Computational Conformal Geometry: Theories, Algorithms and Applications

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Workshop on Geometric Methods for Analyzing Discrete Shapes, Davis, CA

Thanks

Thanks for the invitation.

Collaborators

The work is collaborated with Shing-Tung Yau, Kefeng Liu, Feng Luo, Wen-Wei Lin, Tony Chan, Paul Thompson, Yalin Wang, Ronald Lok Ming Lui, Zhongxuan Luo, Na Lei, Xiaopeng Zheng, Hong Qin, Dimitris Samaras, Jie Gao, Arie Kaufman, and many other mathematicians, computer scientists and medical doctors.

Klein's Program

Klein's Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion

Motivation

Conformal geometry lays down the theoretic foundation for

- Surface mapping
- Geometry classification
- Shape analysis

Applied in computer graphics, computer vision, geometric modeling, wireless sensor networking and medical imaging, and many other engineering, medical fields.

History

- In pure mathematics, conformal geometry is the intersection of complex analysis, algebraic topology, Riemann surface theory, algebraic curves, differential geometry, partial differential equation.
- In applied mathematics, computational complex function theory has been developed, which focuses on the conformal mapping between planar domains.
- Recently, computational conformal geometry has been developed, which focuses on the conformal mapping between surfaces.

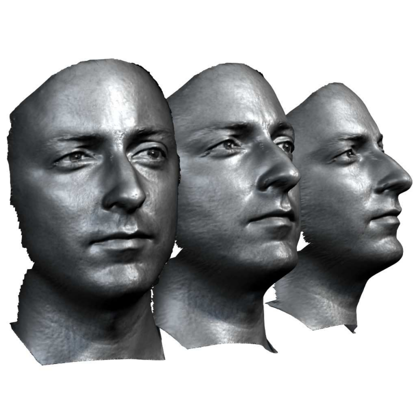
History

Conventional conformal geometric method can only handle the mappings among planar domains.

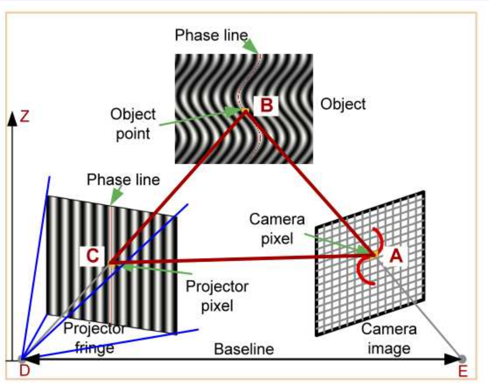
- Applied in thin plate deformation (biharmonic equation)
- Membrane vibration
- Electro-magnetic field design (Laplace equation)
- Fluid dynamics
- Aerospace design

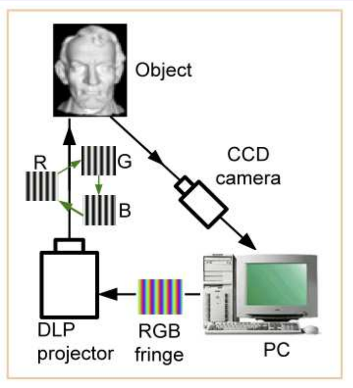
Data Acquisition

3D scanning technology becomes mature, it is easier to obtain surface data.

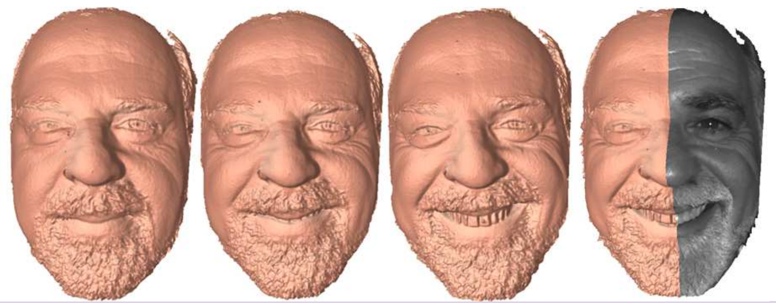


System Layout

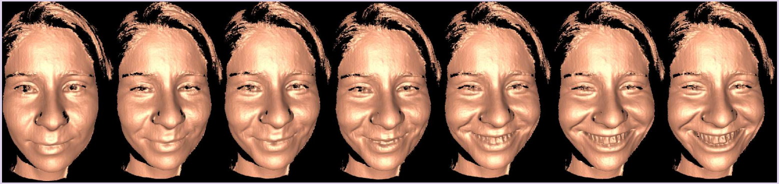




3D Scanning Results



3D Scanning Results



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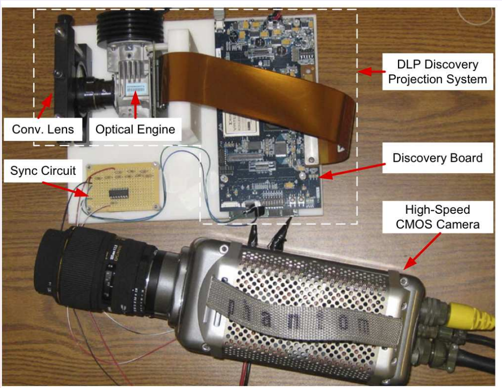
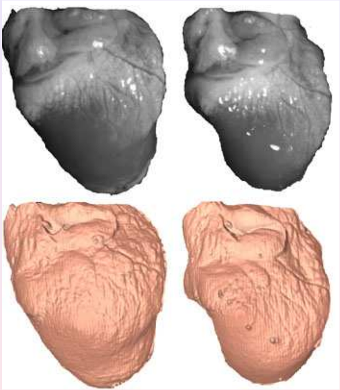




Reasons for Booming

Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

Computational Power

Computational power has been increased tremendously. With the incentive in graphics, GPU becomes mature, which makes numerical methods for solving PDE's much easier.

Fundamental Problems

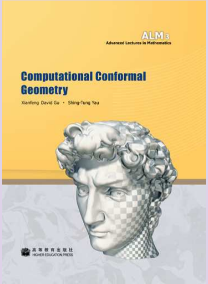
1. Given a Riemannian metric on a surface with an arbitrary topology, determine the corresponding conformal structure.
2. Compute the complete conformal invariants (conformal modules), which are the coordinates of the surface in the Teichmuller shape space.
3. Fix the conformal structure, find the simplest Riemannian metric among all possible Riemannian metrics
4. Given desired Gaussian curvature, compute the corresponding Riemannian metric.
5. Given the distortion between two conformal structures, compute the quasi-conformal mapping.
6. Compute the extremal quasi-conformal maps.
7. Conformal welding, glue surfaces with various conformal modules, compute the conformal module of the glued surface.

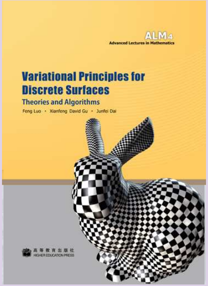
Complete Tools

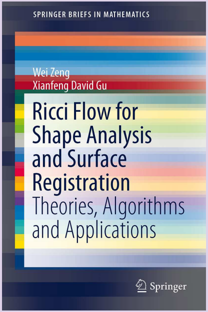
Computational Conformal Geometry Library

1. Compute conformal mappings for surfaces with arbitrary topologies
2. Compute conformal modules for surfaces with arbitrary topologies
3. Compute Riemannian metrics with prescribed curvatures
4. Compute quasi-conformal mappings by solving Beltrami equation

The theory, algorithms and sample code can be found in the following books.



