Ricci Flow in Engineering Fields

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Numerical Ricci Flow in Computer Science, Geometry and Physics, UBC

Thanks

Thanks for the invitation.

Collaborators

The work is collaborated with Shing-Tung Yau, Feng Luo, Tony Chan, Paul Thompson, Yalin Wang, Ronald Lok Ming Lui, Hong Qin, Dimitris Samaras, Jie Gao, Arie Kaufman, and many other mathematicians, computer scientists and medical doctors.

Klein's Program

Klein's Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion

Motivation

Conformal Geometry lays down the theoretic foundation for

- Surface mapping
- Shape classification
- Shape analysis

Applied in computer graphics, computer vision, geometric modeling, wireless sensor networking and medical imaging, and many other engineering, medical fields.

Motivation

Surface Ricci flow plays a fundamental role in conformal geometry.

History

- In pure mathematics, conformal geometry is the intersection of complex analysis, algebraic topology, Riemann surface theory, algebraic curves, differential geometry, partial differential equation.
- In applied mathematics, computational complex function theory has been developed, which focuses on the conformal mapping between planar domains.
- Recently, computational conformal geometry has been developed, which focuses on the conformal mapping between surfaces.

History of Computational Conformal Geometry

History

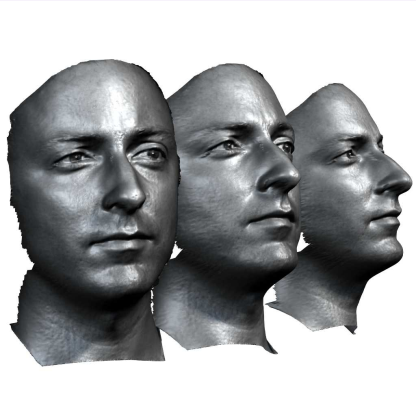
Conventional conformal geometric method can only handle the mappings among planar domains.

- Applied in thin plate deformation (biharmonic equation)
- Membrane vibration
- Electro-magnetic field design (Laplace equation)
- Fluid dynamics
- Aerospace design

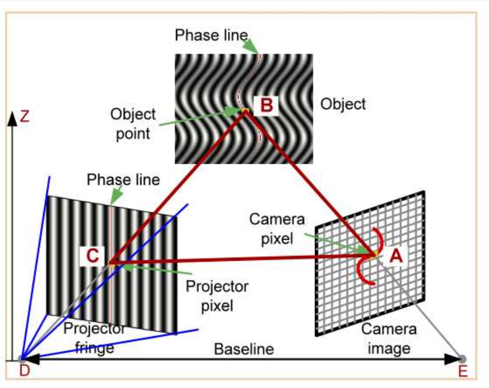
Reasons for Booming

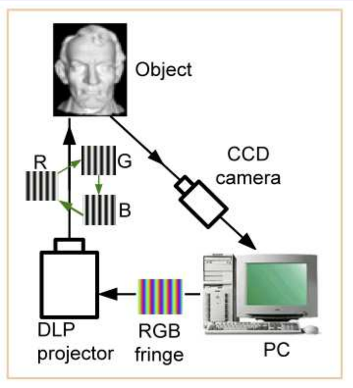
Data Acquisition

3D scanning technology becomes mature, it is easier to obtain surface data.

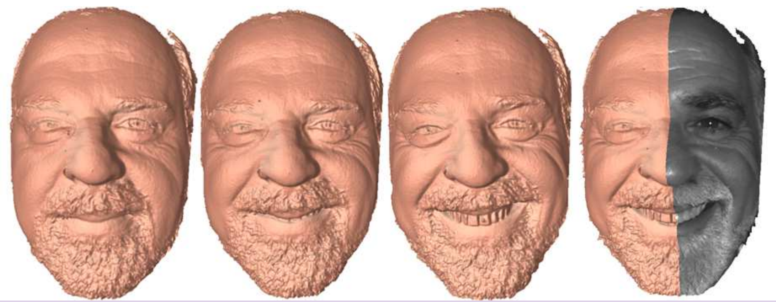


System Layout

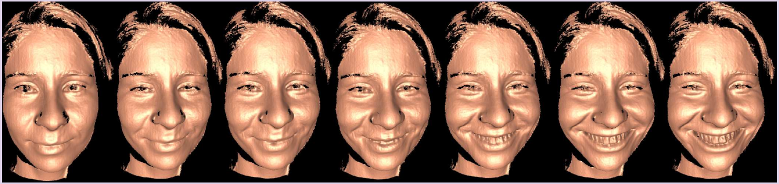




3D Scanning Results



3D Scanning Results



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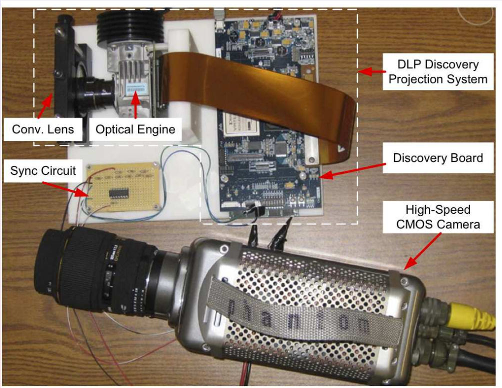
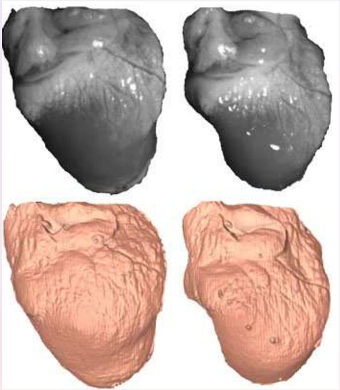




Reasons for Booming

Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

Computational Power

Computational power has been increased tremendously. With the incentive in graphics, GPU becomes mature, which makes numerical methods for solving PDE's much easier.

Fundamental Problems

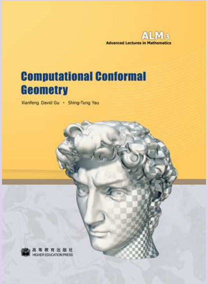
1. Given a Riemannian metric on a surface with an arbitrary topology, determine the corresponding conformal structure.
2. Compute the complete conformal invariants (conformal modules), which are the coordinates of the surface in the Teichmuller shape space.
3. Fix the conformal structure, find the simplest Riemannian metric among all possible Riemannian metrics
4. Given desired Gaussian curvature, compute the corresponding Riemannian metric.
5. Given the distortion between two conformal structures, compute the quasi-conformal mapping.
6. Compute the extremal quasi-conformal maps.
7. Conformal welding, glue surfaces with various conformal modules, compute the conformal module of the glued surface.

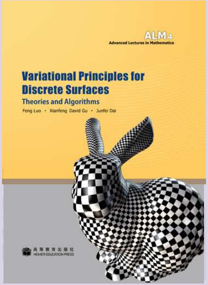
Complete Tools

Computational Conformal Geometry Library

1. Compute conformal mappings for surfaces with arbitrary topologies
2. Compute conformal modules for surfaces with arbitrary topologies
3. Compute Riemannian metrics with prescribed curvatures
4. Compute quasi-conformal mappings by solving Beltrami equation

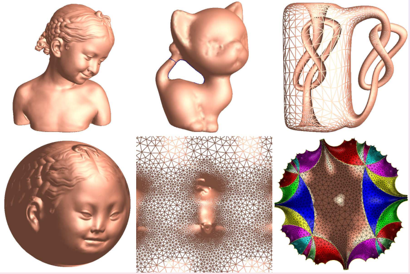
The theory, algorithms and sample code can be found in the following books.

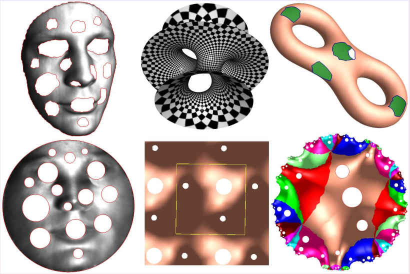




You can find them in the book store.

Please email me gu@cs.sunysb.edu for updated code library on computational conformal geometry.





Conformal Structure

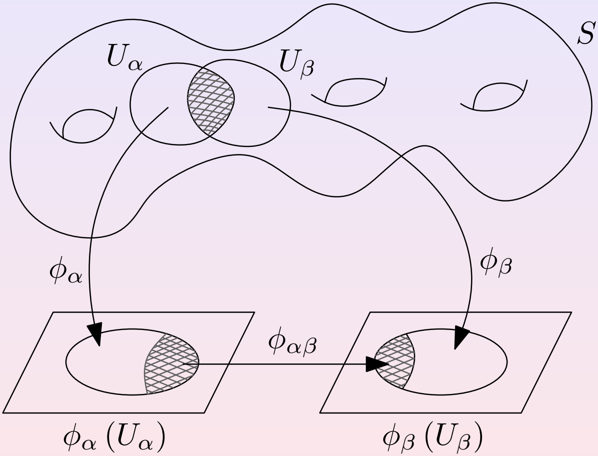
Definition (Manifold)

M is a topological space, $\{U_\alpha\}$ $\alpha \in I$ is an open covering of M, $M \subset \cup_\alpha U_\alpha$. For each U_α, $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. The pair (U_α, ϕ_α) is a chart. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$$

then M is called a smooth manifold, $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas.

Manifold


S
U_α
U_β
ϕ_α
ϕ_β
$\phi_{\alpha\beta}$
$\phi_\alpha(U_\alpha)$
$\phi_\beta(U_\beta)$

Definition (Holomorphic Function)

Suppose $f:\mathbb{C}\to\mathbb{C}$ is a complex function, $f: x+iy \to u(x,y)+iv(x,y)$, if f satisfies Riemann-Cauchy equation

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

then f is a holomorphic function.

Denote

$$dz = dx+idy, d\bar{z} = dx-idy, \frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}), \frac{\partial}{\partial \bar{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})$$

then if $\frac{\partial f}{\partial \bar{z}}=0$, then f is holomorphic.

Conformal Atlas

Definition (Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}$ is an atlas, such that all the chart transition functions $\phi_{\alpha\beta}: \mathbb{C} \to \mathbb{C}$ are bi-holomorphic, then A is called a conformal atlas.

Definition (Compatible Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two conformal atlases. If their union $A_1 \cup A_2$ is still a conformal atlas, we say $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are compatible.

Conformal Structure

The compatible relation among conformal atlases is an equivalence relation.

Definition (Conformal Structure)

Suppose S is a topological surface, consider all the conformal atlases on M, classified by the compatible relation

$$\{\textit{all conformal atlas}\}/\sim$$

each equivalence class is called a conformal structure.

In other words, each maximal conformal atlas is a conformal structure.

Definition (Conformal equivalent metrics)

Suppose g_1, g_2 are two Riemannian metrics on a manifold M, if

$$g_1 = e^{2u} g_2, u : M \to \mathbb{R}$$

then g_1 and g_2 are conformal equivalent.

Definition (Conformal Structure)

Consider all Riemannian metrics on a topological surface S, which are classified by the conformal equivalence relation,

$$\{Riemannian\ metrics\ on\ S\} / \sim,$$

each equivalence class is called a conformal structure.

Relation between Riemannian metric and Conformal Structure

Definition (Isothermal coordinates)

Suppose (S,g) is a metric surface, (U_α,ϕ_α) is a coordinate chart, (x,y) are local parameters, if

$$g = e^{2u}(dx^2+dy^2),$$

then we say (x,y) are isothermal coordinates.

Theorem

Suppose S is a compact metric surface, for each point p, there exits a local coordinate chart (U,ϕ), such that $p \in U$, and the local coordinates are isothermal.

Corollary

For any compact metric surface, there exists a natural conformal structure.

Definition (Riemann surface)

A topological surface with a conformal structure is called a Riemann surface.

Theorem

All compact metric surfaces are Riemann surfaces.

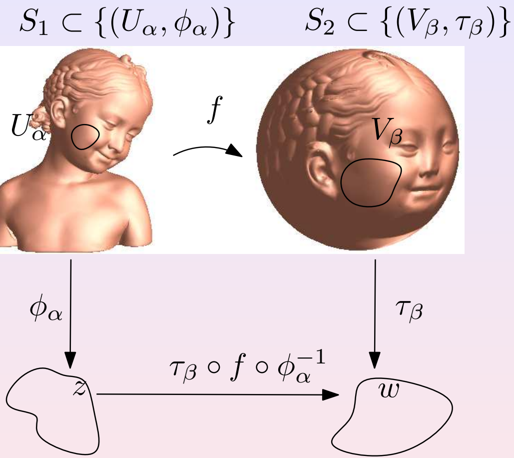
Conformal Mapping

biholomorphic Function

Definition (biholomorphic Function)

Suppose $f:\mathbb{C}\to\mathbb{C}$ is invertible, both f and f^{-1} are holomorphic, then then f is a biholomorphic function.

Conformal Map



The restriction of the mapping on each local chart is biholomorphic, then the mapping is conformal.

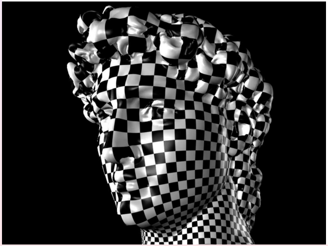
Conformal Mapping

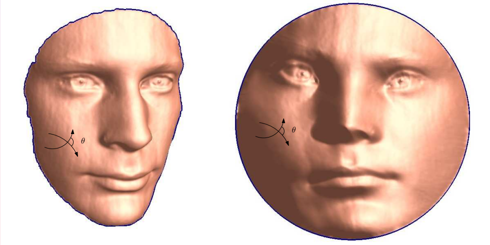
Definition (Conformal Map)

Let $\phi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$ is a homeomorphism, ϕ is conformal if and only if

$$\phi^* \mathbf{g}_2 = e^{2u} \mathbf{g}_1.$$

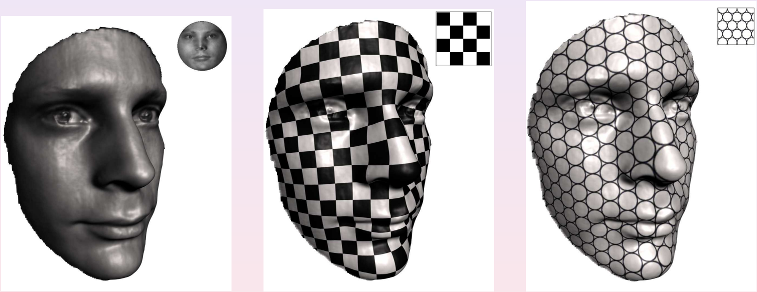
Conformal Mapping preserves angles.





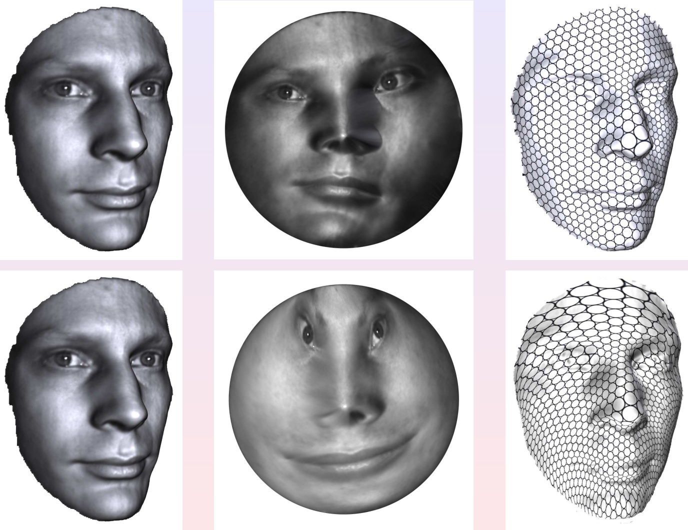
Conformal maps Properties

Map a circle field on the surface to a circle field on the plane.



Quasi-Conformal Map

Diffeomorphisms: maps ellipse field to circle field.

