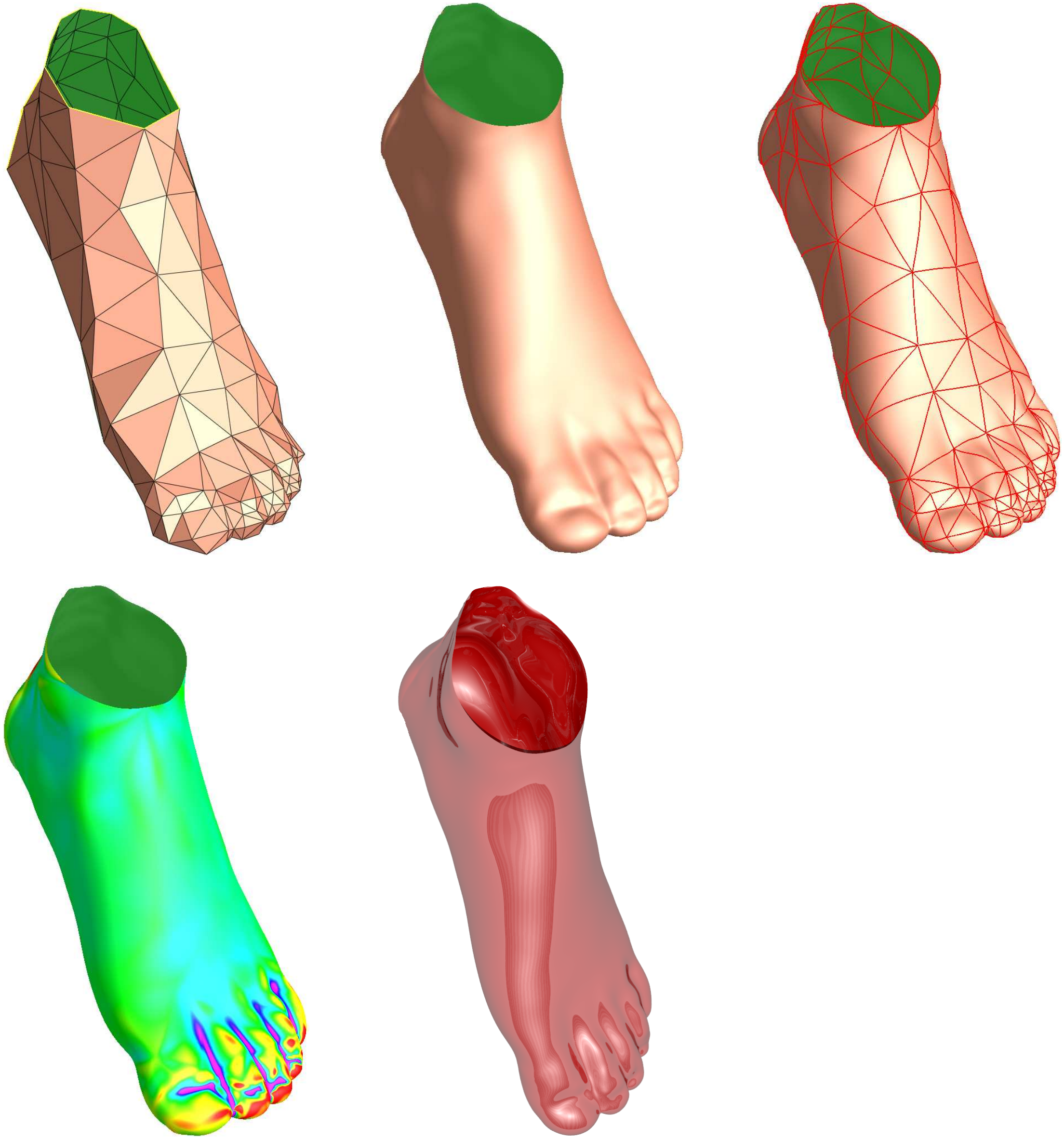


# *Genus 3 Manifold DMS Spline*

# *Manifold DMS Spline*

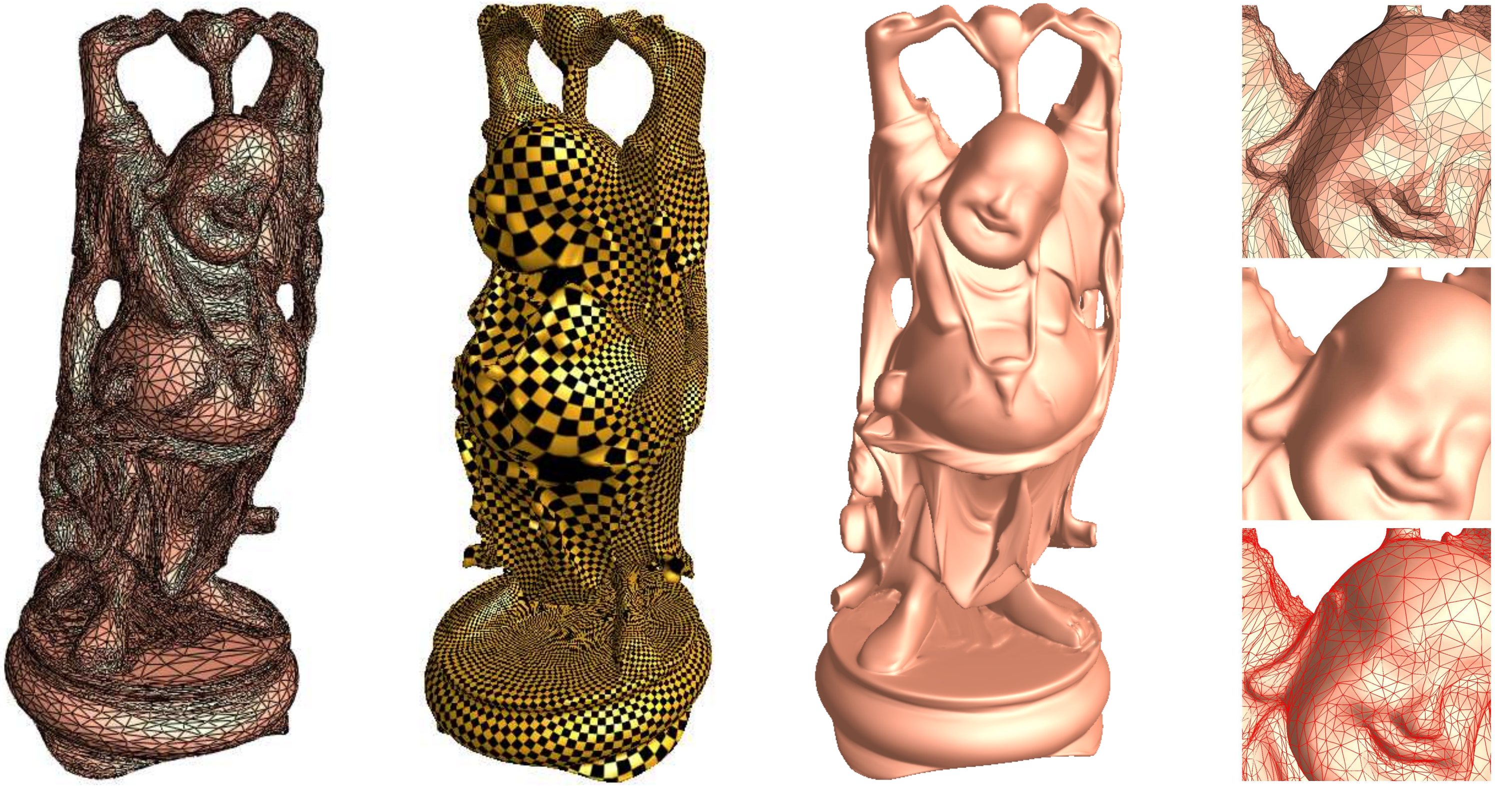


# *Manifold DMS Spline*



# *Manifold Powell-Sabin Spline*

# *Manifold Powell-Sabin Spline*

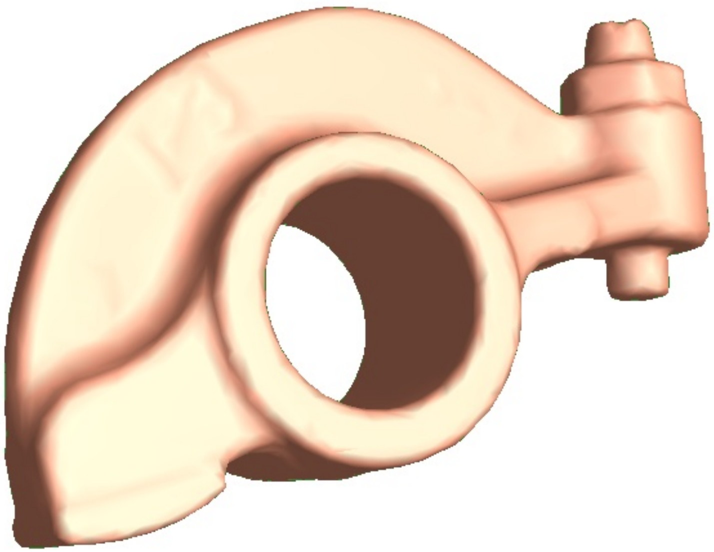


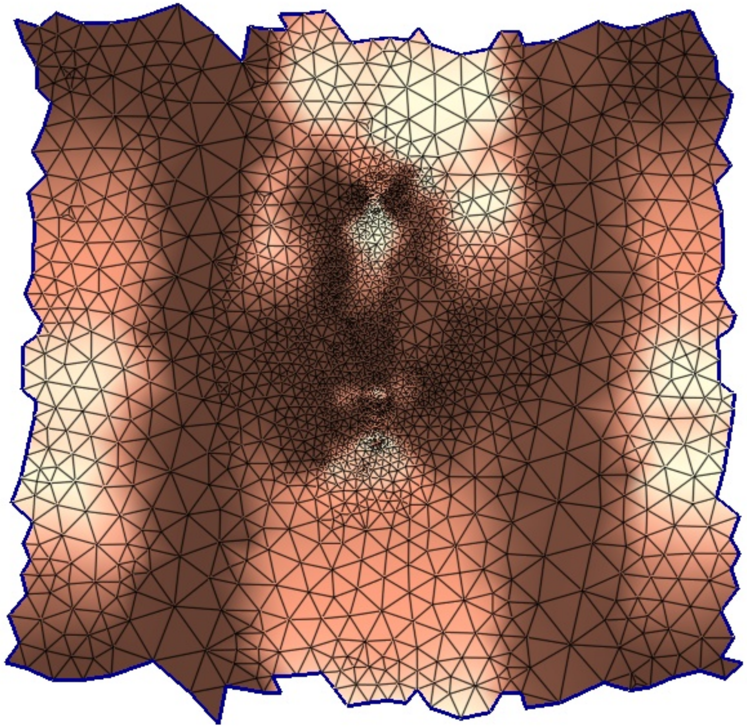
# *Geometric Structures*

Spherical Structure

# *Geometric Structures*

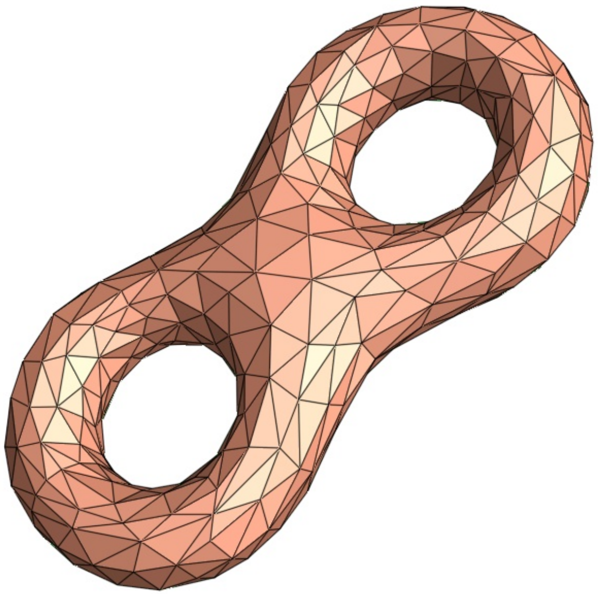