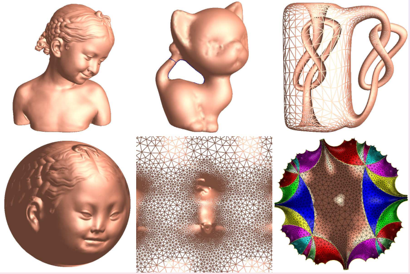


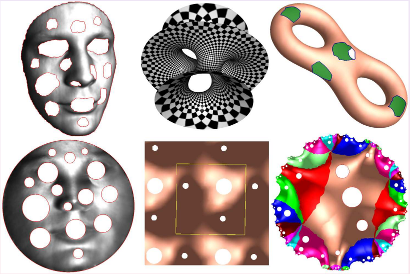


You can find them in the book store.

Source Code Library

Please email me gu@cs.sunysb.edu for updated code library on computational conformal geometry.





Conformal Structure

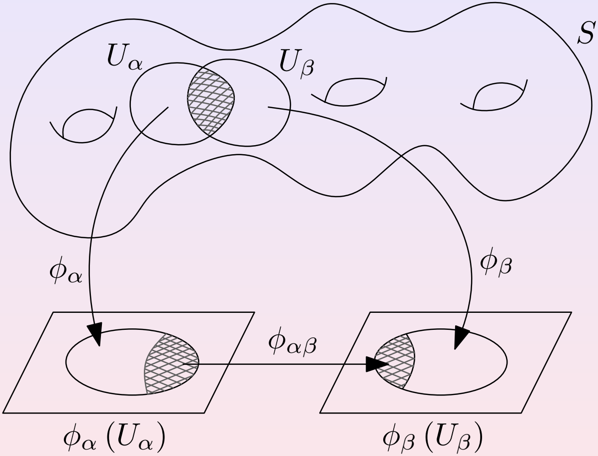
Definition (Manifold)

M is a topological space, $\{U_\alpha\}$ $\alpha \in I$ is an open covering of M, $M \subset \cup_\alpha U_\alpha$. For each U_α, $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. The pair (U_α, ϕ_α) is a chart. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$$

then M is called a smooth manifold, $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas.

Manifold


S
U_α
U_β
ϕ_α
ϕ_β
$\phi_{\alpha\beta}$
$\phi_\alpha(U_\alpha)$
$\phi_\beta(U_\beta)$

Holomorphic Function

Definition (Holomorphic Function)

Suppose $f : \mathbb{C} \to \mathbb{C}$ is a complex function, $f : x + iy \to u(x,y) + iv(x,y)$, if f satisfies Riemann-Cauchy equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then f is a holomorphic function.

Denote

$$dz = dx + idy, d\bar{z} = dx - idy, \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

then if $\frac{\partial f}{\partial \bar{z}} = 0$, then f is holomorphic.

Conformal Atlas

Definition (Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}$ is an atlas, such that all the chart transition functions $\phi_{\alpha\beta}: \mathbb{C} \to \mathbb{C}$ are bi-holomorphic, then A is called a conformal atlas.

Definition (Compatible Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two conformal atlases. If their union $A_1 \cup A_2$ is still a conformal atlas, we say $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are compatible.

Conformal Structure

The compatible relation among conformal atlases is an equivalence relation.

Definition (Conformal Structure)

Suppose S is a topological surface, consider all the conformal atlases on M, classified by the compatible relation

$$\{all\ conformal\ atlas\}/\sim$$

each equivalence class is called a conformal structure.

In other words, each maximal conformal atlas is a conformal structure.

Conformal Structure

Definition (Conformal equivalent metrics)

Suppose g_1, g_2 are two Riemannian metrics on a manifold M, if

$$g_1 = e^{2u} g_2, u : M \to \mathbb{R}$$

then g_1 and g_2 are conformal equivalent.

Definition (Conformal Structure)

Consider all Riemannian metrics on a topological surface S, which are classified by the conformal equivalence relation,

$$\{Riemannian\ metrics\ on\ S\}/\sim,$$

each equivalence class is called a conformal structure.

Relation between Riemannian metric and Conformal Structure

Definition (Isothermal coordinates)

Suppose (S,g) is a metric surface, (U_α,ϕ_α) is a coordinate chart, (x,y) are local parameters, if

$$g = e^{2u}(dx^2+dy^2),$$

then we say (x,y) are isothermal coordinates.

Theorem

Suppose S is a compact metric surface, for each point p, there exits a local coordinate chart (U,ϕ), such that $p\in U$, and the local coordinates are isothermal.

Corollary

For any compact metric surface, there exists a natural conformal structure.

Definition (Riemann surface)

A topological surface with a conformal structure is called a Riemann surface.

Theorem

All oriented compact metric surfaces are Riemann surfaces.

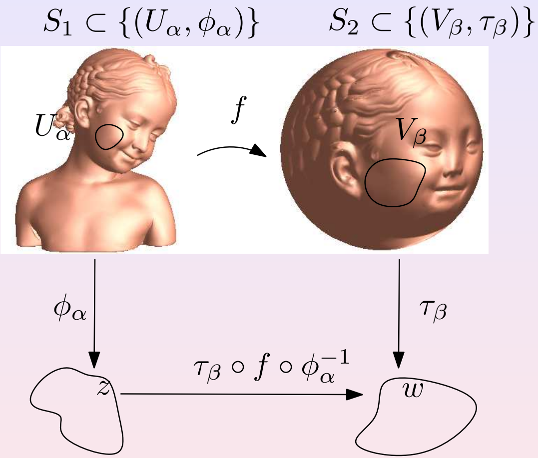
Conformal Mapping

biholomorphic Function

Definition (biholomorphic Function)

Suppose $f : \mathbb{C} \to \mathbb{C}$ is invertible, both f and f^{-1} are holomorphic, then then f is a biholomorphic function.

Conformal Map



The restriction of the mapping on each local chart is biholomorphic, then the mapping is conformal.

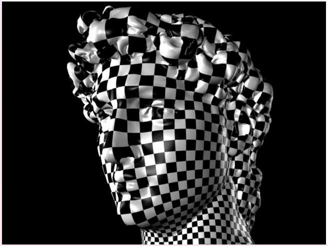
Conformal Mapping

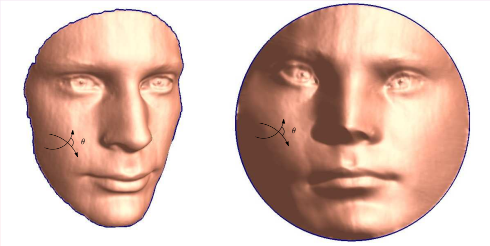
Definition (Conformal Map)

Let $\phi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$ is a homeomorphism, ϕ is conformal if and only if

$$\phi^* \mathbf{g}_2 = e^{2u} \mathbf{g}_1.$$

Conformal Mapping preserves angles.