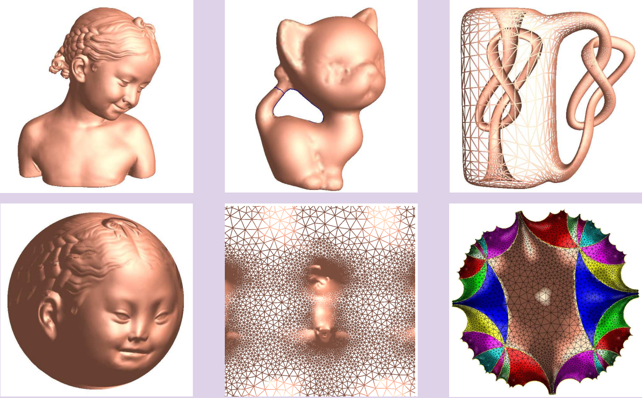


Uniformization

Conformal Canonical Representations

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



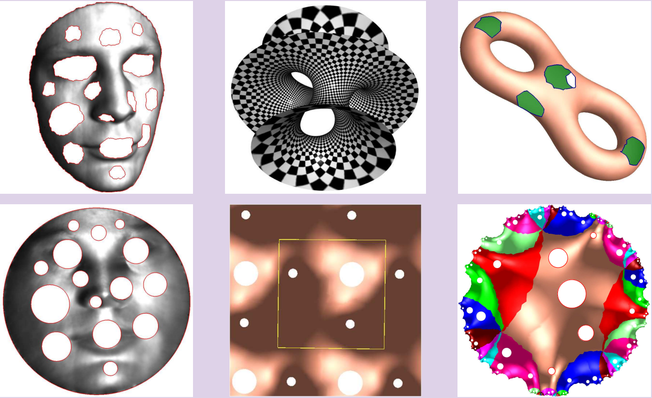
Definition (Circle Domain)

A domain in the Riemann sphere $\hat{\mathbb{C}}$ is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem

Any domain Ω in $\hat{\mathbb{C}}$, whose boundary $\partial\Omega$ has at most countably many components, is conformally homeomorphic to a circle domain Ω^* in $\hat{\mathbb{C}}$. Moreover Ω^* is unique upto Möbius transformations, and every conformal automorphism of Ω^* is the restriction of a Möbius transformation.

Uniformization of Open Surfaces

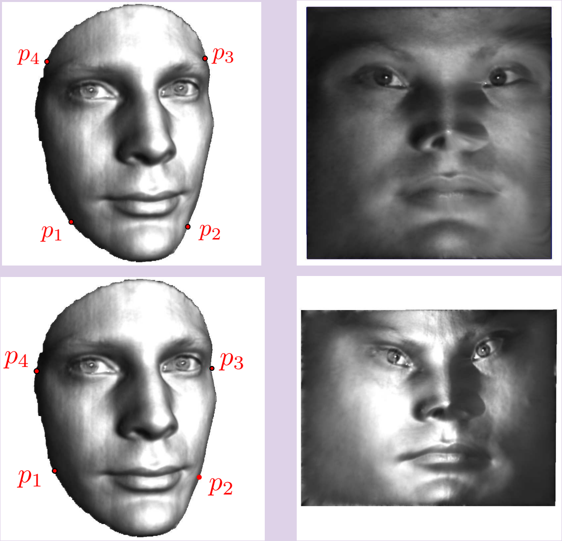


Spherical Euclidean Hyperbolic

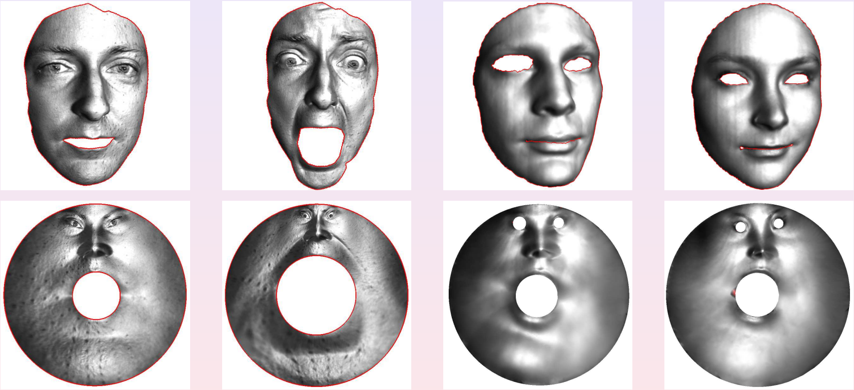
Conformal Canonical Representation

Simply Connected Domains

Topological Quadrilateral

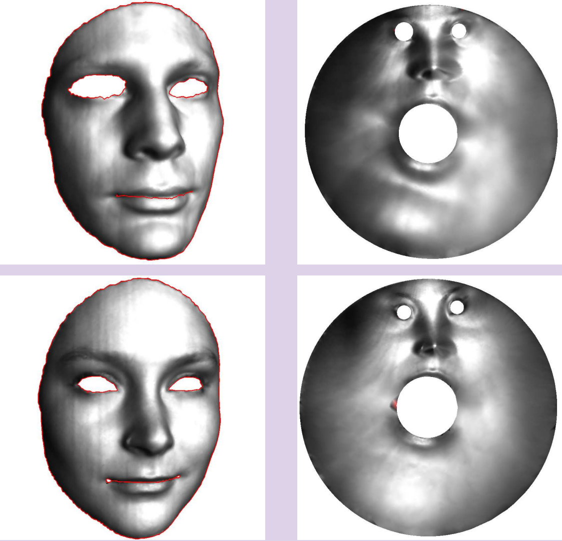


Multiply Connected Domains

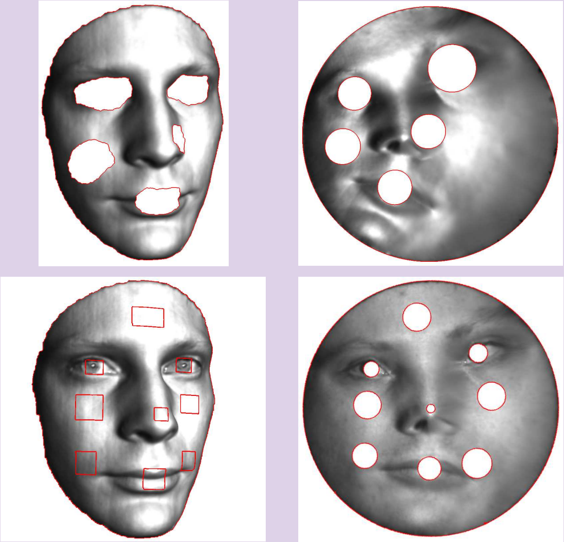


Conformal Canonical Forms

Multiply Connected Domains

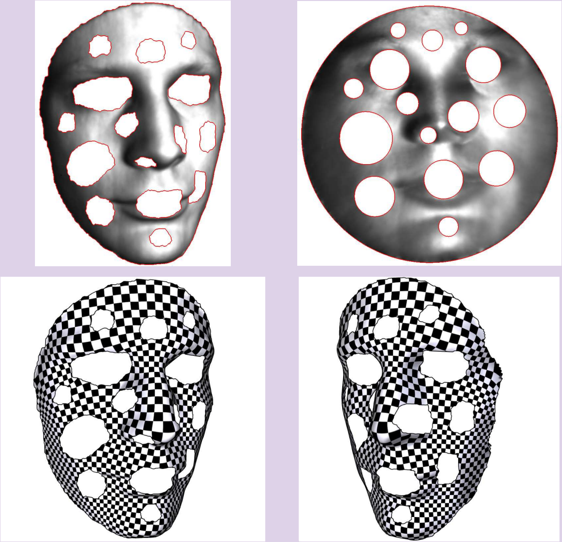


Multiply Connected Domains



Conformal Canonical Forms

Multiply Connected Domains



Conformal Canonical Representations

Definition (Circle Domain in a Riemann Surface)

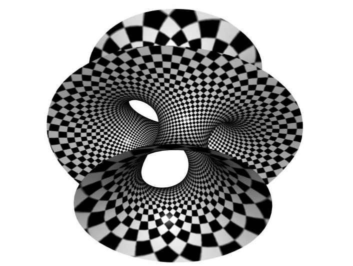
A circle domain in a Riemann surface is a domain, whose complement's connected components are all closed geometric disks and points. Here a geometric disk means a topological disk, whose lifts in the universal cover or the Riemann surface (which is $\mathbb{H}^2$, $\mathbb{R}^2$ or $\mathbb{S}^2$ are round.

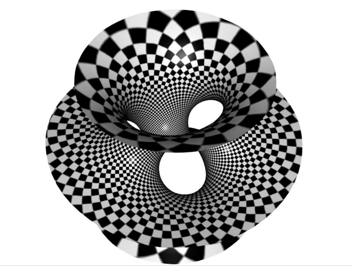
Theorem

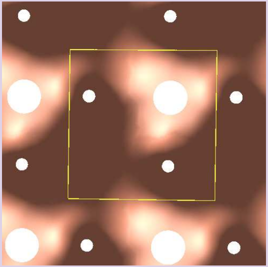
Let Ω be an open Riemann surface with finite genus and at most countably many ends. Then there is a closed Riemann surface R^* such that Ω is conformally homeomorphic to a circle domain Ω^* in R^*. More over, the pair (R^*, Ω^*) is unique up to conformal homeomorphism.

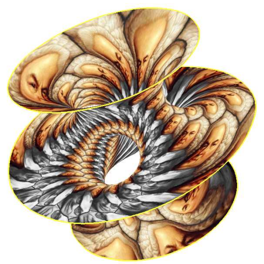
Conformal Canonical Form

Tori with holes



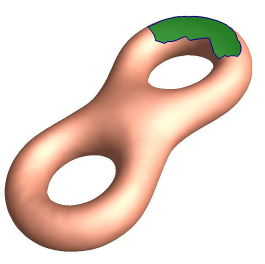


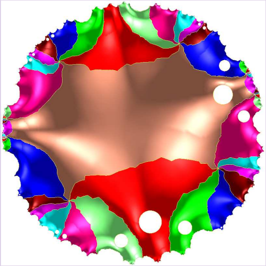




Conformal Canonical Form

High Genus Surface with holes





Teichmüller Space

Definition (Conformal Mapping)

Suppose $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ are two metric surfaces, $\phi : S_1 \to S_2$ is conformal, if on S_1

$$\mathbf{g}_1 = e^{2\lambda}\phi^*\mathbf{g}_2,$$

where $\phi^*\mathbf{g}_2$ is the pull-back metric induced by ϕ.

Definition (Conformal Equivalence)

Suppose two surfaces S_1, S_2 with marked homotopy group generators, $\{a_i, b_i\}$ and $\{\alpha_i, \beta_i\}$. If there exists a conformal map $\phi : S_1 \to S_2$, such that

$$\phi_*[a_i] = [\alpha_i], \phi_*[b_i] = [\beta_i],$$

then we say two marked surfaces are conformal equivalent.

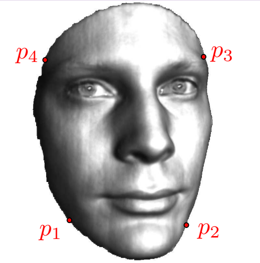
Definition (Teichmüller Space)

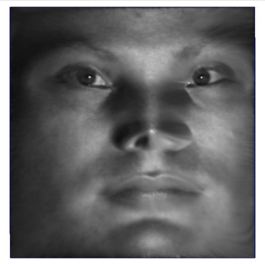
Fix the topology of a marked surface S, all conformal equivalence classes sharing the same topology of S, form a manifold, which is called the Teichmüller space of S. Denoted as T_S.

- Each point represents a class of surfaces.
- A path represents a deformation process from one shape to the other.
- The Riemannian metric of Teichmüller space is well defined.

Teichmüller Space

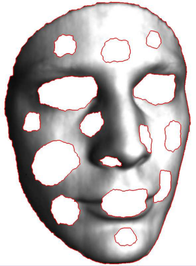
Topological Quadrilateral

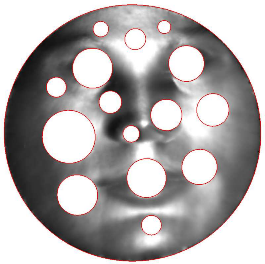




Conformal module: $\frac{h}{w}$. The Teichmüller space is 1 dimensional.

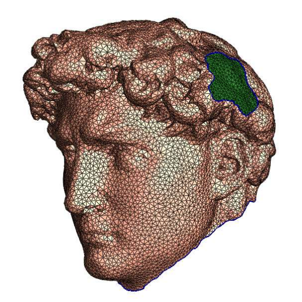
Multiply Connected Domains

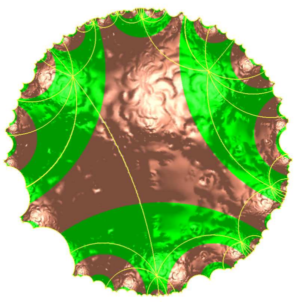




Conformal Module : centers and radii, with Möbius ambiguity. The Teichmüller space is $3n-3$ dimensional, n is the number of holes.

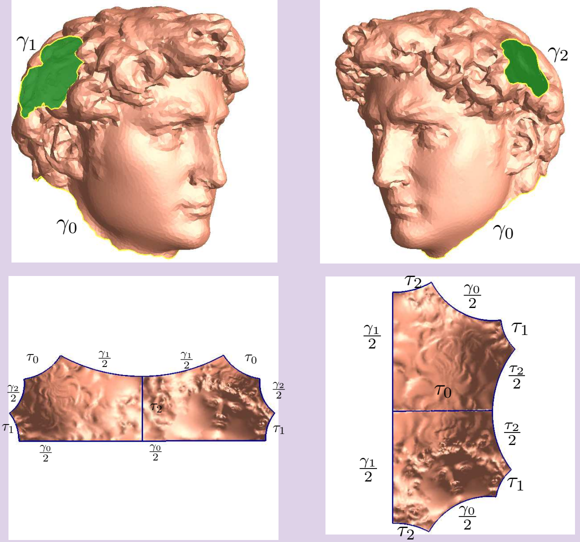
Topological Pants



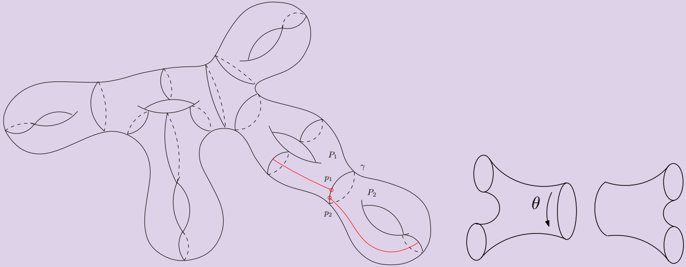


Genus 0 surface with 3 boundaries is conformally mapped to the hyperbolic plane, such that all boundaries become geodesics.

Topological Pants

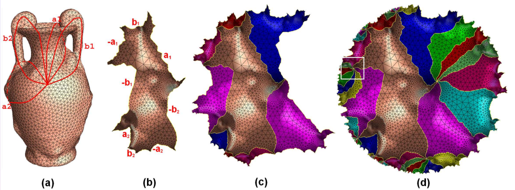


Topological Pants Decomposition - $2g - 2$ pairs of Pants

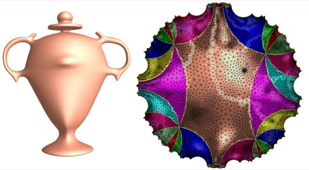


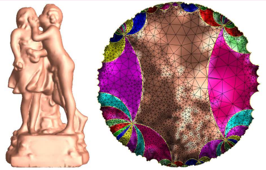
Compute Teichmüller coordinates

Step 1. Compute the hyperbolic uniformization metric.

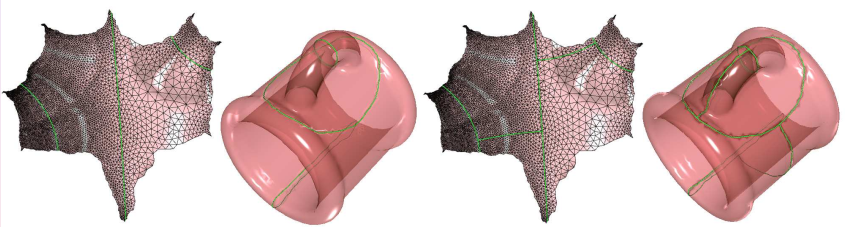


Step 2. Compute the Fuchsian group generators.