Euclidean Structure

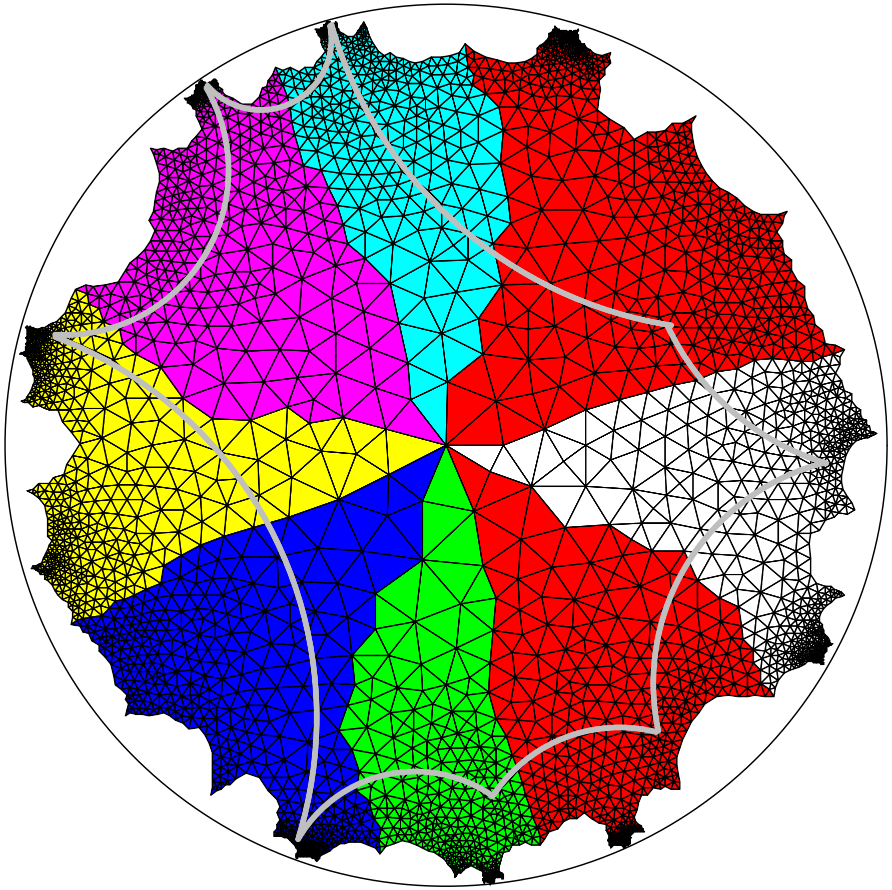




# *Geometric Structures*

Hyperbolic Structure





# *Geometric Structures*

Suppose $M$ is a manifold, $X$ is a topological space, $G$ is the transformation group on $G$, a $(G,X)$ atlas is an atlas $\{(U_\alpha, \phi_\alpha)\}$, such that

1. Local coordinates are in $X$,

$$\phi_\alpha : U_\alpha \longrightarrow X.$$

2. Transition functions are in group $G$,

$$\phi_{\alpha\beta} \in G.$$

Two $(G,X)$ atlas are equivalent, if their union is still a $(G,X)$ atlas. Each equivalent class of $(G,X)$ atlas is a $(G,X)$ structure.

# *Common Geometric Structures*

| Structure | $X$ | $G$ | Surfaces |
|---|---|---|---|
| Topology | $\mathbb{R}^2$ | homeomorphisms | arbitrary |
| Differential | $\mathbb{R}^2$ | diffeomorphisms | arbitrary |
| Spherical | $\mathbb{S}^2$ | rotation | closed genus zero |
| Euclidean | $\mathbb{E}^2$ | rigid motion | closed genus one |
| Hyperbolic | $\mathbb{H}^2$ | Möbius Transformation | high genus |
| Conformal | $\mathbb{C}$ | Holomorphic functions | Riemann surface |
| Affine | $\mathbb{R}^2$ | Affine transformation | Zero Euler Class |
| Projective | $\mathbb{RP}^2$ | Projective Transformation | arbitrary oriented |