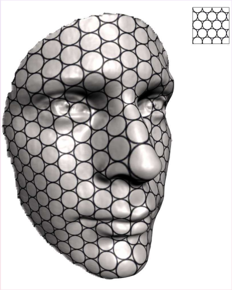




Conformal Mapping

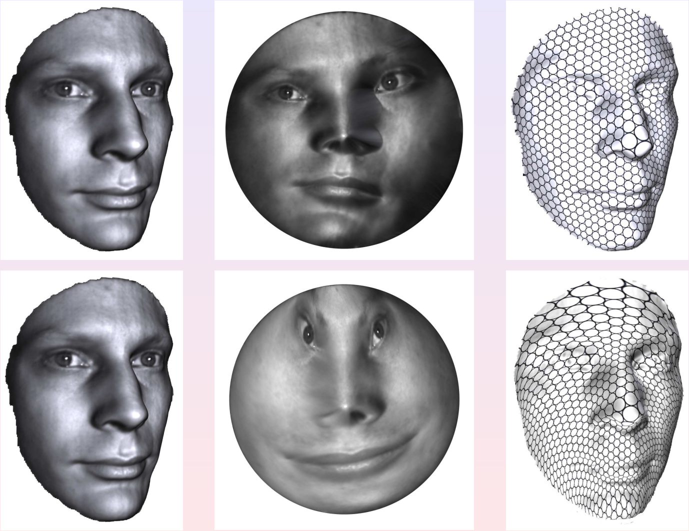
Conformal maps Properties

Map a circle field on the surface to a circle field on the plane.



Quasi-Conformal Map

Diffeomorphisms: maps ellipse field to circle field.

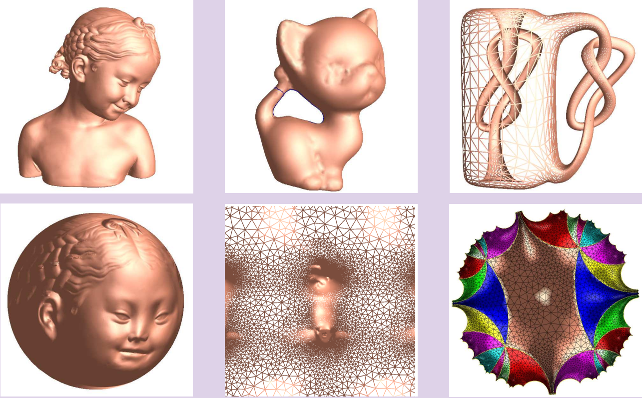


Uniformization

Conformal Canonical Representations

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

Uniformization of Open Surfaces

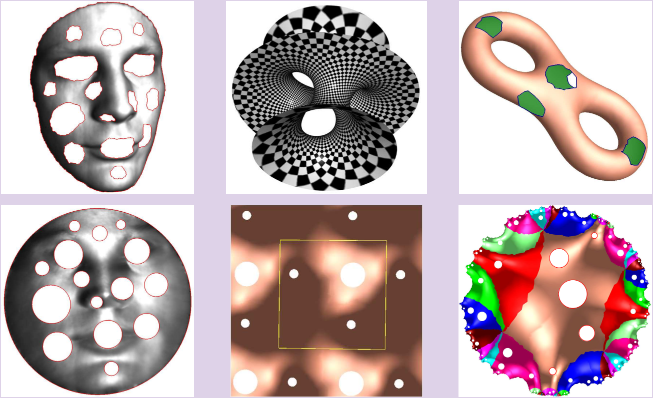
Definition (Circle Domain)

A domain in the Riemann sphere $\hat{\mathbb{C}}$ is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem

Any domain Ω in $\hat{\mathbb{C}}$, whose boundary $\partial\Omega$ has at most countably many components, is conformally homeomorphic to a circle domain Ω^* in $\hat{\mathbb{C}}$. Moreover Ω^* is unique upto Möbius transformations, and every conformal automorphism of Ω^* is the restriction of a Möbius transformation.