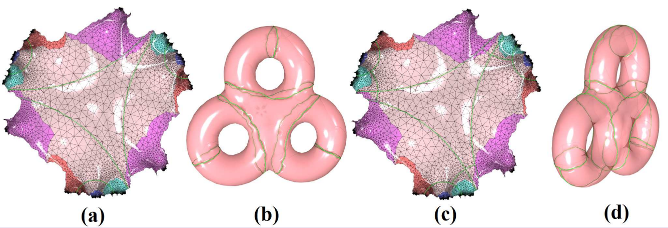
Step 3. Pants decomposition using geodesics and compute the twisting angle.



Teichmüller Coordinates

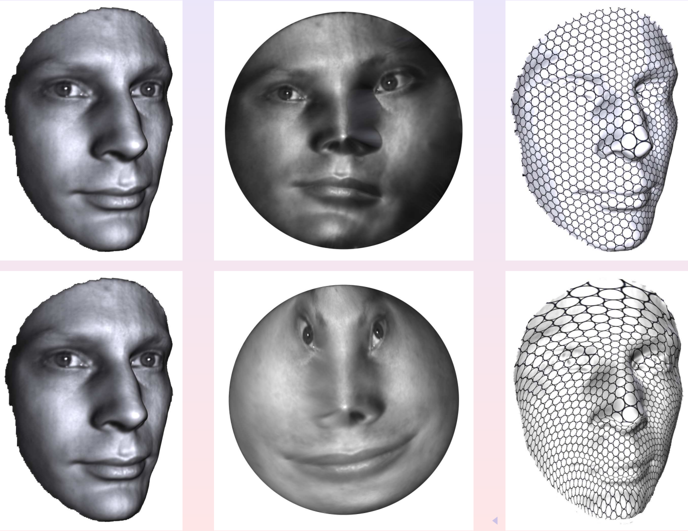
Compute Teichmüller coordinates

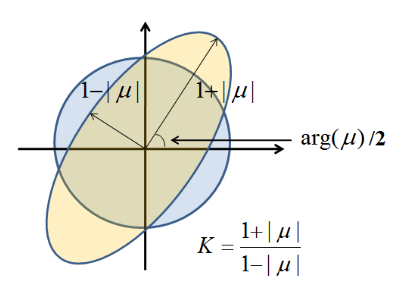
Compute the pants decomposition using geodesics and compute the twisting angle.



Quasi-Conformal Maps

Most homeomorphisms are quasi-conformal, which maps infinitesimal circles to ellipses.





Beltrami Coefficient

Let $\phi : S_1 \to S_2$ be the map, z, w are isothermal coordinates of S_1, S_2, Beltrami equation is defined as $\|\mu\|_\infty < 1$

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z}$$

Diffeomorphism Space

Theorem

Given two genus zero metric surface with a single boundary,

$$\{Diffeomorphisms\} \cong \frac{\{Beltrami\ Coefficient\}}{\{Mobius\}}.$$

Solving Beltrami Equation

The problem of computing Quasi-conformal map is converted to compute a conformal map.

Solveing Beltrami Equation

Given metric surfaces $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$, let z, w be isothermal coordinates of S_1, S_2,$w = \phi(z)$.

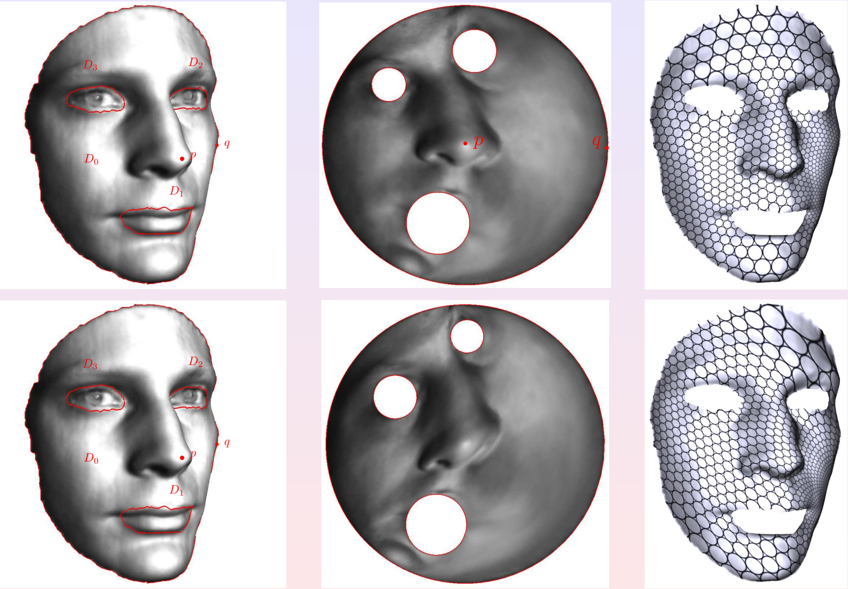
$$\mathbf{g}_1 = e^{2u_1} dz d\bar{z} \quad (1)$$

$$\mathbf{g}_2 = e^{2u_2} dw d\bar{w}, \quad (2)$$

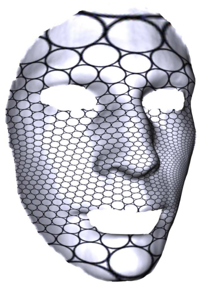
Then

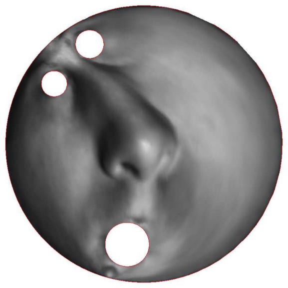
- $\phi : (S_1, \mathbf{g}_1) \rightarrow (S_2, \mathbf{g}_2)$, quasi-conformal with Beltrami coefficient μ.
- $\phi : (S_1, \phi^* \mathbf{g}_2) \rightarrow (S_2, \mathbf{g}_2)$ is isometric
- $\phi^* \mathbf{g}_2 = e^{u_2} |dw|^2 = e^{u_2} |dz + \mu d\bar{dz}|^2$.
- $\phi : (S_1, |dz + \mu d\bar{dz}|^2) \rightarrow (S_2, \mathbf{g}_2)$ is conformal.

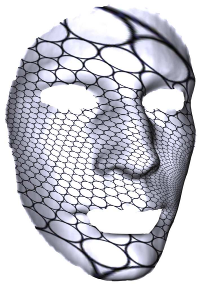
Quasi-Conformal Map Examples

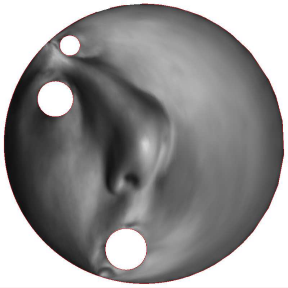


Quasi-Conformal Map Examples



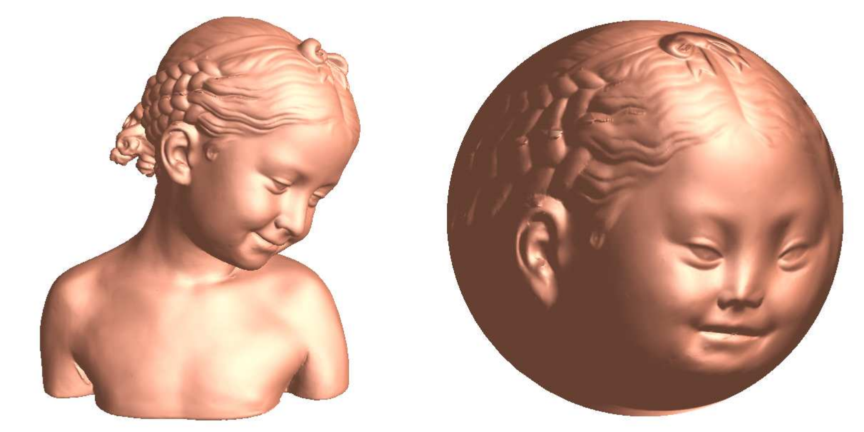






Computational Method - Harmonic Mapping

Spherical harmonic map



Let (M,g) and (N,h) be Riemannian manifolds, $u: M \to N$ is a C^1 mapping.

$$ds_M^2 = \sum g_{\alpha\beta} dx^\alpha dx^\beta, ds_N^2 = \sum h_{ij}(u(x)) du^i du^j.$$

The pull back metric of h induced by u is $u^*(ds_N^2)$ is a symmetric bilinear form

$$u^*(dS_N^2) = \sum_{\alpha,\beta} (\sum_{i,j} h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}) dx^\alpha dx^\beta.$$

The *energy density* of mapping u is defined as

$$|du|^2 = \sum_{i,j,\alpha,\beta} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Energy of the mapping

Equivalently, choose an orthogonal frame field under $u^*(ds_N^2)$, each basis vector field is of unit length under **g**, the dual 1-forms are $\{\omega_1, \omega_2, \cdots, \omega_n\}$, such that

$$u^*(ds_N^2) = \sum_{\alpha=1}^{n} \lambda_\alpha (\omega_\alpha)^2.$$

The the energy density of the mapping u is given by

$$|du|^2 = \sum_{\alpha=1}^{n} \lambda_\alpha.$$

Definition (Harmonic Energy)

The harmonic energy functional $E(u)$ is defined as

$$E(u) = \int_M |du|^2 dv_M,$$

where $dv_M = (detg)^{\frac{1}{2}} dx$ is the volume element of M.

Definition (Harmonic Mapping)

In the space of mappings, the critical points of $E(u)$ are called harmonic mappings.

Harmonic Energy Conformal Invariant

Suppose u is a mapping from a surface (S,g) to (N,h). Suppose $\tilde{g} = e^{2\lambda}g$ is another metric of S, conformal to g, then

$$|\tilde{d}u|^2 = e^{-2\lambda}|du|^2, \sqrt{det\tilde{g}} = e^{2\lambda}\sqrt{detg},$$

Then $\tilde{g} = g$. Harmonic energy is invariant under conformal metric transformation.

Theorem

Harmonic energy only depends on the conformal structure of the surface, independent of the choice of Riemannian metric.

Harmonic Mapping

Suppose N is embedded in $\mathbb{R}^3$, $u : S \to N$ is a harmonic mapping, then

$$\Delta_g u^{T_u N} \equiv 0.$$

where $\Delta_g u = (\Delta_g u_1, \Delta_g u_2, \Delta_g u_3)$. Namely, $\Delta_g u$ is orthogonal to the tangent plane at the target space.

Heat Flow method

Definition (Heat flow)

Let $u : S \to N \subset \mathbb{R}^3$, the heat flow is given by

$$\frac{du(x,t)}{dt} = -(\Delta_g u)^{T_{u(x)}N}$$

The heat flow method will deform a mapping to the harmonic mapping under special normalization conditions.

Heat Flow method

Theorem

Harmonic mapping from a genus zero closed surface to the unit sphere must be a conformal mapping.

Proof.

Let $u : S \to \mathbb{S}^2$. Choose isothermal coordinates of both surfaces, define

$$\phi(z) = \langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \rangle$$

then

$$\phi(z) = \frac{1}{4}(|\frac{\partial u}{\partial x}|^2 - |\frac{\partial u}{\partial y}|^2 - \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle).$$

if $\phi(z) = 0$, then the mapping is conformal.
On the other hand, $\frac{\partial \phi(z)}{\partial \bar{z}} = 0$, then $\phi(z)$ is holomorphic.
$\phi(z)dz^2$ is globally defined, the so-called Hopf differential.
Sphere has no non-zero holomorphic quadratic differentials.

Möbius transformation

Theorem

The conformal automorphism from a sphere to itself must be a Möbius transformation

$$z \to \frac{az+b}{cz+d}, ad-bc=1, a,b,c,d \in \mathbb{C}.$$

Rado's theorem

Theorem (Rado)

Let $\Omega \subset \mathbb{R}^2$ is a convex domain with smooth boundary. For any homeomorphism $\phi : S^1 \to \partial\Omega$, there exists a unique harmonic mapping $u : D \to \Omega$, such that $u|_{\partial}D = \phi$, furthermore, u is a diffeomorphism.

Discrete Approximation

We use piecewise linear triangle mesh to approximate the original surface. suppose $u : M \to \mathbb{R}$ the harmonic energy is given by

$$E(u) = \frac{1}{2}\sum_{[v_i,v_j]\in M} w_{ij}(f(v_i)-f(v_j))^2.$$

The discrete Laplace-Beltrami operator is given by

$$\Delta f(v_i) = \sum_j w_{ij}(f(v_j)-f(v_i)).$$

where w_{ij} is the cotangent formula.

Computational Method - Holomorphic Form

Harmonic 1-form

Each cohomologous class has a unique harmonic 1-form, which represents a vortex free, source-sink free flow field.

Theorem (Hodge)

All the harmonic 1-forms form a group, which is isomorphic to $H^1(M)$.

Theorem (Hodge Decomposition)

$$\Omega^k(M) = Imgd^{k-1} \oplus Img\delta^{k+1} \oplus H^k_\Delta(M).$$

Compute Harmonic 1-forms

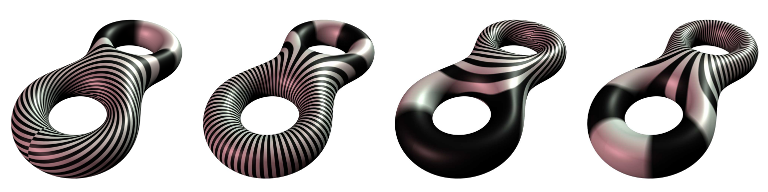
Harmonic 1-form

Let ω be a closed 1-form. Compute a function $f \in C^0(M, \mathbb{R})$, such that

$$\delta^1(\omega + df) = 0,$$

then $\omega + df$ is the unique harmonic 1-form, cohomologous to ω.

Harmonic 1-form Basis



Compute Harmonic 1-forms

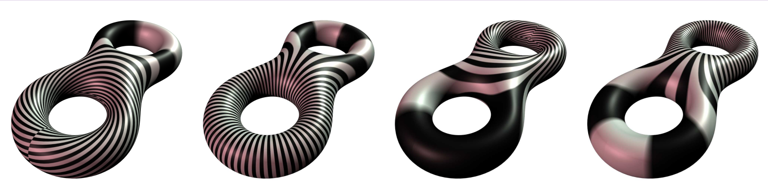
Harmonic 1-form

Let ω be a closed 1-form. Compute a function $f : V \to \mathbb{R}$, such that

$$\sum_j w_{ij}(\omega + df)([v_i, v_j]) = 0, \forall v_i \in V.$$

Then $\omega + df$ is the unique harmonic 1-form, cohomologous to ω.

Harmonic 1-form Basis



Hodge Star Operator

Let $(S,\mathbf{g})$ be a metric surface, $\{e_1,e_2\}$ be an orthonormal frame field, $\{\frac{\partial}{\partial u},\frac{\partial}{\partial v}\}$ be the base vector fields, $\{du,dv\}$ be the dual differential 1-form fields.

$$^*du = dv, ^*dv = -du.$$

Conjugate harmonic 1-forms $\omega + \sqrt{-1}^*\omega$

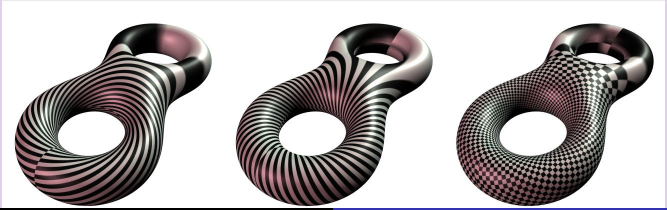
Hodge Star Operator

If ω is a harmonic 1-form, so is ${}^*\omega$. Suppose $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$ is the set basis of harmonic 1-forms, then ${}^*\omega = \sum_k \lambda_k \omega_k$. Locally, on each triangle ${}^*(adx + bdy) = ady - bdx$. Solve linear system

$$\int_M \omega_i \wedge {}^*\omega = \sum_k \lambda_k \int_M \omega_i \wedge \omega_k, i = 1, 2, \cdots, 2g.$$

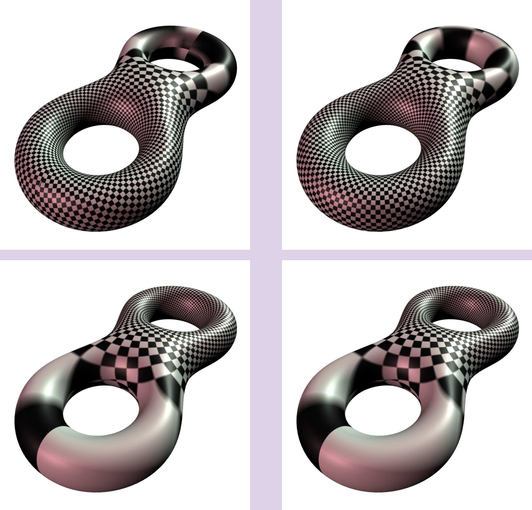
to solve λ_k's, where ${}^*\omega$ on the left hand side is locally evaluated.