# *Computing Geometric Structures*

The computational algorithms for geometric structures are categorized to 3 classes, (sorted by difficulty levels)

1. Topological: Solely depends on the topologies of the surface, has nothing to do with connectivity and geometry.
2. Combinatorial: Solely depends on the connectivity, independent of the geometry.
3. Geometric: Solely depends on the geometry, independent of connectivity. Most intrinsic method, the metric of the surface in local coordinates is represented as

$$ds^2 = \lambda(u,v)^2(du^2 + dv^2).$$

# *Three levels of Computing Geometric Structures*

For example, we want to compute the affine structure of a torus,

1. Topological: Find two canonical homology basis generators, slice the surface open to a fundamental domain, map the domain to a square in a brute-force way.
2. Combinatorial: Circle packing, when one changes the triangulations, the atlas changes; when one changes the position of vertices, the atlas doesn't change.
3. Geometric: Holomorphic 1-form, when one changes the triangulations, the atlas is invariant; when one changes the positions of vertices, the atlas changes.

## *Computing Geometric Structures*

| Structure | Level | Inventors |
|---|---|---|
| Conformal | Combinatorial | Thurston 1980 |
| Conformal | Combinatorial | Ying, Zorin, 2004 |
| Conformal | Geometric | Gu-Yau, 2002 |
| Affine | Combinatorial | Catmull, Clark , 1978 |
| Affine | Combinatorial | Khodakovsky, Litke, Schröder, 2003 |
| Affine | Geometric | Gu,He,Qin ,2005 |
| Hyperbolic | Topological | Ferguson, Rockwood, Cox, 1992 |
| Projective | Topological | Wallner, Pottman, 1997 |

Klein's Erlangen Program (1872): Different geometries study the invariants under different transformation groups.

Boston Program (2005): Different manifold splines study the functional basis defined on different $(G, X)$ structures which are invariant under the transformations in $G$.

# *General Manifold Splines*

The general manifold splines problems is to

1. Discover a $(G,X)$ structure of the domain manifold $M$.
2. Compute the $(G,X)$ structure of $M$.
3. Construct spline scheme on $X$, which is invariant under the transformations in $G$, and generalize it to $M$ via $(G,X)$ structure.

# *General Manifold Splines*

| Structure | Level | Basis | Inventors |
|---|---|---|---|
| Affine | Combinatorial | Polar form | Catmull, Clark,1978 |
| Hyperbolic | Topological | Gaussian | Ferguson, Rockwood, Cox,1992 |
| Differential | Combinatorial | Gaussian | Grimm, Hughes,1995 |
| Projective | Topological | rational | Wallner, Pottman,1997 |
| Conformal | Combinatorial | Gaussian | Ying, Zorin, 2004 |
| Affine | Geometric | Polar form | Gu,He,Qin,2005 |

## *Conclusion*

After years' effort, we have fully understood the problem and found the solutions.

- Rigorously, precisely formulate the manifold spline concept. The manifold spline concept unifies traditional splines and subdivision surfaces.
- Build the connection between manifold spline and its intrinsic geometric structure - Affine structure.
- Prove the non-existence of manifold spline is due to the topological obstruction.
- Build the connection between affine structure and the Riemann surface structure ( conformal structure ).
- Develop practical algorithms to construct manifold splines with Euler number of singularities.
- Formulate the general manifold spline problem. Point out the future directions.

# *Acknowledgements*

We deeply appreciate all the encouragements and advices from colleagues and friends, especially,

- Reviewers for their encouragements and valuable comments.
- Shing-Tung Yau, for his profound geometric insights, and encouragements.

## *Links*

The geometric models, the holomorphic 1-forms, the control nets, manifold splines can be downloaded form

$$http://www.cs.sunysb.edu/~gu/manifold\_spline/$$

Please send your questions and comments to

$$\{gu|yhe|qin\}@cs.sunysb.edu$$

# Thank You!