Uniformization of Open Surfaces



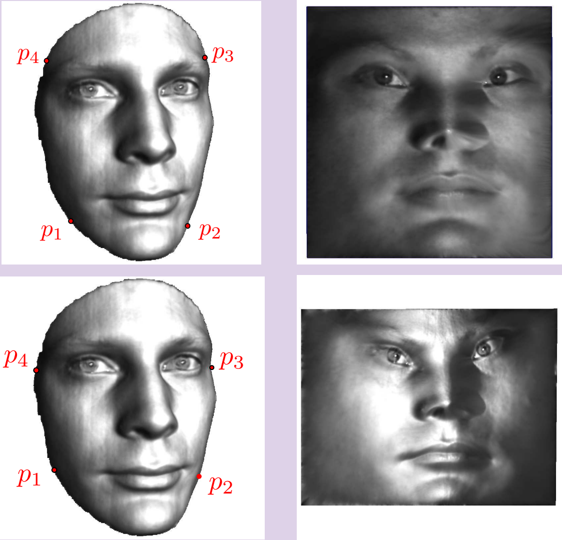
Spherical Euclidean Hyperbolic

Conformal Canonical Representation

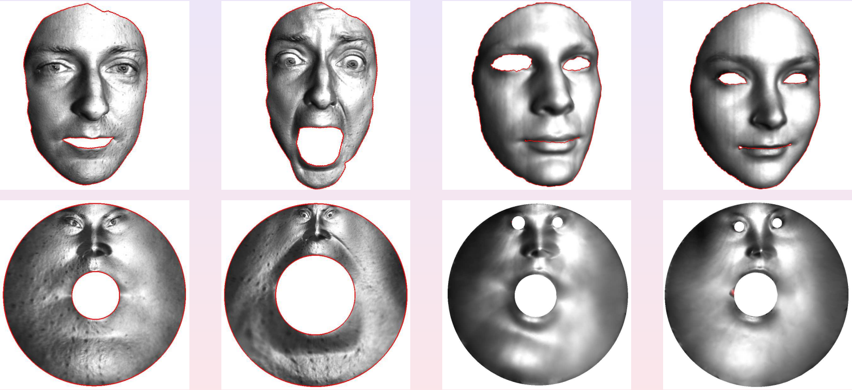
Simply Connected Domains

Conformal Canonical Forms

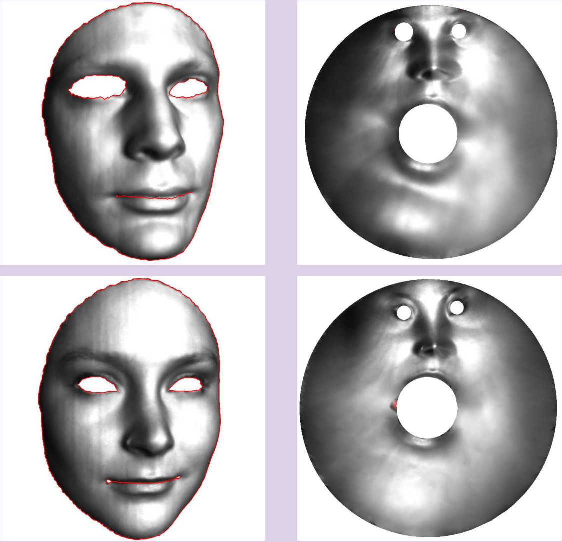
Topological Quadrilateral



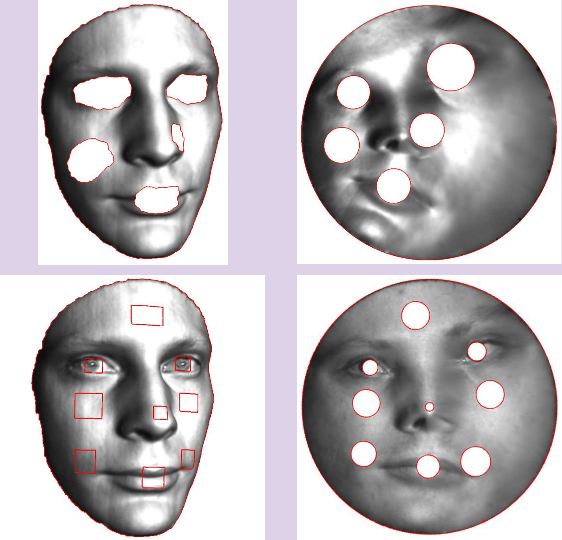
Multiply Connected Domains



Multiply Connected Domains

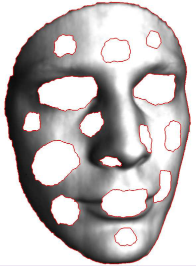
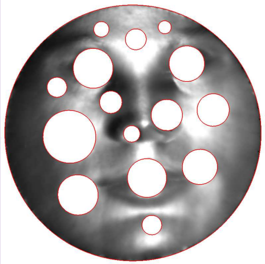
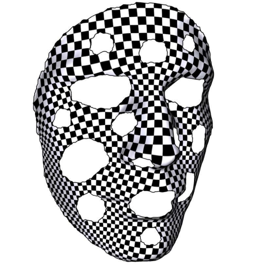
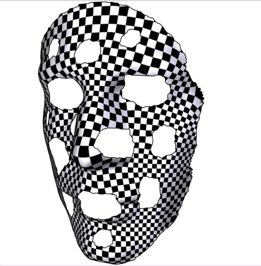


Multiply Connected Domains



Conformal Canonical Forms

Multiply Connected Domains

Conformal Canonical Representations

Definition (Circle Domain in a Riemann Surface)

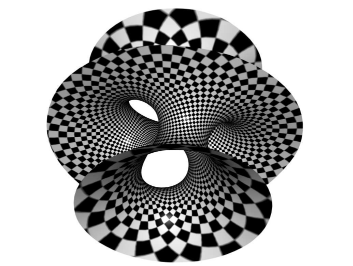
A circle domain in a Riemann surface is a domain, whose complement's connected components are all closed geometric disks and points. Here a geometric disk means a topological disk, whose lifts in the universal cover or the Riemann surface (which is $\mathbb{H}^2$, $\mathbb{R}^2$ or $\mathbb{S}^2$ are round.

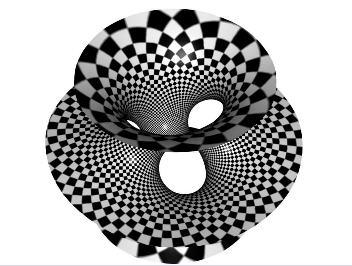
Theorem

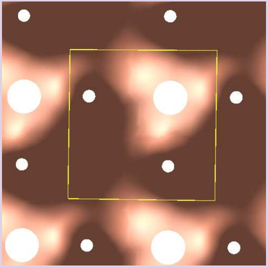
Let Ω be an open Riemann surface with finite genus and at most countably many ends. Then there is a closed Riemann surface R^* such that Ω is conformally homeomorphic to a circle domain Ω^* in R^*. More over, the pair (R^*, Ω^*) is unique up to conformal homeomorphism.

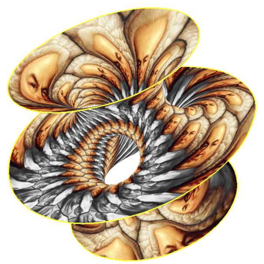
Conformal Canonical Form

Tori with holes



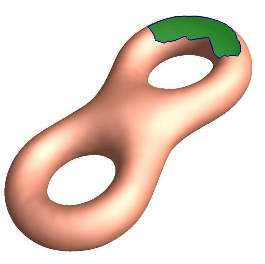


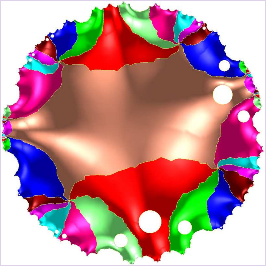




Conformal Canonical Form

High Genus Surface with holes





Teichmüller Space

Definition (Conformal Mapping)

Suppose $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ are two metric surfaces, $\phi : S_1 \to S_2$ is conformal, if on S_1

$$\mathbf{g}_1 = e^{2\lambda} \phi^* \mathbf{g}_2,$$

where $\phi^* \mathbf{g}_2$ is the pull-back metric induced by ϕ.

Definition (Conformal Equivalence)

Suppose two surfaces S_1, S_2 with marked homotopy group generators, $\{a_i, b_i\}$ and $\{\alpha_i, \beta_i\}$. If there exists a conformal map $\phi : S_1 \to S_2$, such that

$$\phi_*[a_i] = [\alpha_i], \phi_*[b_i] = [\beta_i],$$

then we say two marked surfaces are conformal equivalent.

Teichmüller Space

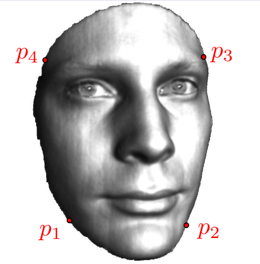
Definition (Teichmüller Space)

Fix the topology of a marked surface S, all conformal equivalence classes sharing the same topology of S, form a manifold, which is called the Teichmüller space of S. Denoted as T_S.

- Each point represents a class of surfaces.
- A path represents a deformation process from one shape to the other.
- The Riemannian metric of Teichmüller space is well defined.

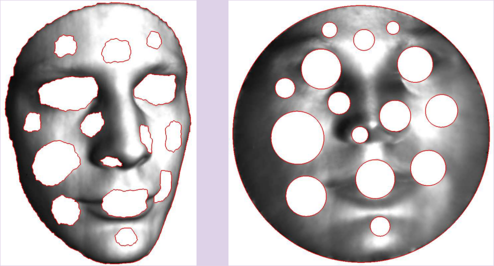
Teichmüller Space

Topological Quadrilateral



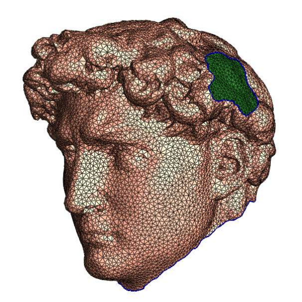
Conformal module: $\frac{h}{w}$. The Teichmüller space is 1 dimensional.

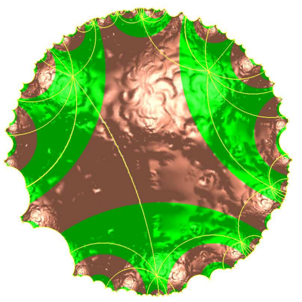
Multiply Connected Domains



Conformal Module : centers and radii, with Möbius ambiguity. The Teichmüller space is $3n-3$ dimensional, n is the number of holes.