Conjugate harmonic 1-forms $\omega + \sqrt{-1}^*\omega$



Holomorphic 1-form

Holomorphic 1-form Basis



Topological Quadrilateral

Topological Quadrilateral

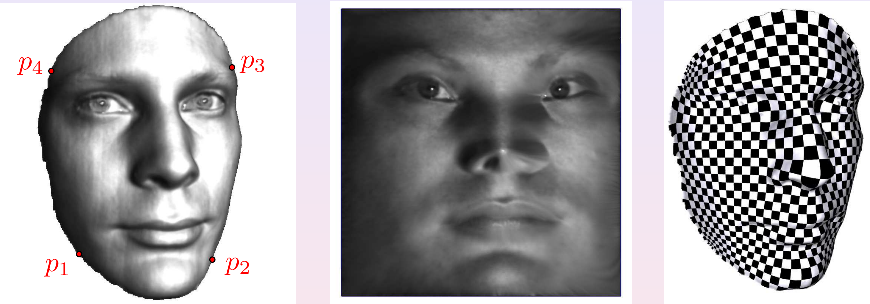


Figure: Topological quadrilateral.

Topological Quadrilateral

Definition (Topological Quadrilateral)

Suppose S is a surface of genus zero with a single boundary, and four marked boundary points $\{p_1, p_2, p_3, p_4\}$ sorted counter-clock-wisely. Then S is called a topological quadrilateral, and denoted as $Q(p1, p2, p3, p4)$.

Theorem

Suppose $Q(p1, p2, p3, p4)$ is a topological quadrilateral with a Riemannian metric $\mathbf{g}$, then there exists a unique conformal map $\phi : S \to \mathbb{C}$, such that ϕ maps Q to a rectangle, $\phi(p_1) = 0$, $\phi(p_2) = 1$. The height of the image rectangle is the conformal module of the surface.

Assume the boundary of Q consists of four segments $\partial Q = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, such that

$$\partial \gamma_1 = p_2 - p_1, \partial \gamma_2 = p_3 - p_2, \partial \gamma_3 = p_4 - p_3, \gamma_4 = p_1 - p_4.$$

We compute two harmonic functions $f_1, f_2 \to \mathbb{R}$, such that

$$\left\{ \begin{array}{rcl} \Delta f_1 &=& 0 \\ f_1|_{\gamma_1} &=& 0 \\ f_1|_{\gamma_3} &=& 1 \\ \frac{\partial f_1}{\partial \mathbf{n}}|_{\gamma_2 \cup \gamma_4} &=& 0 \end{array} \right. \left\{ \begin{array}{rcl} \Delta f_2 &=& 0 \\ f_2|_{\gamma_2} &=& 0 \\ f_2|_{\gamma_4} &=& 1 \\ \frac{\partial f_2}{\partial \mathbf{n}}|_{\gamma_1 \cup \gamma_3} &=& 0 \end{array} \right.$$

The df_1 and df_2 are two exact harmonic 1-forms. We need to find a scalar λ, such that ${}^*df_1 = \lambda df_2$, this can be achieved by solving the following equation,

$$\int_S df_1 \wedge {}^*df_2 = \lambda \int_S df_1 \wedge df_2.$$

Then the desired holomorphic 1-form $\omega = df_1 + i\lambda df_2$. The conformal mapping is given by

$$\phi(p) = \int_q^p \omega,$$

where q is the base point, the path from q to p is arbitrarily chosen.

Topological Annulus

Topological Annulus

Figure: Topological annulus.

Topological Annulus

Definition (Topological Annulus)

Suppose S is a surface of genus zero with two boundaries, the S is called a topological annulus.

Theorem

Suppose S is a topological annulus with a Riemannian metric $\mathbf{g}$*, the boundary of S are two loops $\partial S = \gamma_1 - \gamma_2$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical annulus, $\phi(\gamma_1)$ is the unit circle, $\phi(\gamma_2)$ is another concentric circle with radius γ. Then $-\log\gamma$ is the conformal module of S. The mapping ϕ is unique up to a planar rotation.*

Topological Annulus

First, we compute a harmonic function $f : S \to \mathbb{R}$, such that

$$\begin{cases} f|_{\gamma_1} & = & 0 \\ f|_{\gamma_2} & = & 1 \\ \Delta f & = & 0 \end{cases}$$

Then df is an exact harmonic 1-form. Then we compute a harmonic 1-form τ, such that $\int_{\gamma_1} \tau = 1$.

Then we compute a constant λ, such that ${}^*df = \lambda\tau$, by solving the following equation,

$$\int_S df \wedge {}^*df = \lambda \int_S df \wedge \tau.$$

Then $\omega = df + i\lambda\tau$ is a holomorphic 1-form. Let $Img(\int_{\gamma_1} \omega) = k$. The conformal mapping is given by

$$\phi(p) = exp^{\frac{2\pi}{k}\int_q^p \omega}.$$

Topological Annulus

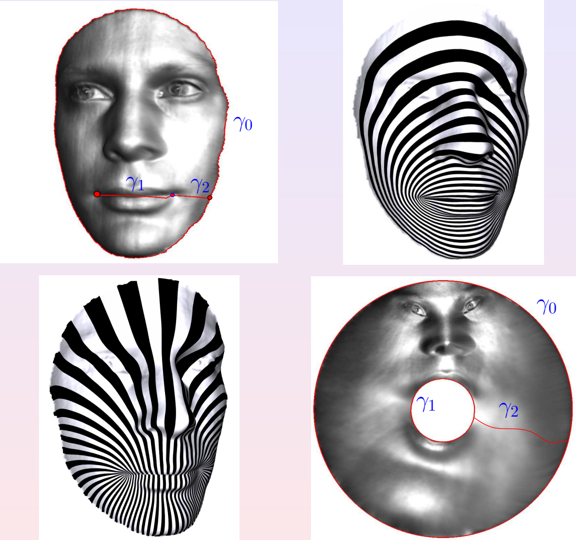


Figure: Topological annulus.

Riemann Mapping

Simply Connected Domains

Topological Disk

Definition (Topological Disk)

Suppose S is a surface of genus zero with one boundary, the S is called a topological disk.

Theorem

Suppose S is a topological disk with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical disk. The mapping ϕ is unique up to a Möbius transformation,

$$z \to e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 z}.$$

Topological Disk

Punch a small hole in the disk, then use the algorithm for topological annulus to compute the conformal mapping. The punched hole will be mapped to the center.

Multiply connected domains

Multiply-Connected Annulus

Definition (Multiply-Connected Annulus)

Suppose S is a surface of genus zero with multiple boundaries, then S is called a multiply connected annulus.

Theorem

Suppose S is a multiply connected annulus with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the unit disk with circular holes. The radii and the centers of the inner circles are the conformal module of S. Such kind of conformal mapping are unique up to Möbius transformations.

Conformal Slit Mapping

Figure: Harmonic forms and holomorphic forms.

Slit Mapping

Suppose there are $n+1$ boundary components$\{\gamma_0,\gamma_1,\gamma_2,\cdots,\gamma_n\}$. $\{\omega_1,\omega_2,\cdots,\omega_n\}$ are the holomorphic 1-form basis. Choose two boundary components, γ_0,γ_1, solve linear equation $\omega = \sum_{k=1}^{n} \lambda_k \omega_k$,

$$img(\int_{\gamma_0} \omega) = 2\pi, img(\int_{\gamma_1} \omega) = -2\pi, img(\int_{\gamma_k} \omega) = 0, 2 \le k \le n.$$

Then the mapping is given by

$$p \to \exp \int_q^p \omega,$$

where q is the base point on the surface.

Conformal Circular Slit Mapping

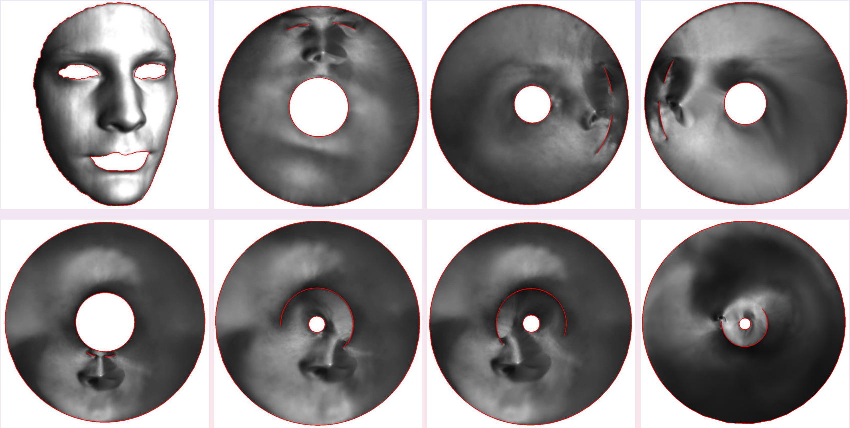


Figure: Conformal circular slit mapping.

Hole Filling

Adding sample points in the center hole, use Delaunay triangulation to fill in with boundary constraints.

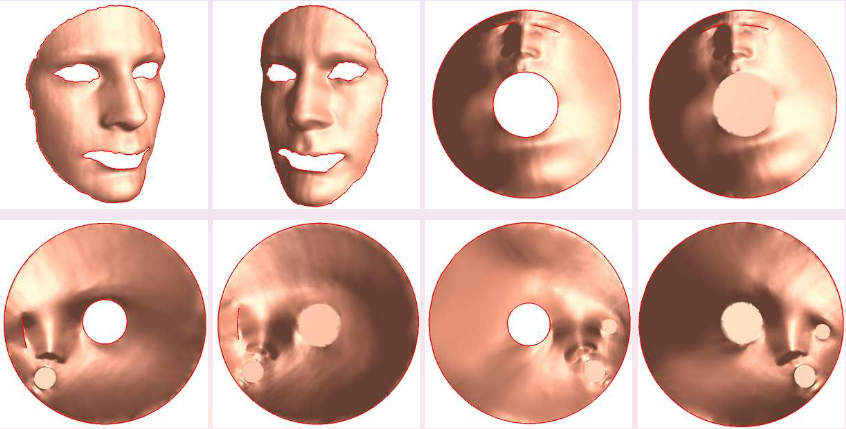


Figure: Fill interior holes.

Koebe's Iteration - I

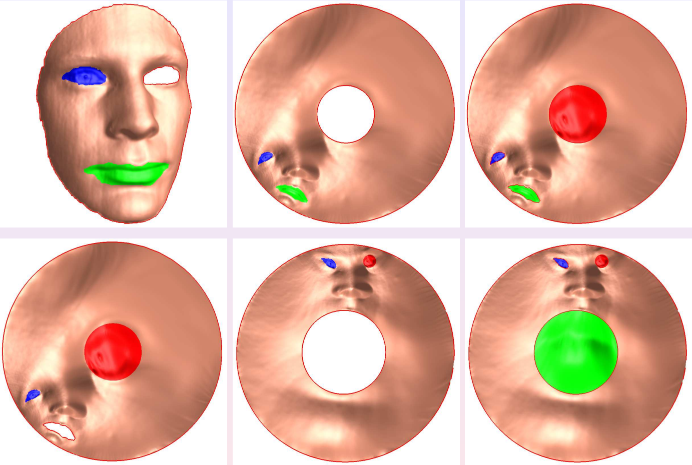


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - II

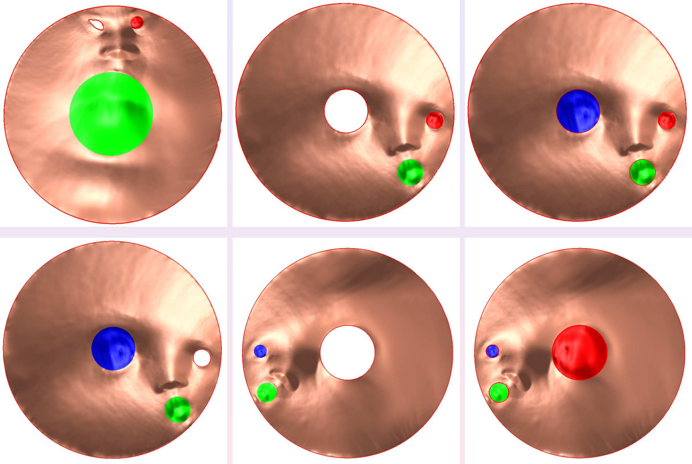


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - III

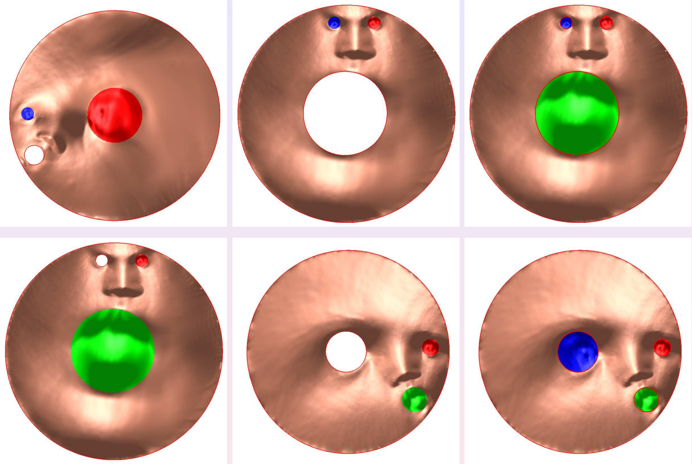


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Convergence Analysis

Theorem (Gu and Luo 2009)

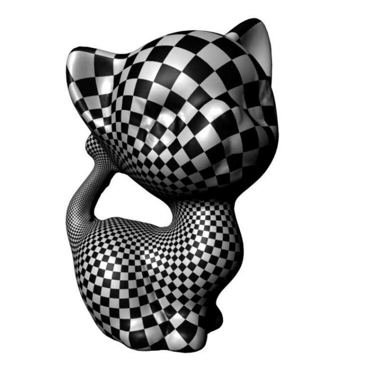
Suppose genus zero surface has n boundaries, then there exists constants $C_1 > 0$ and $0 < C_2 < 1$, for step k, for all $z \in \mathbb{C}$,

$$|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2[\frac{k}{n}]},$$

where f is the desired conformal mapping.

Topological Torus

Topological torus



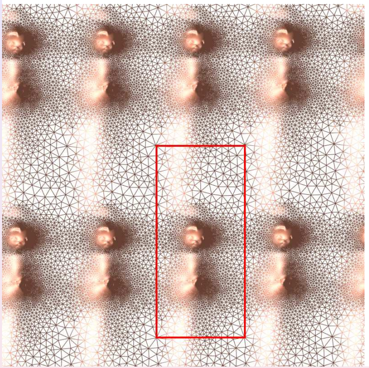


Figure: Genus one closed surface.

Topological Torus

1. We compute a basis for the fundamental group $\pi_1(S)$, $\{\gamma_1, \gamma_2\}$.
2. Compute the holomorophic 1-form basis ω_1, ω_2, such that $\int_{\gamma_i} \omega_j = \delta_{ij}$.
3. Slice the surface along γ_1, γ_2 to get a fundamental domain $\tilde{S}$,
4. The conformal mapping $\phi : \tilde{S} \to \mathbb{C}$ is given by

$$\phi(p) = \int_q^p \omega_1,$$

where q is the base point, the path from q to p in $\tilde{S}$ can be arbitrarily chosen.

Topological Torus

Suppose $a + ib = \int_{\gamma_2} \omega_1$, then $a + ib$ is the conformal module of the torus. The deck transformation group generators are

$$T_1(z) = z + 1, T_2(z) = z + a + ib.$$

By using all deck transformations to translate $\phi(\tilde{S})$, we can conformally map the universal covering space of S onto the whole complex plane $\mathbb{C}$, each fundamental domain is a parallelogram.