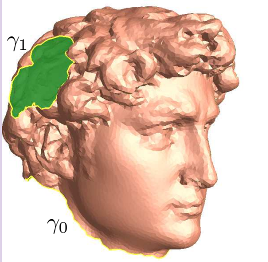
Topological Pants

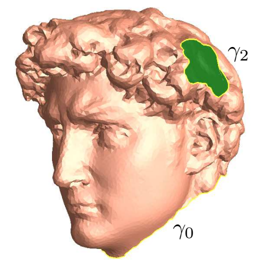


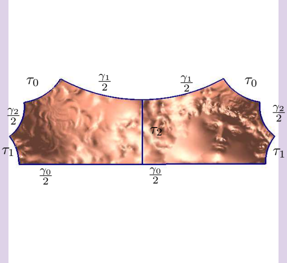


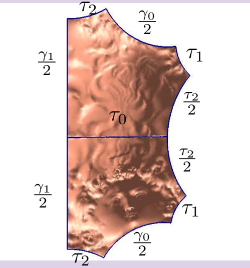
Genus 0 surface with 3 boundaries is conformally mapped to the hyperbolic plane, such that all boundaries become geodesics.

Topological Pants



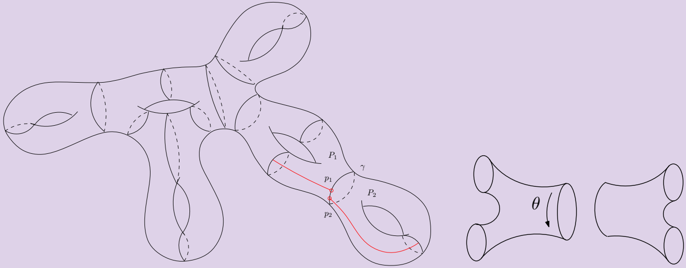






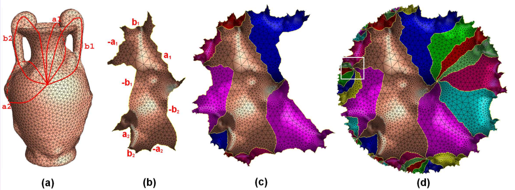
Teichmüller Space

Topological Pants Decomposition - $2g - 2$ pairs of Pants

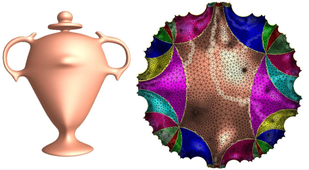


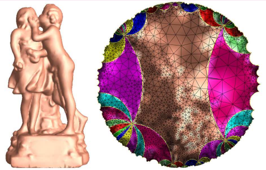
Compute Teichmüller coordinates

Step 1. Compute the hyperbolic uniformization metric.

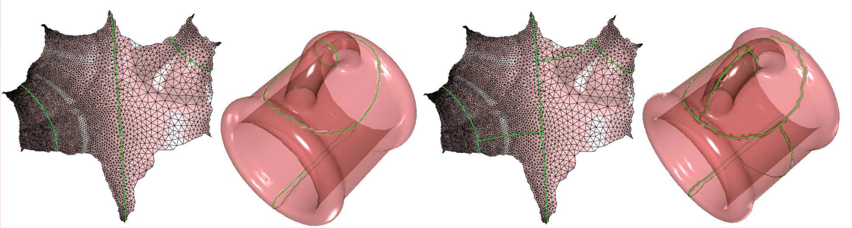


Step 2. Compute the Fuchsian group generators.





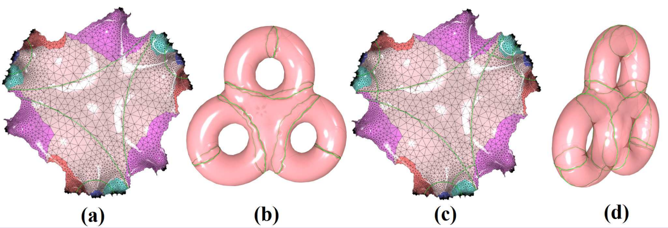
Step 3. Pants decomposition using geodesics and compute the twisting angle.



Teichmüller Coordinates

Compute Teichmüller coordinates

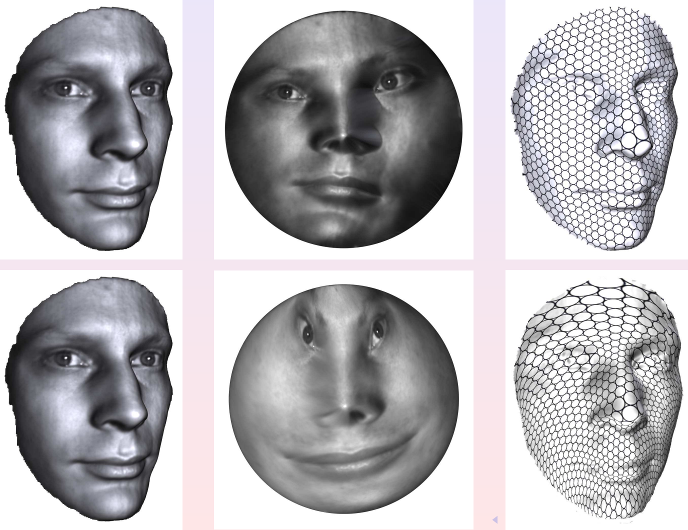
Compute the pants decomposition using geodesics and compute the twisting angle.

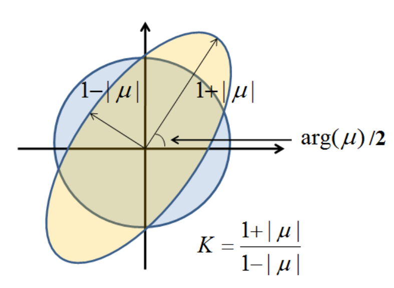


Quasi-Conformal Maps

Quasi-Conformal Map

Most homeomorphisms are quasi-conformal, which maps infinitesimal circles to ellipses.





Beltrami Coefficient

Let $\phi : S_1 \to S_2$ be the map, z, w are isothermal coordinates of S_1, S_2, Beltrami equation is defined as $\|\mu\|_\infty < 1$

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z}$$

Diffeomorphism Space

Theorem

Given two genus zero metric surface with a single boundary,

$$\{Diffeomorphisms\} \cong \frac{\{Beltrami\ Coefficient\}}{\{Mobius\}}.$$

Solving Beltrami Equation

The problem of computing Quasi-conformal map is converted to compute a conformal map.

Solveing Beltrami Equation

Given metric surfaces $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$, let z, w be isothermal coordinates of S_1, S_2,$w = \phi(z)$.

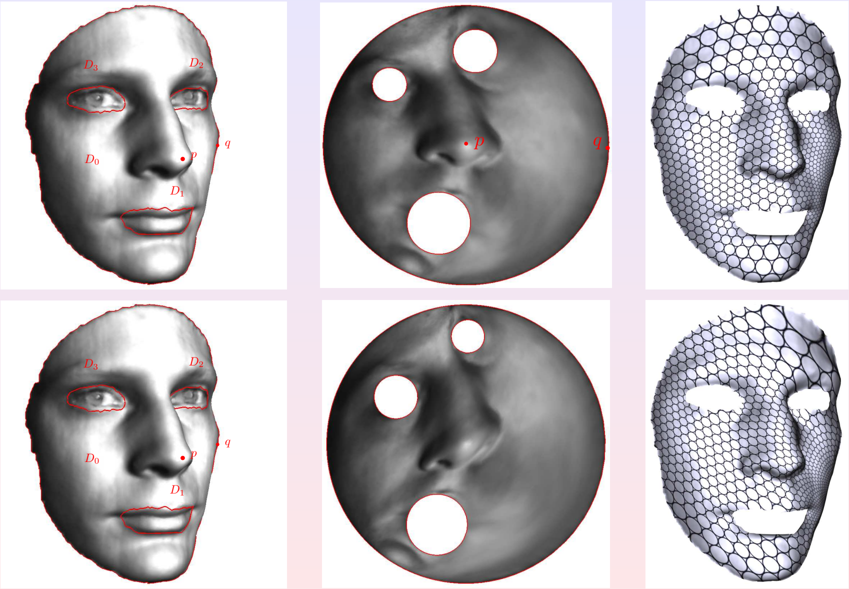
$$\mathbf{g}_1 = e^{2u_1} dz d\bar{z} \quad (1)$$

$$\mathbf{g}_2 = e^{2u_2} dw d\bar{w}, \quad (2)$$

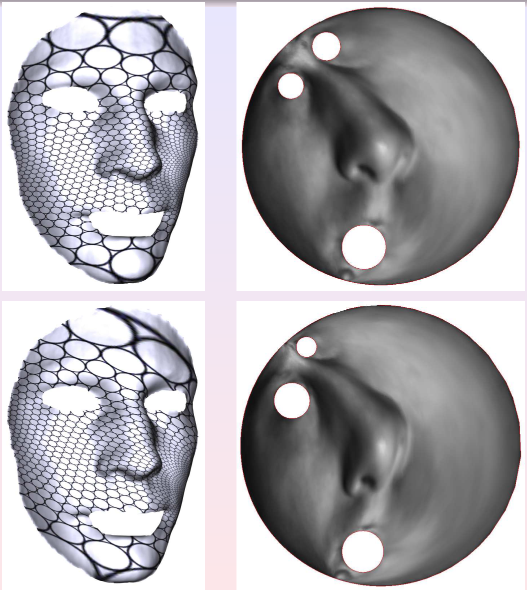
Then

- $\phi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$, quasi-conformal with Beltrami coefficient μ.
- $\phi : (S_1, \phi^* \mathbf{g}_2) \to (S_2, \mathbf{g}_2)$ is isometric
- $\phi^* \mathbf{g}_2 = e^{u_2} |dw|^2 = e^{u_2} |dz + \mu d\bar{z}|^2$.
- $\phi : (S_1, |dz + \mu d\bar{z}|^2) \to (S_2, \mathbf{g}_2)$ is conformal.

Quasi-Conformal Map Examples

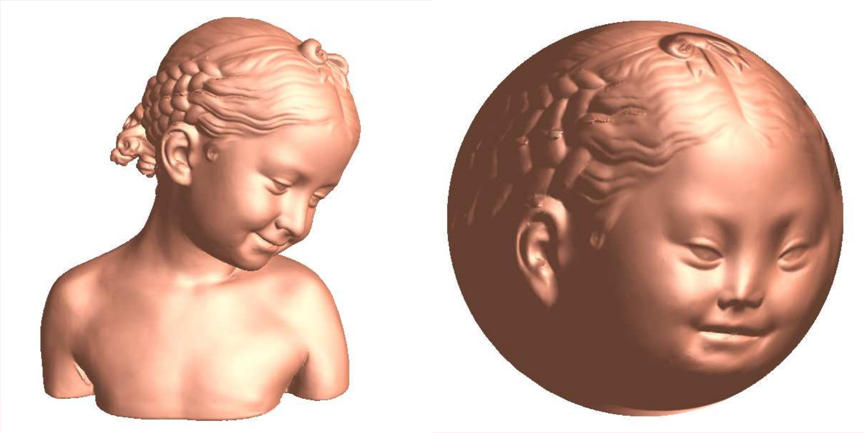


Quasi-Conformal Map Examples



Computational Method - Harmonic Mapping

Spherical harmonic map



Let (M,g) and (N,h) be Riemannian manifolds, $u: M \to N$ is a C^1 mapping.

$$ds_M^2 = \sum g_{\alpha\beta} dx^\alpha dx^\beta, ds_N^2 = \sum h_{ij}(u(x)) du^i du^j.$$

The pull back metric of h induced by u is $u^*(ds_N^2)$ is a symmetric bilinear form

$$u^*(dS_N^2) = \sum_{\alpha,\beta} (\sum_{i,j} h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}) dx^\alpha dx^\beta.$$

The *energy density* of mapping u is defined as

$$|du|^2 = \sum_{i,j,\alpha,\beta} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Energy of the mapping

Equivalently, choose an orthogonal frame field under $u^*(ds_N^2)$, each basis vector field is of unit length under **g**, the dual 1-forms are $\{\omega_1, \omega_2, \cdots, \omega_n\}$, such that

$$u^*(ds_N^2) = \sum_{\alpha=1}^{n} \lambda_\alpha (\omega_\alpha)^2.$$

The the energy density of the mapping u is given by

$$|du|^2 = \sum_{\alpha=1}^{n} \lambda_\alpha.$$

Harmonic Energy and Harmonic Mapping

Definition (Harmonic Energy)

The harmonic energy functional $E(u)$ is defined as

$$E(u) = \int_M |du|^2 dv_M,$$

where $dv_M = (detg)^{\frac{1}{2}}dx$ is the volume element of M.

Definition (Harmonic Mapping)

In the space of mappings, the critical points of $E(u)$ are called harmonic mappings.

Harmonic Energy Conformal Invariant

Suppose u is a mapping from a surface (S,g) to (N,h). Suppose $\tilde{g} = e^{2\lambda}g$ is another metric of S, conformal to g, then

$$|\tilde{d}u|^2 = e^{-2\lambda}|du|^2, \sqrt{det\tilde{g}} = e^{2\lambda}\sqrt{detg},$$

Then $\tilde{g} = g$. Harmonic energy is invariant under conformal metric transformation.

Theorem

Harmonic energy only depends on the conformal structure of the surface, independent of the choice of Riemannian metric.

Suppose N is embedded in $\mathbb{R}^3$, $u : S \to N$ is a harmonic mapping, then

$$\Delta_g u^{T_u N} \equiv 0.$$

where $\Delta_g u = (\Delta_g u_1, \Delta_g u_2, \Delta_g u_3)$. Namely, $\Delta_g u$ is orthogonal to the tangent plane at the target space.

Heat Flow method

Definition (Heat flow)

Let $u : S \to N \subset \mathbb{R}^3$, the heat flow is given by

$$\frac{du(x,t)}{dt} = -(\Delta_g u)^{T_{u(x)}N}$$

The heat flow method will deform a mapping to the harmonic mapping under special normalization conditions.

Heat Flow method

Theorem

Harmonic mapping from a genus zero closed surface to the unit sphere must be a conformal mapping.

Proof.