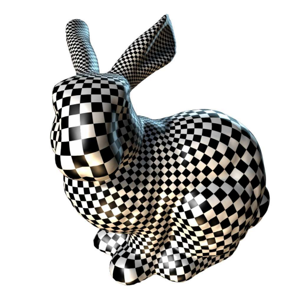
Computational Method - Surface Ricci Flow

Discrete Curvature Flow

Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

The Gaussian curvature is given by

$$K(u,v) = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda(u,v)}}\Delta\lambda(u,v),$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Conformal Metric Deformation

Definition

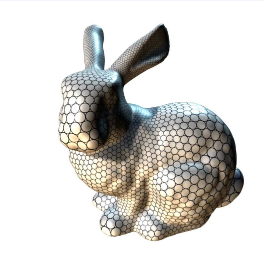
Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Yamabe Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

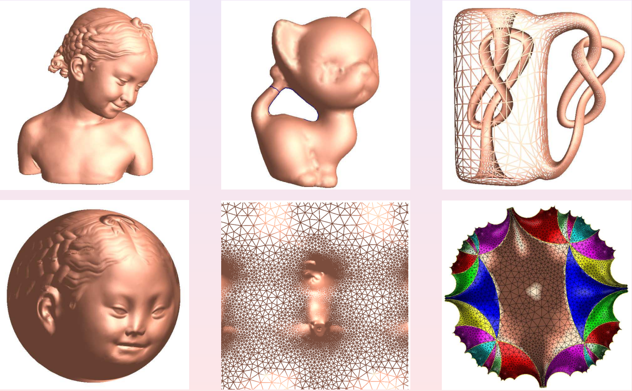
geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(-\partial_n\lambda + k_g).$$

Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

Definition (Hamilton's Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -Kg_{ij}.$$

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant every where.

Key Idea

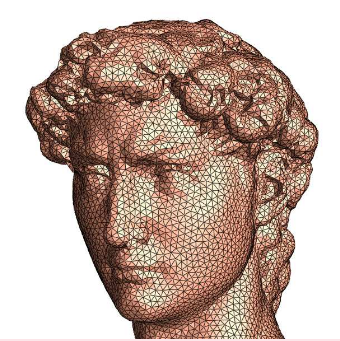
$K = -\Delta_{\mathbf{g}}\lambda$, Roughly speaking, $\frac{dK}{dt} = \Delta_{\mathbf{g}}\frac{d\lambda}{dt}$. Let $\frac{d\lambda}{dt} = -K$, then

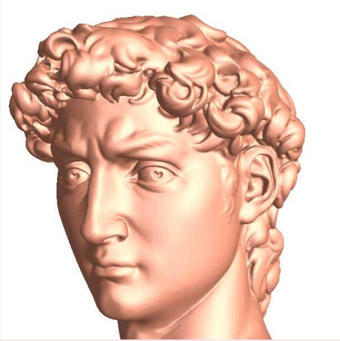
$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K$$

Heat equation!

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.



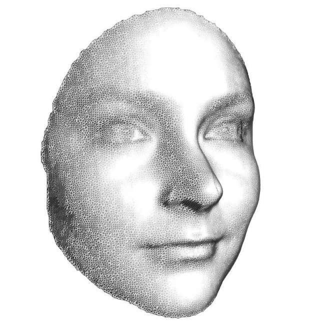


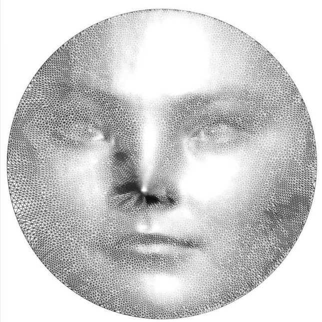
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

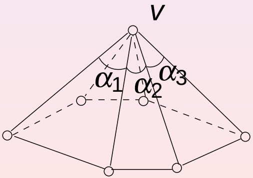
Definition (Discrete Curvature)

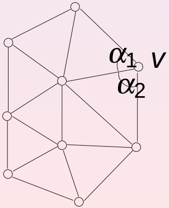
Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

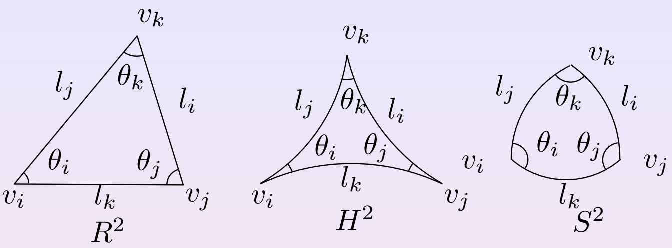
Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$





Discrete Metrics Determines the Curvatures



Angle and edge length relations: cosine laws $\mathbb{R}^2, \mathbb{H}^2, \mathbb{S}^2$

$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \tag{3}$$

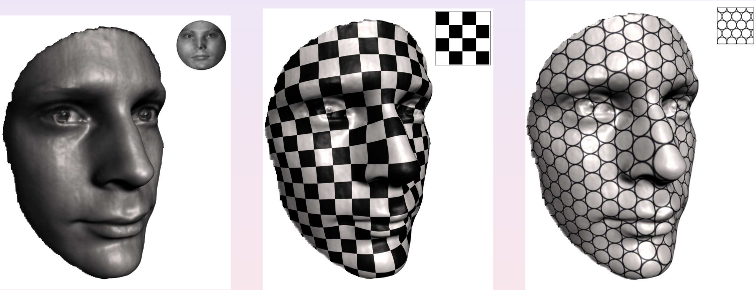
$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} \tag{4}$$

$$1 = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \tag{5}$$

Discrete Conformal Metric Deformation

Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

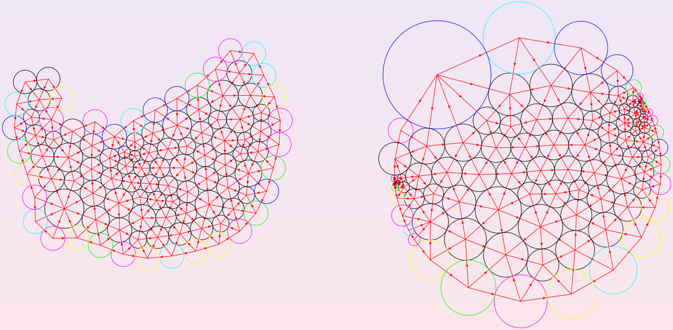


Idea - Approximate conformal metric deformation

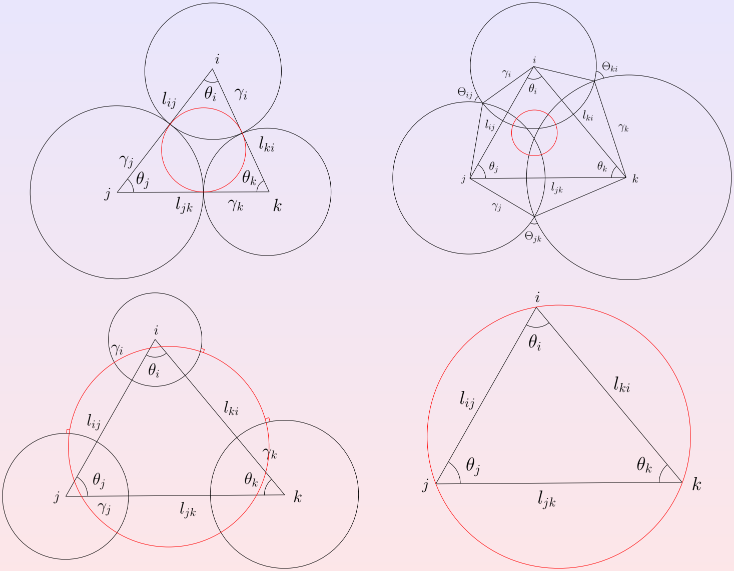
Replace infinitesimal circles by circles with finite radii.

Circle Patterns

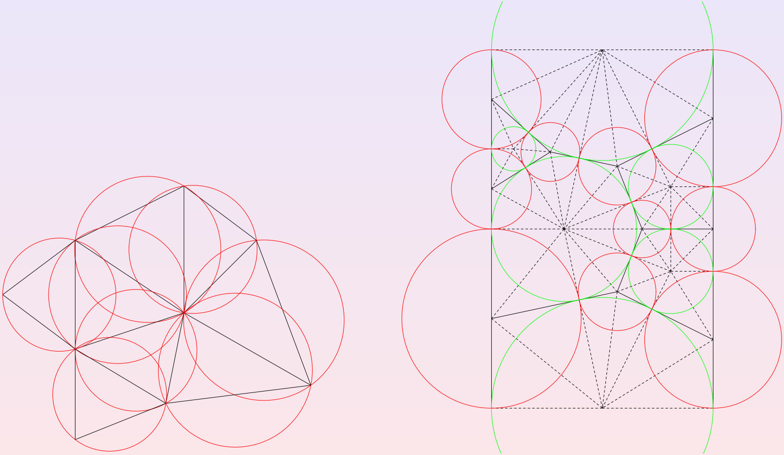
There are many local settings for circle patterns. The radius is variable, the intersection angles do not change.



Circle Patterns



Circle Patterns



Discrete Approach

Circle Packing

- Thurston introduced circle packing metric for studying 3-manifolds in 1978.
- Sullivan and Rodin proved Thurston's circle packing conjecture in 1987.
- The first variational principle for CP metrics was founded by Colin de Verdiere (1991).
- Zheng-Xu He and O. Schramm proved the classical Riemannian mapping theorem in 1994.
- Chow and Luo built the connection between Ricci flow and circle packing in 2003.
- Springborn, Bobenko and Schroder's circle pattern in 2005.

Circle Packing Metric

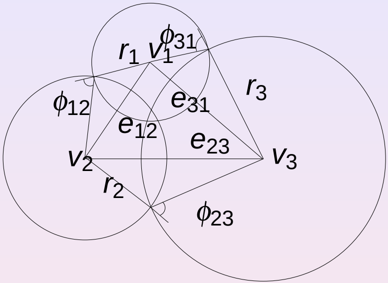
CP Metric

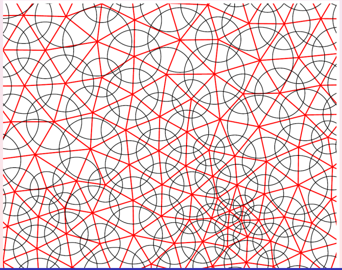
We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$





Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u}: V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation

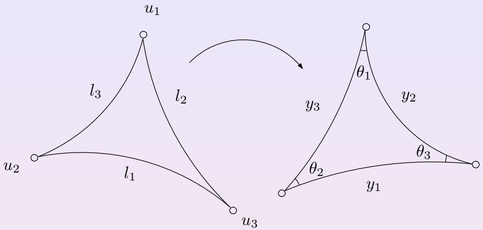
$$d\mathbf{K} = \Delta d\mathbf{u},$$

Δ is a discrete Lapalce-Beltrami operator.

- Feng Luo, *Combinatorial Yamabe Flow on Surfaces*, Commun.Contemp.Math., Vol.6 Num. 5, Pages 765-780, **2004**.
- Boris, Springborn and Peter Schröder and Ulrich Pinkall, *Conformal equivalence of triangle meshes*, ACM Trans. Graph., vol.27 Num. 3, pages 1-11, **2008**.

Discrete Conformal Factor for Yamabe Flow

Discrete conformal metric deformation:



conformal factor

$$\begin{aligned} \frac{y_k}{2} &= e^{u_i} \frac{l_k}{2} e^{u_j} && \mathbb{R}^2 \\ \sinh \frac{y_k}{2} &= e^{u_i} \sinh \frac{l_k}{2} e^{u_j} && \mathbb{H}^2 \\ \sin \frac{y_k}{2} &= e^{u_i} \sin \frac{l_k}{2} e^{u_j} && \mathbb{S}^2 \end{aligned}$$

Properties: $\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$ and $d\mathbf{K} = \Delta d\mathbf{u}$.

Unified Framework of Discrete Curvature Flow

Unified framework for both Discrete Ricci flow and Yamabe flow

- Curvature flow

$$\frac{du}{dt} = \bar{K} - K,$$

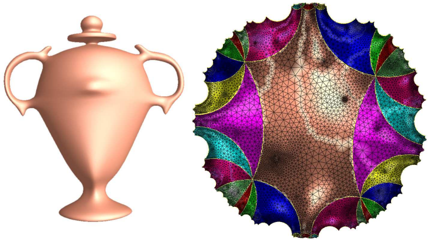
- Energy

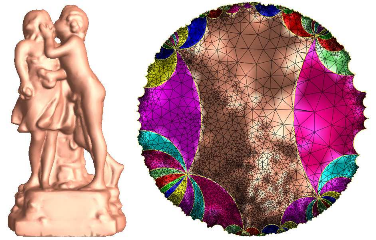
$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$

- Hessian of E denoted as Δ,

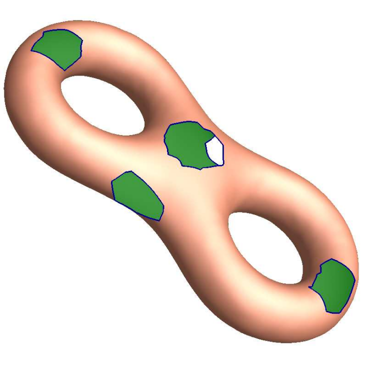
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

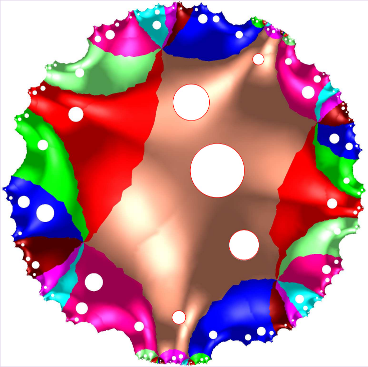
Computational results for genus 2 and genus 3 surfaces.





Hyperbolic Koebe's Method





Volumetric Parameterization

Based on surface parameterization method

1. Volumetric harmonic map
2. Green's function on Star Shape
3. Direct product decomposition
4. Volumetric curvature flow

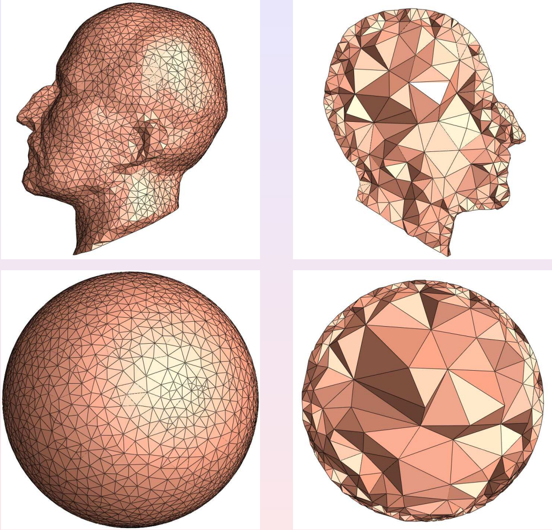
Volumetric Harmonic Map

Suppose we want to compute a volumetric mapping $\phi : V \to D$,

1. Compute a conformal mapping between the boundary surfaces
$$\psi : \partial V \to \partial D$$
2. Compute a volumetric harmonic mapping with Dirichlet boundary condition

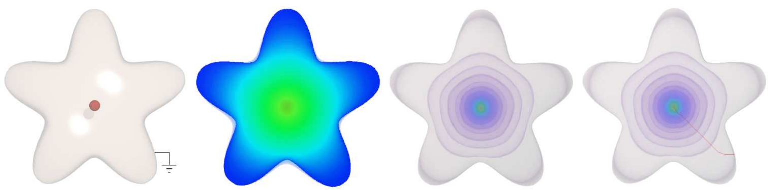
$$\Delta\phi = 0, \phi|_{\partial V} = \psi.$$

Volumetric Harmonic Mapping



Green's Function on Star Shape

Suppose V is a star shape with the center o, then each ray from the center intersects the boundary of V only once.



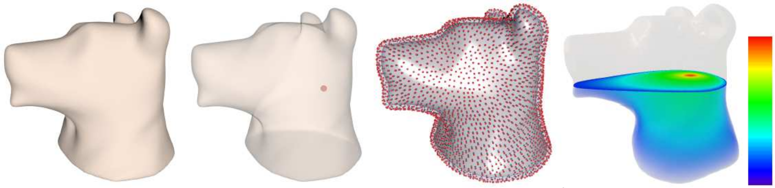
Green's Function on Star Shape

1. Compute the Green's function G on V, centered at o, such that $G|_{\partial V} = 0$.
2. Trace the gradient line from the center to the boundary.
3. Conformal map the boundary to the unit sphere.
4. The level sets of G are mapped to concentric spheres, the gradient lines are mapped to the radii of the unit ball.

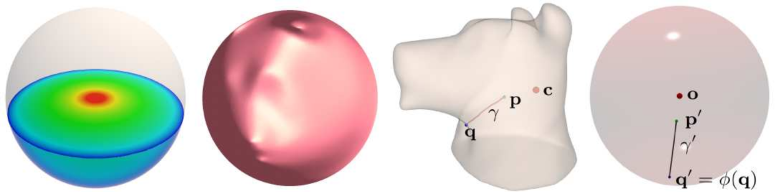
Theorem

The mapping must be a diffeomorphism.

Green's Function on Star Shape



Green's Function on Star Shape



Green's Function on Star Shape

Direct Product Method

The volume can be decomposed as the direct product of a surface and a curve (or circle), $V = S \times C$,

1. Decompose the boundary surface of V to top T, bottom B and wall W.
2. Conformal parameterize the top B, $\phi : B \to S$.
3. Compute a harmonic function f, such that

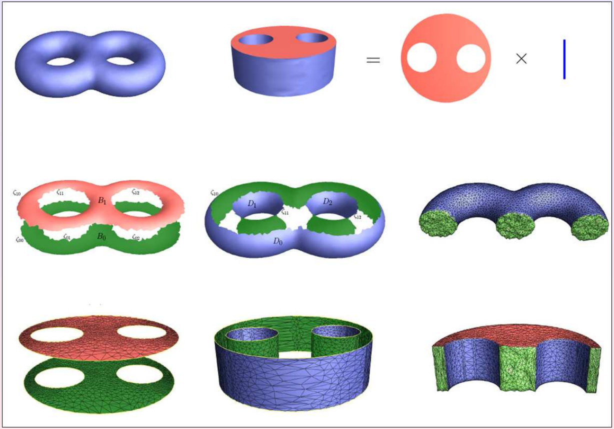
$$\begin{cases} f(p) = 1 & p \in T \\ f(p) = 0 & p \in B \\ \Delta f = 0 & \end{cases}$$

4. Trace the gradient line of f, then maps the volume to $S \times [0,1]$.

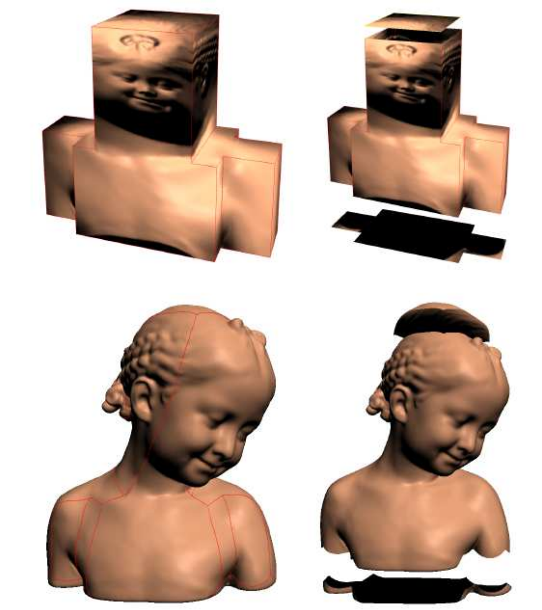
Theorem

The mapping must be a diffeomorphism.

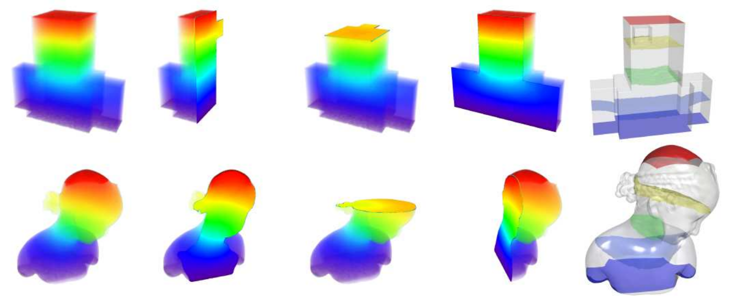
Direct Product



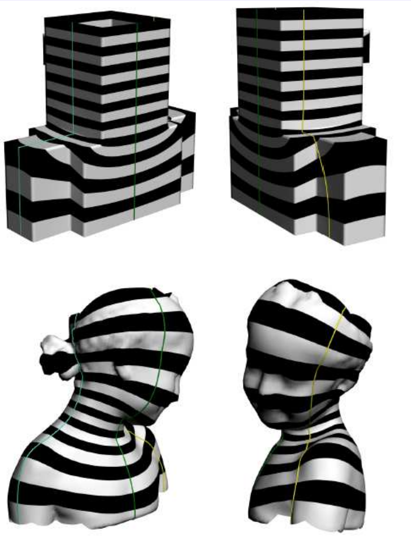
Direct Product



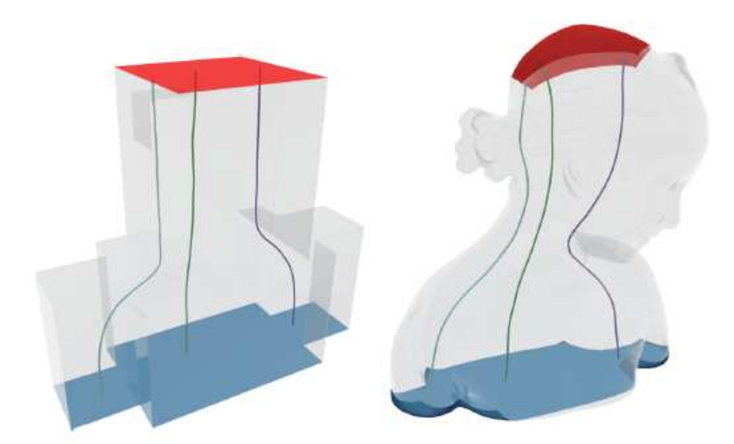
Direct Product



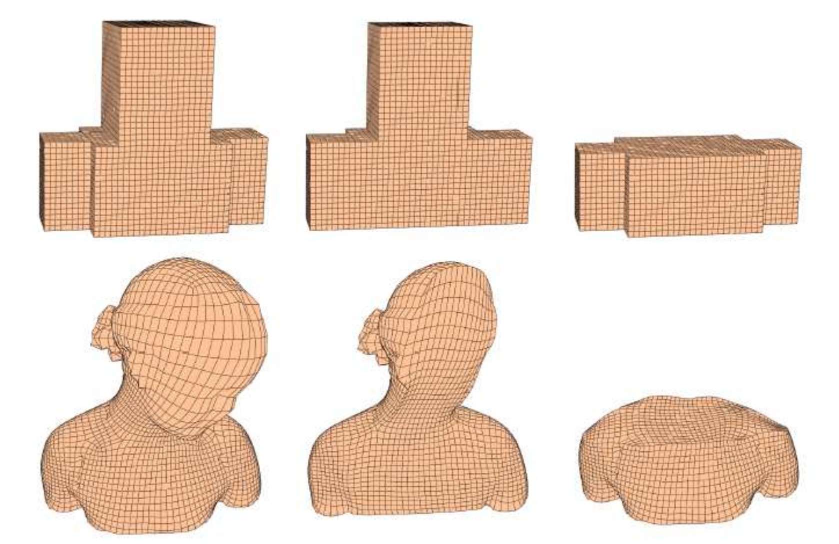
Direct Product



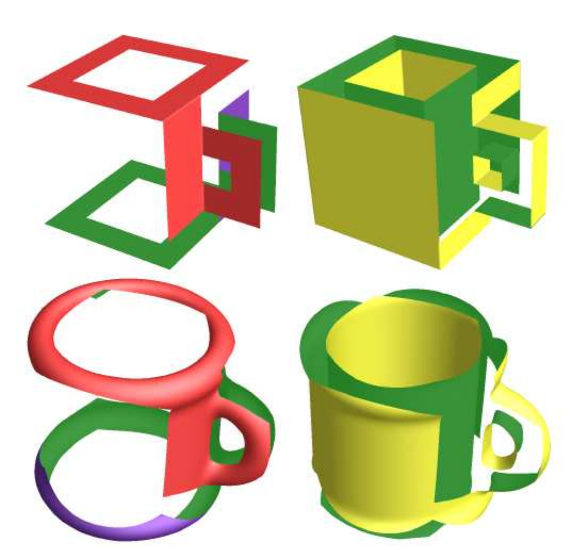
Direct Product



Direct Product

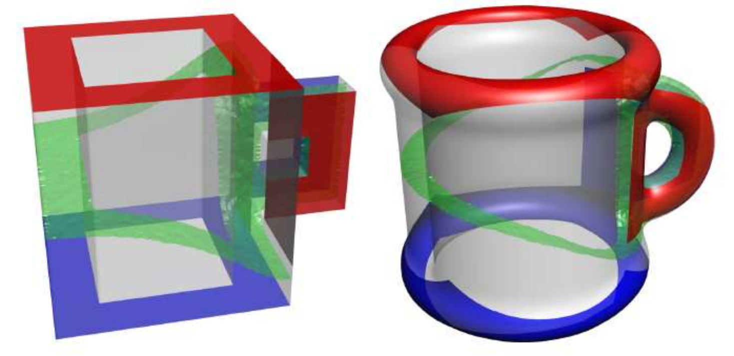


Direct Product



Direct Product

Direct Product



Direct Product

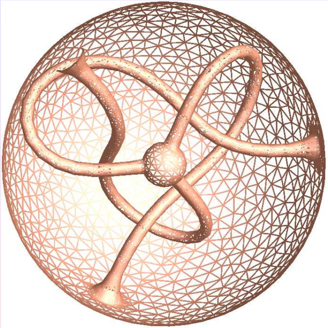
Suppose the given volume has complicated topology, such that the boundary surfaces are with high genus. Then we can compute the canonical hyperbolic Riemannian metric of the volume, and embed the universal covering space of the volume in three dimensional hyperbolic space $\mathbb{H}^3$.

Hyperbolic Volumetric Curvature Flow

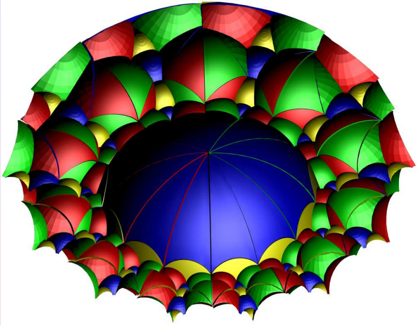
1. Triangulate the volume to truncated tetrahedra.
2. Compute the curvature on each edge of the tetrahedra mesh
$$K(e_{ij}) = 2\pi - \sum_{kl} \theta_{ij}^{kl},$$
where θ_{ij}^{kl} is the dihedral angle on the edge (e_{ij} in the tetrahedron $[v_i, v_j, v_k, v_l]$.
3. Run curvature flow,
$$\frac{dl_{ij}}{dt} = K_{ij}$$
where l_{ij} is the edge length of e_{ij}.

Theorem

If the input 3-manifold is a hyperbolic 3-manifold with complete geodesic boundary, then the curvature flow will converge to the canoical hyperbolic metric.



a. input 3-manifold



b. embedding of its UCS in $\mathbb{H}^3$

Applications

Medical Imaging Application

Medical Imaging

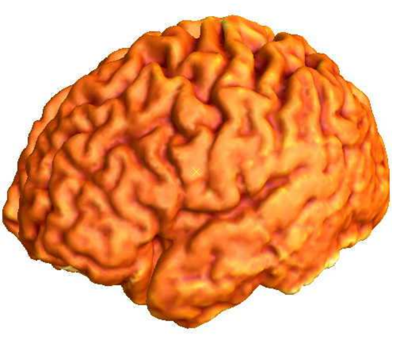
Quantitatively measure and analyze the surface shapes, to detect potential abnormality and illness.

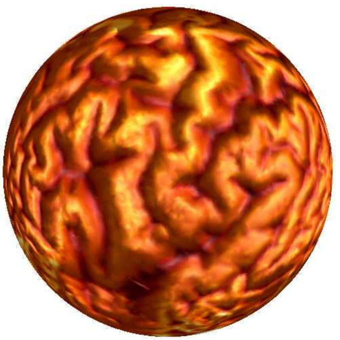
- Shape reconstruction from medical images.
- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Shape retrieval.

Conformal Brain Mapping

Brain Cortex Surface

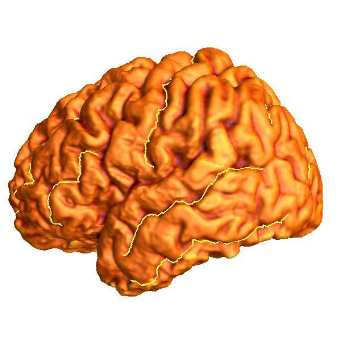
Conformal Brain Mapping for registration, matching, comparison.

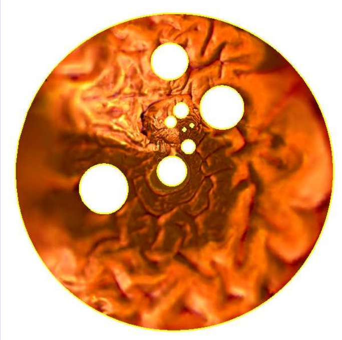




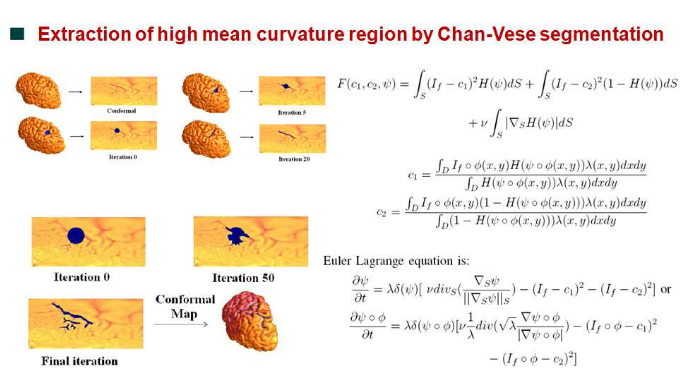
Using conformal module to analyze shape abnormalities.

Brain Cortex Surface



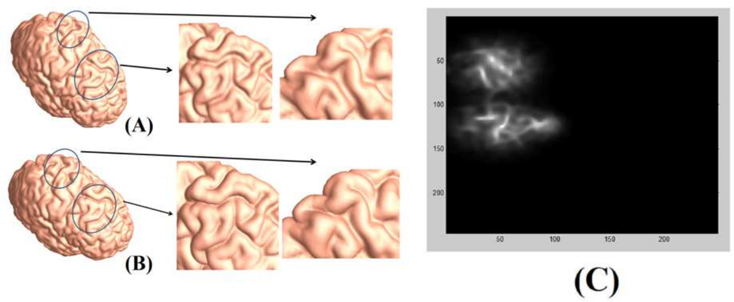


- With the conformal structure, PDE on Riemann surfaces can be easily solved.
- Chan-Vese segmentation model is generalized to Riemann surfaces to detect sulcal landmarks on the cortical surfaces automatically

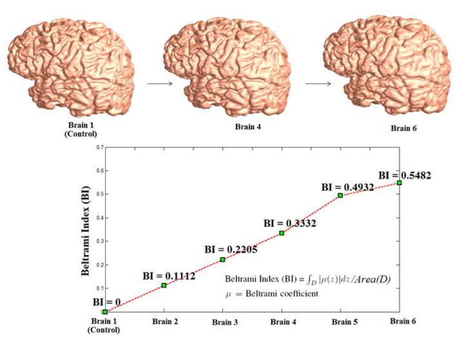


Abnormality detection on brain surfaces

The Beltrami coefficient of the deformation map detects the abnormal deformation on the brain.

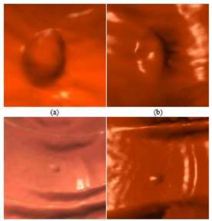


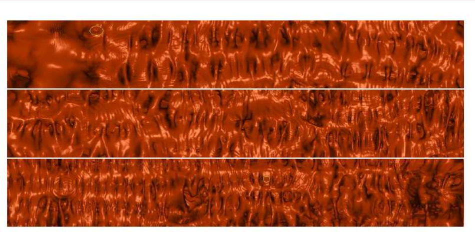
The brain is undergoing gyri thickening (commonly observed in Williams Syndrome) The Beltrami index can effectively measure the gyrification pattern of the brain surface for disease analysis.