Let $u : S \to \mathbb{S}^2$. Choose isothermal coordinates of both surfaces, define

$$\phi(z) = \langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \rangle$$

then

$$\phi(z) = \frac{1}{4}(|\frac{\partial u}{\partial x}|^2 - |\frac{\partial u}{\partial y}|^2 - \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle).$$

if $\phi(z) = 0$, then the mapping is conformal.
On the other hand, $\frac{\partial \phi(z)}{\partial \bar{z}} = 0$, then $\phi(z)$ is holomorphic.
$\phi(z)dz^2$ is globally defined, the so-called Hopf differential.
Sphere has no non-zero holomorphic quadratic differentials.

Theorem

The conformal automorphism from a sphere to itself must be a Möbius transformation

$$z \to \frac{az+b}{cz+d}, ad-bc = 1, a, b, c, d \in \mathbb{C}.$$

Rado's theorem

Theorem (Rado)

Let $\Omega \subset \mathbb{R}^2$ is a convex domain with smooth boundary. For any homeomorphism $\phi : S^1 \to \partial\Omega$, there exists a unique harmonic mapping $u : D \to \Omega$, such that $u|_{\partial} D = \phi$, furthermore, u is a diffeomorphism.

Discrete Approximation

We use piecewise linear triangle mesh to approximate the original surface. suppose $u : M \to \mathbb{R}$ the harmonic energy is given by

$$E(u) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} (f(v_i) - f(v_j))^2.$$

The discrete Laplace-Beltrami operator is given by

$$\Delta f(v_i) = \sum_j w_{ij} (f(v_j) - f(v_i)).$$

where w_{ij} is the cotangent formula.

Computational Method - Holomorphic Form

Harmonic 1-form

Each cohomologous class has a unique harmonic 1-form, which represents a vortex free, source-sink free flow field.

Theorem (Hodge)

All the harmonic 1-forms form a group, which is isomorphic to $H^1(M)$.

Theorem (Hodge Decomposition)

$$\Omega^k(M) = Imgd^{k-1} \oplus Img\delta^{k+1} \oplus H^k_\Delta(M).$$

Compute Harmonic 1-forms

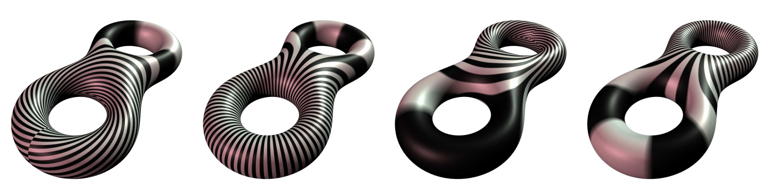
Harmonic 1-form

Let ω be a closed 1-form. Compute a function $f \in C^0(M, \mathbb{R})$, such that

$$\delta^1(\omega + df) = 0,$$

then $\omega + df$ is the unique harmonic 1-form, cohomologous to ω.

Harmonic 1-form Basis



Compute Harmonic 1-forms

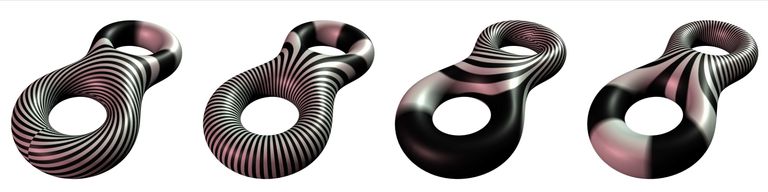
Harmonic 1-form

Let ω be a closed 1-form. Compute a function $f : V \to \mathbb{R}$, such that

$$\sum_j w_{ij}(\omega + df)([v_i, v_j]) = 0, \forall v_i \in V.$$

Then $\omega + df$ is the unique harmonic 1-form, cohomologous to ω.

Harmonic 1-form Basis



Hodge Star Operator

Let $(S,\mathbf{g})$ be a metric surface, $\{e_1,e_2\}$ be an orthonormal frame field, $\{\frac{\partial}{\partial u},\frac{\partial}{\partial v}\}$ be the base vector fields, $\{du,dv\}$ be the dual differential 1-form fields.

$$^*du = dv, ^*dv = -du.$$

Conjugate harmonic 1-forms $\omega+\sqrt{-1}^*\omega$

Hodge Star Operator

If ω is a harmonic 1-form, so is ${}^*\omega$. Suppose $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$ is the set basis of harmonic 1-forms, then ${}^*\omega = \sum_k \lambda_k \omega_k$. Locally, on each triangle ${}^*(adx + bdy) = ady - bdx$. Solve linear system

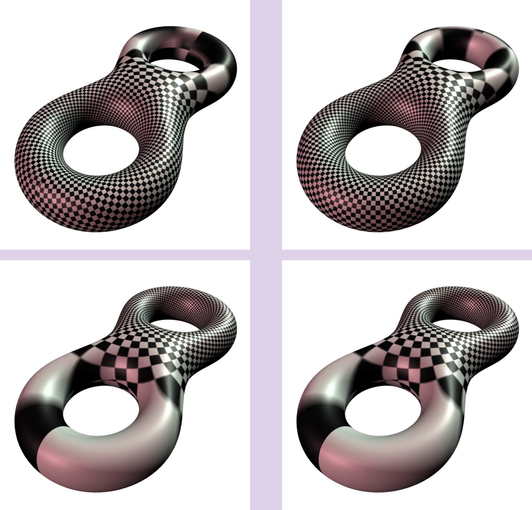
$$\int_M \omega_i \wedge {}^*\omega = \sum_k \lambda_k \int_M \omega_i \wedge \omega_k, i = 1, 2, \cdots, 2g.$$

to solve λ_k's, where ${}^*\omega$ on the left hand side is locally evaluated.

Conjugate harmonic 1-forms $\omega + \sqrt{-1}{}^*\omega$

Holomorphic 1-form

Holomorphic 1-form Basis



Topological Quadrilateral

Topological Quadrilateral

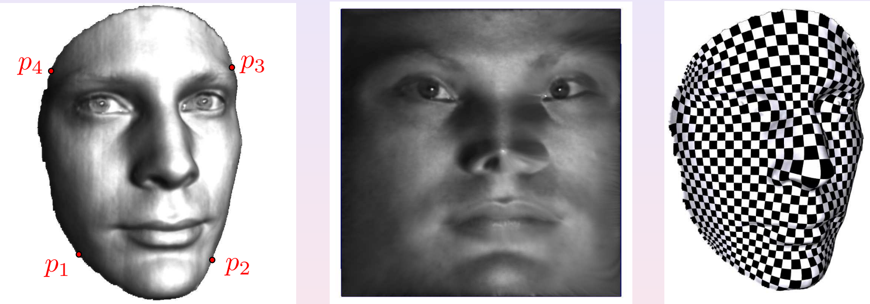


Figure: Topological quadrilateral.

Topological Quadrilateral

Definition (Topological Quadrilateral)

Suppose S is a surface of genus zero with a single boundary, and four marked boundary points $\{p_1, p_2, p_3, p_4\}$ sorted counter-clock-wisely. Then S is called a topological quadrilateral, and denoted as $Q(p1, p2, p3, p4)$.

Theorem

Suppose $Q(p1, p2, p3, p4)$ is a topological quadrilateral with a Riemannian metric $\mathbf{g}$, then there exists a unique conformal map $\phi : S \to \mathbb{C}$, such that ϕ maps Q to a rectangle, $\phi(p_1) = 0$, $\phi(p_2) = 1$. The height of the image rectangle is the conformal module of the surface.