Virtual Colonoscopy

Colon cancer is the 4th killer for American males. Virtual colonosocpy aims at finding polyps, the precursor of cancers. Conformal flattening will unfold the whole surface.

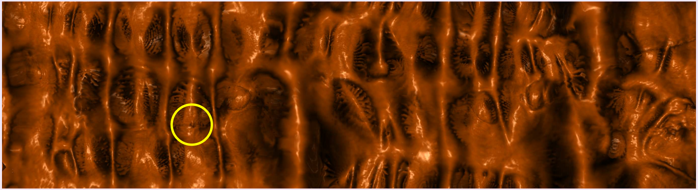


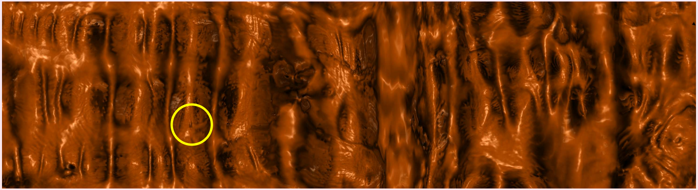


Virtual Colonoscopy

Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.

Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.





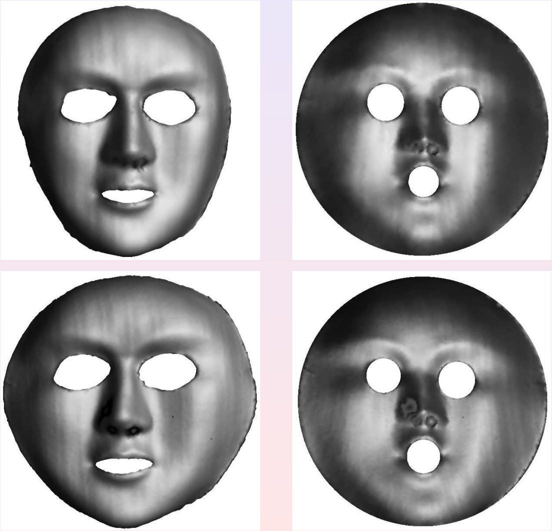
Computer Vision Application

Vision

- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Tracking.

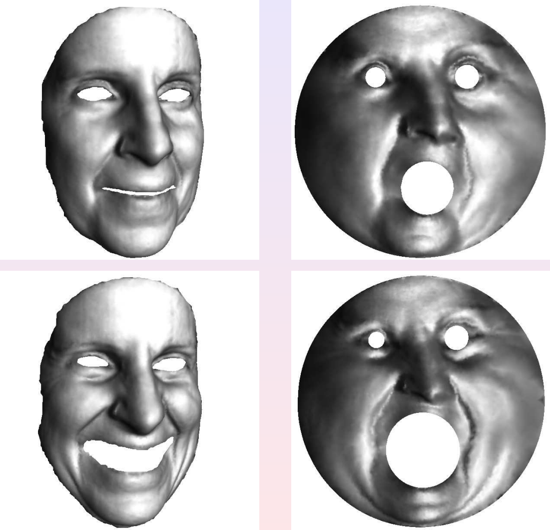
Surface Matching

Isometric deformation is conformal. The mask is bent without stretching.



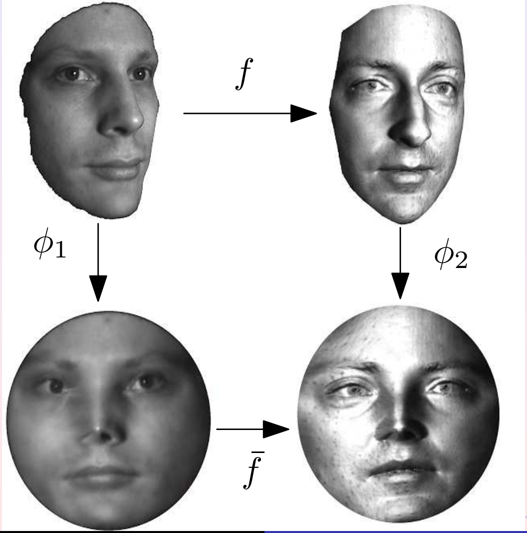
Surface Matching

Facial expression change is not-conformal.

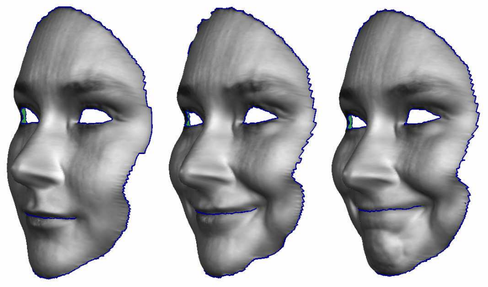


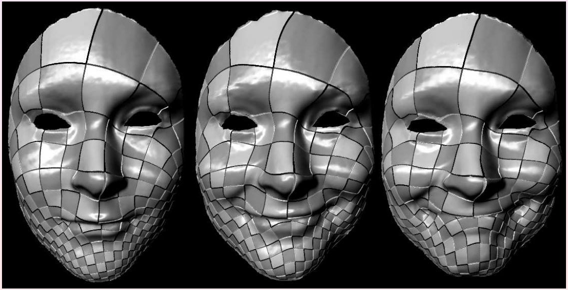
Surface Matching

3D surface matching is converted to image matching by using conformal mappings.

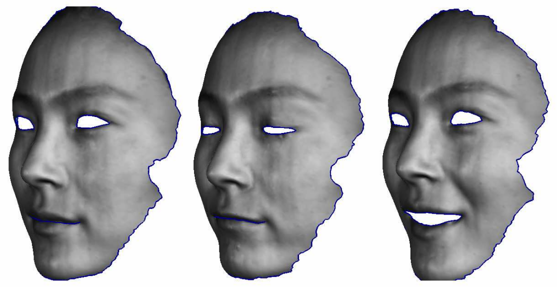


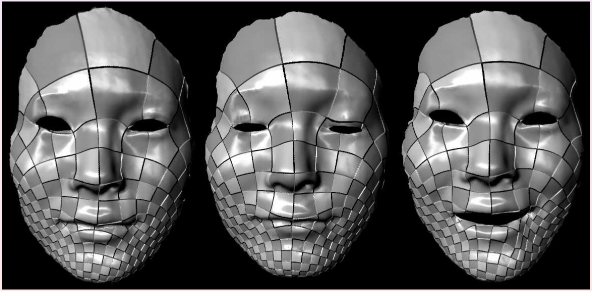
Face Surfaces with Different Expressions are Matched



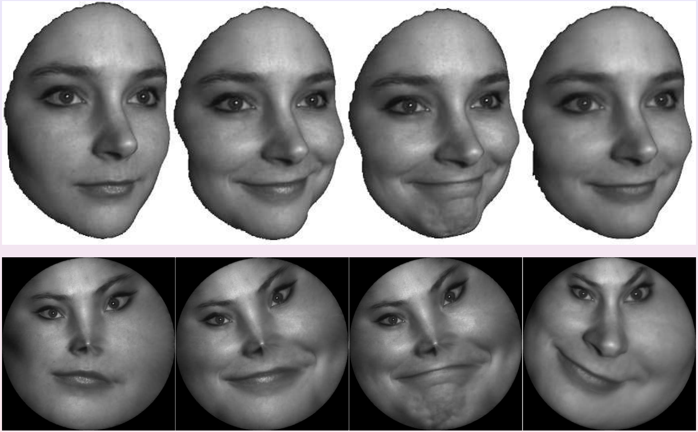


Face Surfaces with Different Expressions are Matched

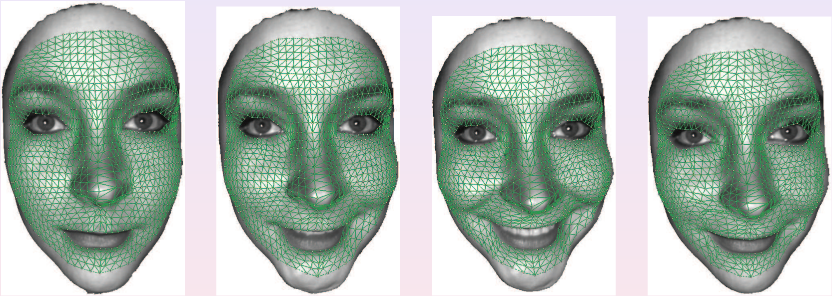




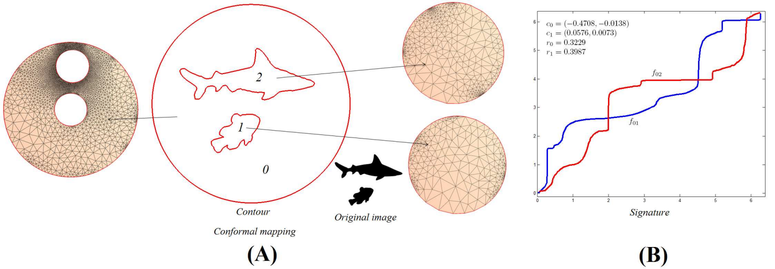
Face Expression Tracking



Face Expression Tracking



$$\{2D\ Contours\} \cong \frac{\{Diffeomorphism\ on\ S^1\} \cup \{Conformal\ Module\}}{\{Mobius\ Transformation\}}$$



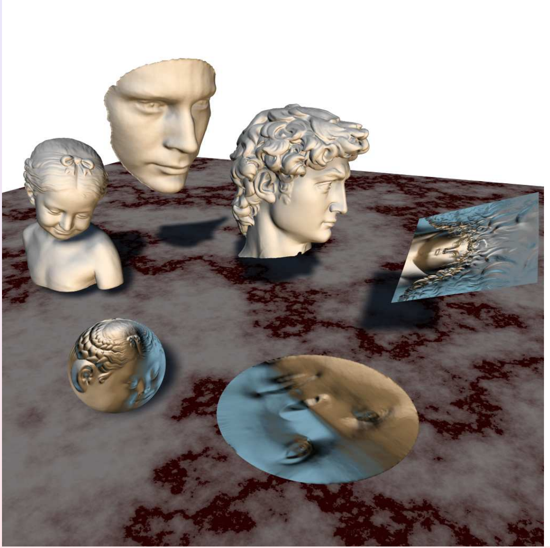
Computer Graphics Application

Graphics

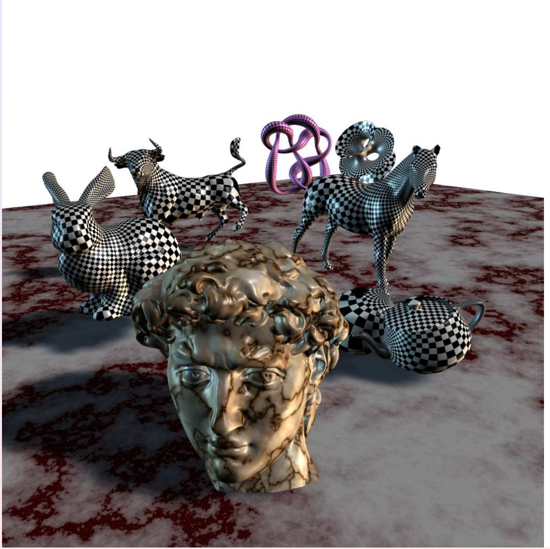
- Surface Parameterization, texture mapping
- Texture synthesis, transfer
- Vector field design
- Shape space and retrieval.

Surface Parameterization

Map the surfaces onto canonical parameter domains

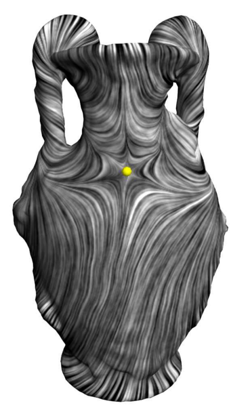


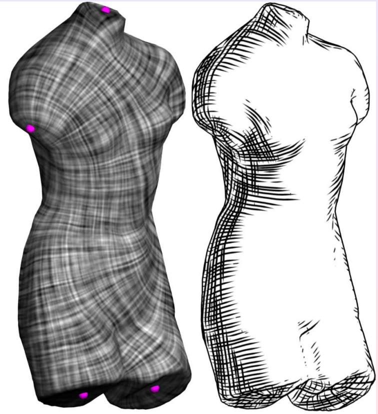
Applied for texture mapping.



n-Rosy Field Design

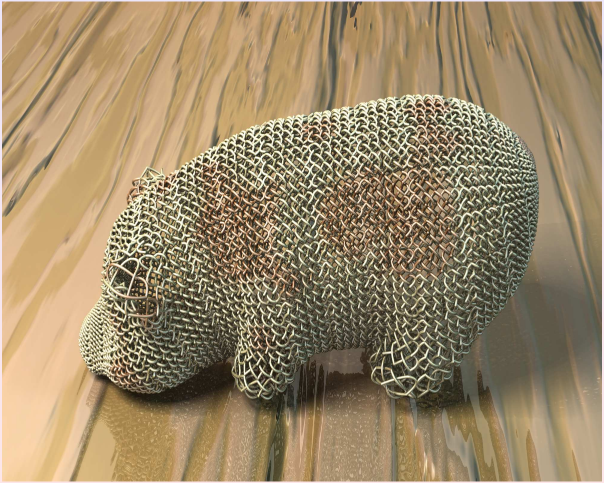
Design vector fields on surfaces with prescribed singularity positions and indices.





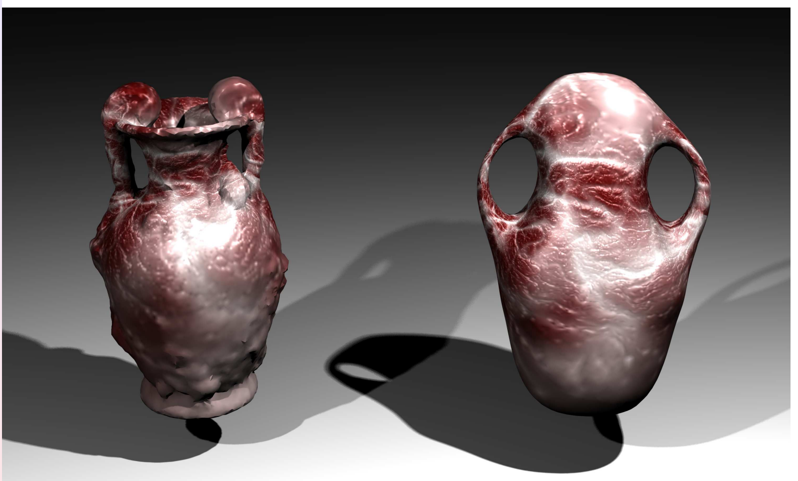
n-Rosy Field Design

Convert the surface to knot structure using smooth vector fields.



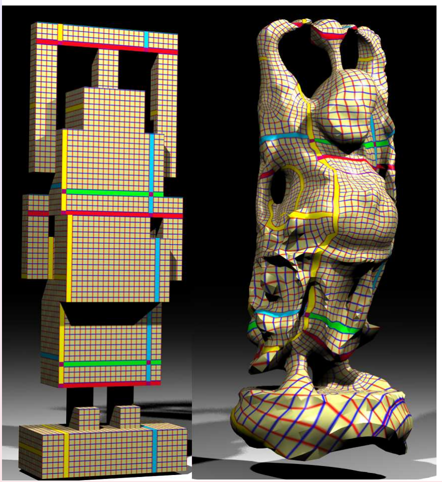
Texture Transfer

Transfer the texture between high genus surfaces.



Polycube Map

Compute polycube maps for high genus surfaces.



Manifold Spline

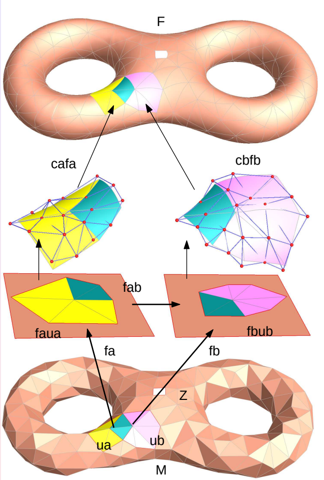
- Convert scanned polygonal surfaces to smooth spline surfaces.
- Conventional spline scheme is based on affine geometry. This requires us to define affine geometry on arbitrary surfaces.
- This can be achieved by designing a metric, which is flat everywhere except at several singularities (extraordinary points).
- The position and indices of extraordinary points can be fully controlled.

Manifold Spline

Extraordinary Points

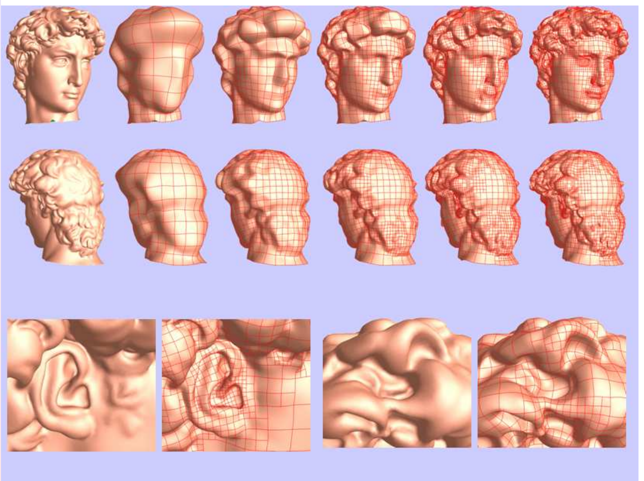
- Fully control the number, the index and the position of extraordinary points.
- For surfaces with boundaries, splines without extraordinary point can be constructed.
- For closed surfaces, splines with only one singularity can be constructed.

Manifold Spline



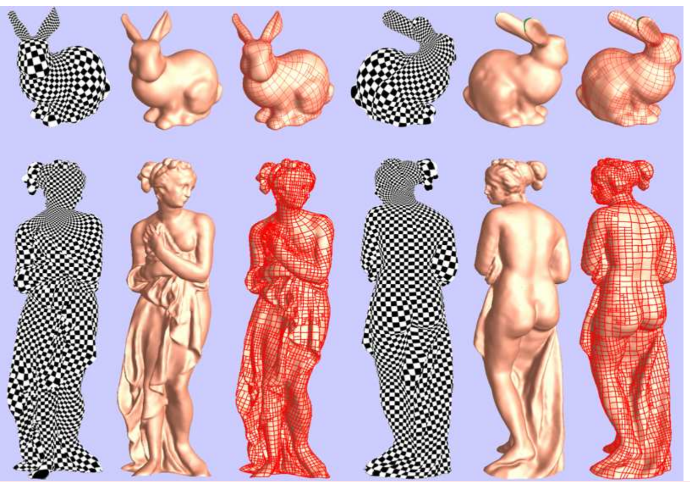
Manifold Spline

Converting a polygonal mesh to TSplines with multiple resolutions.



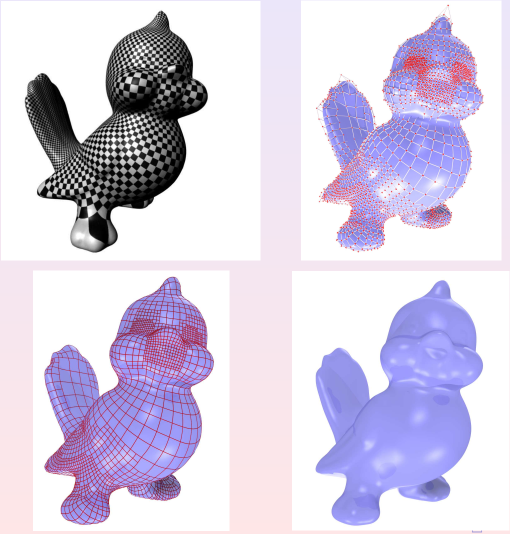
Manifold Spline

Converting scanned data to spline surfaces.



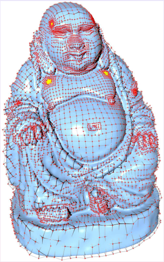
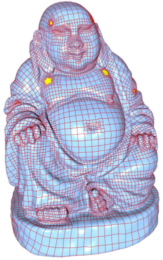
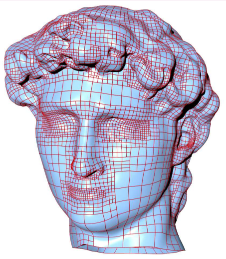
Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.

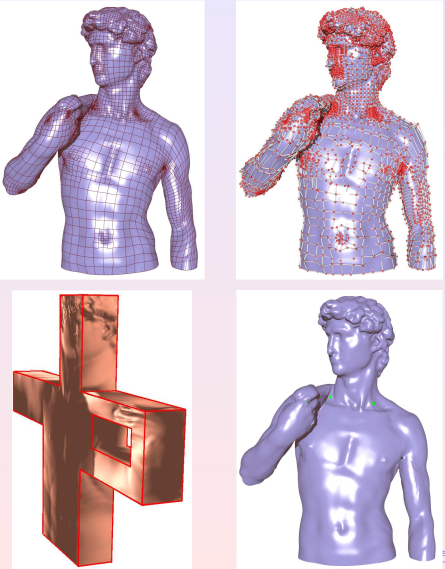


Manifold Spline

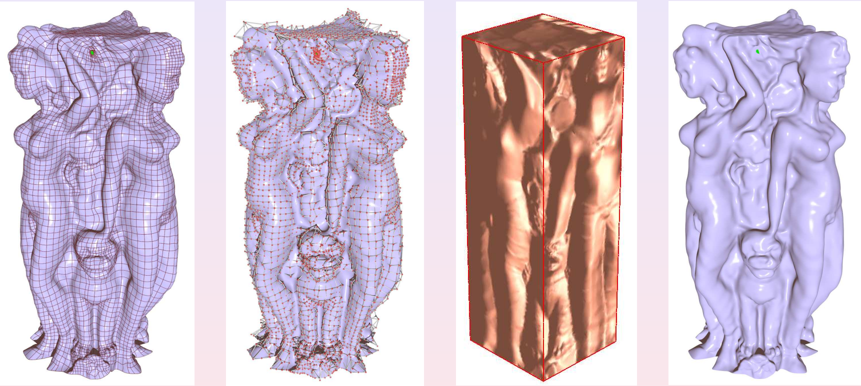
Converting scanned data to spline surfaces, the control points, knot structure are shown.


Polygonal mesh to spline, control net and the knot structure.

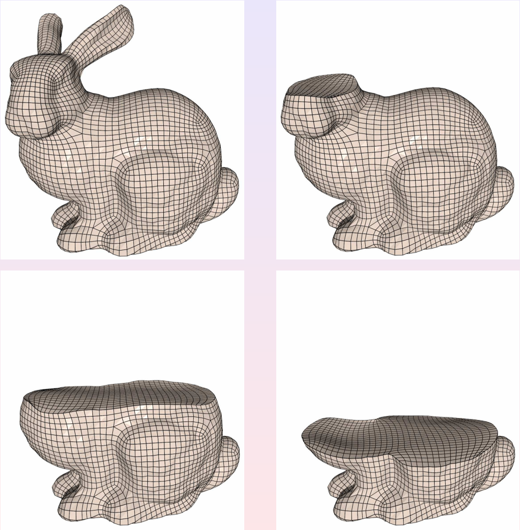


Manifold Spline



Manifold Spline

volumetric spline.



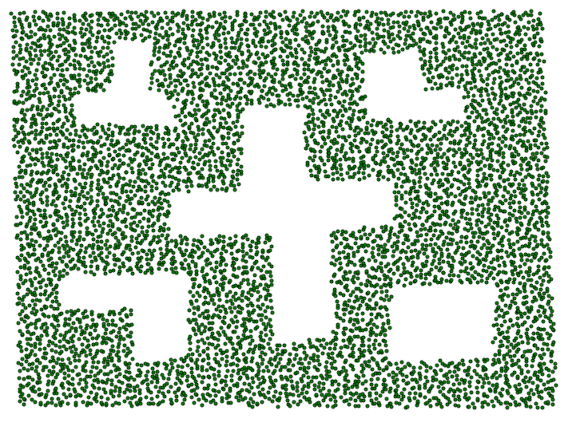
Wireless Sensor Network Application

Wireless Sensor Network

- Detecting global topology.
- Routing protocol.
- Load balancing.
- Isometric embedding.

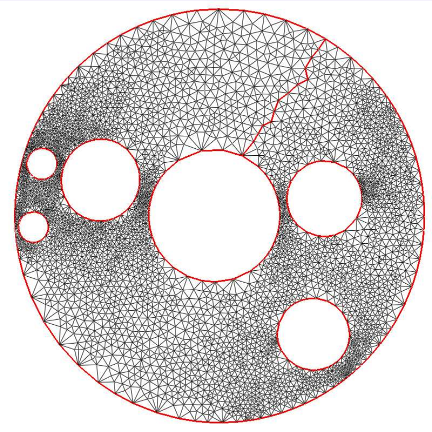
Greedy Routing

Given sensors on the ground, because of the concavity of the boundaries, greedy routing doesn't work.



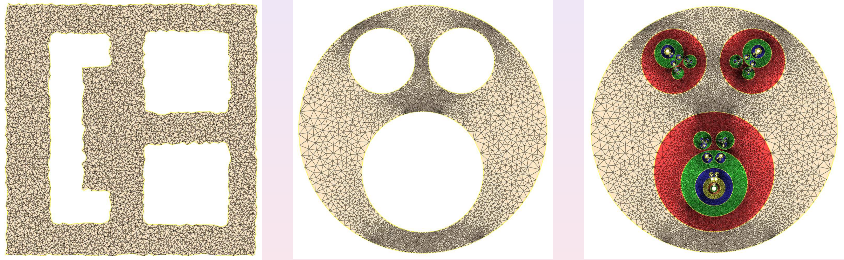
Greedy Routing

Map the network to a circle domain, all boundaries are circles, greedy routing works.



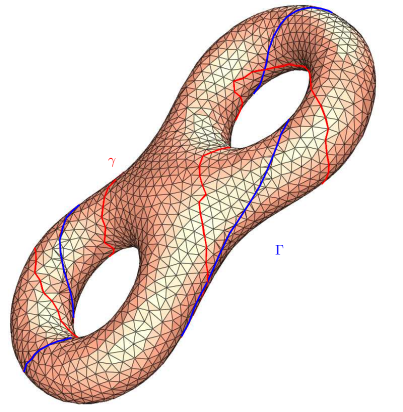
Load Balancing

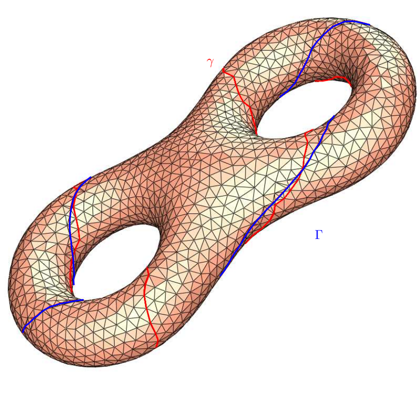
Schoktty Group - Circular Reflection



Canonical Homotopy Class Representative

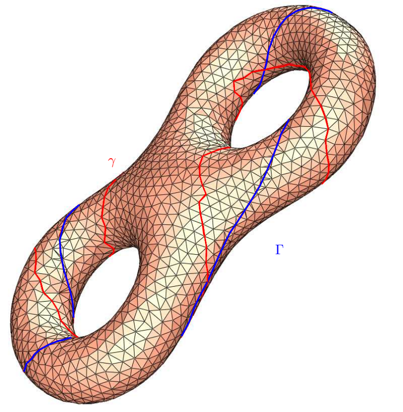
Under hyperbolic metric, each homotopy class has a unique geodesic, which is the representative of the homotopy class.

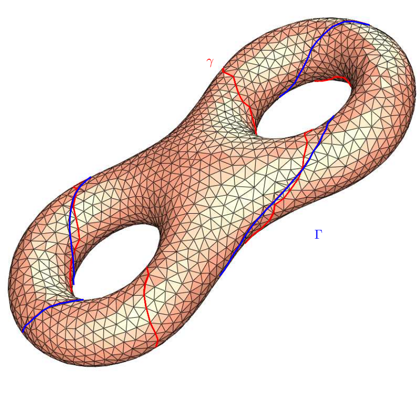




Homotopy Detection Problem

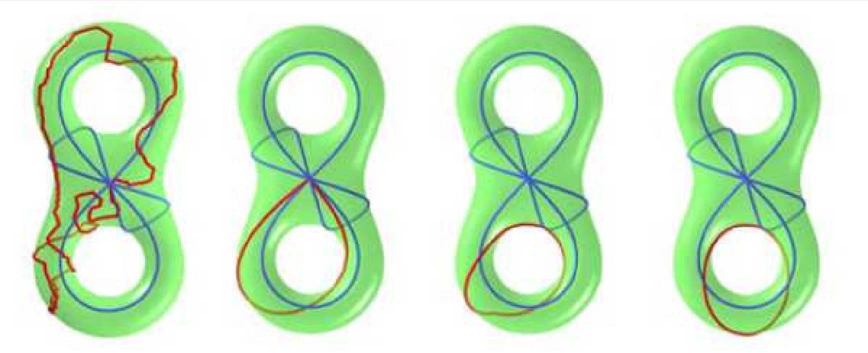
Verify whether two loops are homotopic equivalent.



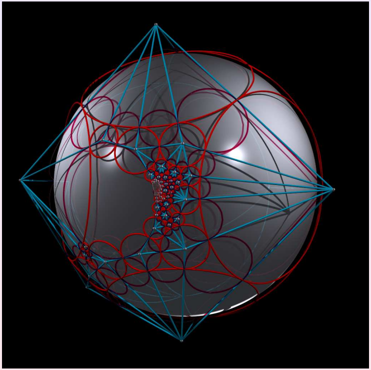


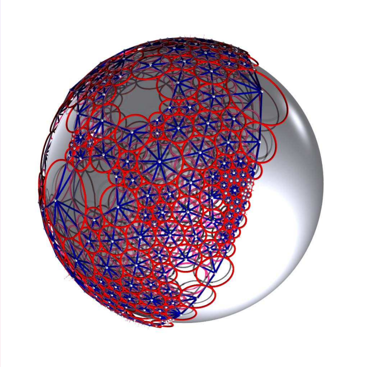
Shortest Word Problem

Find the shortest representation of a homotopy class in the fundamental group.



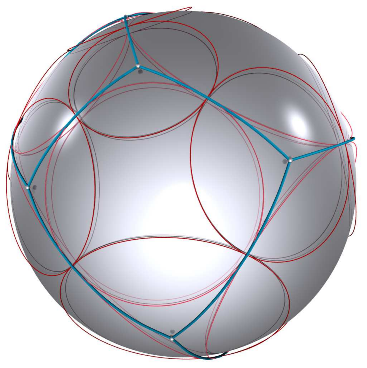
Planar Planar Graph Embedding.

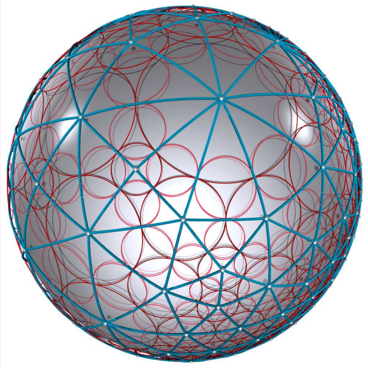




Graph isomorphism

Detect if two planar graphs are isomorphic.





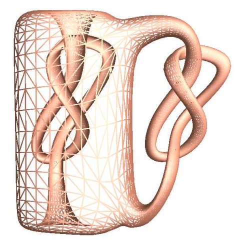
Summary

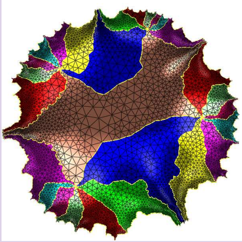
Ricci flow is a powerful tool for designing Riemannian metrics using curvatures, and extremely valuable for many fields in computer science.

- Computer Graphics - Parameterization
- Computer Vision - Surface registration, tracking, analysis
- Geometric Modeling - Spline construction
- Networking - Routing, load balancing
- Medical Imaging - Brain mapping, Virtual Colonoscopy

Thanks

For more information, please email to gu@cs.sunysb.edu.





Thank you!