Topological Quadrilateral

Assume the boundary of Q consists of four segments $\partial Q = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, such that

$$\partial\gamma_1 = p_2 - p_1, \partial\gamma_2 = p_3 - p_2, \partial\gamma_3 = p_4 - p_3, \gamma_4 = p_1 - p_4.$$

We compute two harmonic functions $f_1, f_2 \to \mathbb{R}$, such that

$$\left\{\begin{array}{lcl} \Delta f_1 & = & 0 \\ f_1|_{\gamma_1} & = & 0 \\ f_1|_{\gamma_3} & = & 1 \\ \frac{\partial f_1}{\partial \mathbf{n}}|_{\gamma_2\cup\gamma_4} & = & 0 \end{array}\right. \left\{\begin{array}{lcl} \Delta f_2 & = & 0 \\ f_2|_{\gamma_2} & = & 0 \\ f_2|_{\gamma_4} & = & 1 \\ \frac{\partial f_2}{\partial \mathbf{n}}|_{\gamma_1\cup\gamma_3} & = & 0 \end{array}\right.$$

Topological Quadrilateral

The df_1 and df_2 are two exact harmonic 1-forms. We need to find a scalar λ, such that $^*df_1 = \lambda df_2$, this can be achieved by solving the following equation,

$$\int_S df_1 \wedge {}^*df_2 = \lambda \int_S df_1 \wedge df_2.$$

Then the desired holomorphic 1-form $\omega = df_1 + i\lambda df_2$. The conformal mapping is given by

$$\phi(p) = \int_q^p \omega,$$

where q is the base point, the path from q to p is arbitrarily chosen.

Topological Annulus

Topological Annulus

Figure: Topological annulus.

Definition (Topological Annulus)

Suppose S is a surface of genus zero with two boundaries, the S is called a topological annulus.

Theorem

Suppose S is a topological annulus with a Riemannian metric $\mathbf{g}$, *the boundary of S are two loops $\partial S = \gamma_1 - \gamma_2$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical annulus, $\phi(\gamma_1)$ is the unit circle, $\phi(\gamma_2)$ is another concentric circle with radius γ. Then $-\log\gamma$ is the conformal module of S. The mapping ϕ is unique up to a planar rotation.*

Topological Annulus

First, we compute a harmonic function $f : S \to \mathbb{R}$, such that

$$\begin{cases} f|_{\gamma_1} & = & 0 \\ f|_{\gamma_2} & = & 1 \\ \Delta f & = & 0 \end{cases}$$

Then df is an exact harmonic 1-form. Then we compute a harmonic 1-form τ, such that $\int_{\gamma_1} \tau = 1$.

Topological Annulus

Then we compute a constant λ, such that ${}^*df = \lambda\tau$, by solving the following equation,

$$\int_S df \wedge {}^*df = \lambda \int_S df \wedge \tau.$$

Then $\omega = df + i\lambda\tau$ is a holomorphic 1-form. Let $Img(\int_{\gamma_1} \omega) = k$. The conformal mapping is given by

$$\phi(p) = exp^{\frac{2\pi}{k}\int_q^p \omega}.$$

Topological Annulus

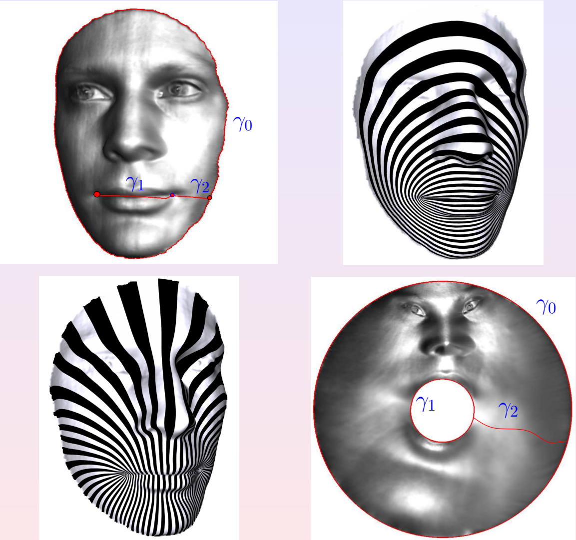


Figure: Topological annulus.

Riemann Mapping

Conformal Module

Simply Connected Domains

Topological Disk

Definition (Topological Disk)

Suppose S is a surface of genus zero with one boundary, the S is called a topological disk.

Theorem

Suppose S is a topological disk with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical disk. The mapping ϕ is unique up to a Möbius transformation,

$$z \to e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Topological Disk

Punch a small hole in the disk, then use the algorithm for topological annulus to compute the conformal mapping. The punched hole will be mapped to the center.

Multiply connected domains

Multiply-Connected Annulus

Definition (Multiply-Connected Annulus)

Suppose S is a surface of genus zero with multiple boundaries, then S is called a multiply connected annulus.

Theorem

Suppose S is a multiply connected annulus with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the unit disk with circular holes. The radii and the centers of the inner circles are the conformal module of S. Such kind of conformal mapping are unique up to Möbius transformations.

Conformal Slit Mapping

Figure: Harmonic forms and holomorphic forms.

Slit Mapping

Suppose there are $n+1$ boundary components$\{\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n\}$. $\{\omega_1, \omega_2, \cdots, \omega_n\}$ are the holomorphic 1-form basis. Choose two boundary components, γ_0, γ_1, solve linear equation $\omega = \sum_{k=1}^{n} \lambda_k \omega_k$,

$$img(\int_{\gamma_0} \omega) = 2\pi, img(\int_{\gamma_1} \omega) = -2\pi, img(\int_{\gamma_k} \omega) = 0, 2 \leq k \leq n.$$

Then the mapping is given by

$$p \to \exp \int_q^p \omega,$$

where q is the base point on the surface.

Conformal Circular Slit Mapping

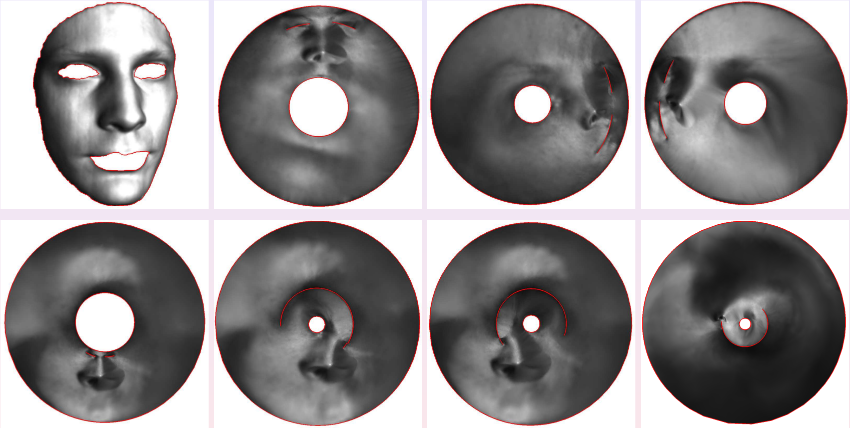


Figure: Conformal circular slit mapping.

Hole Filling

Adding sample points in the center hole, use Delaunay triangulation to fill in with boundary constraints.

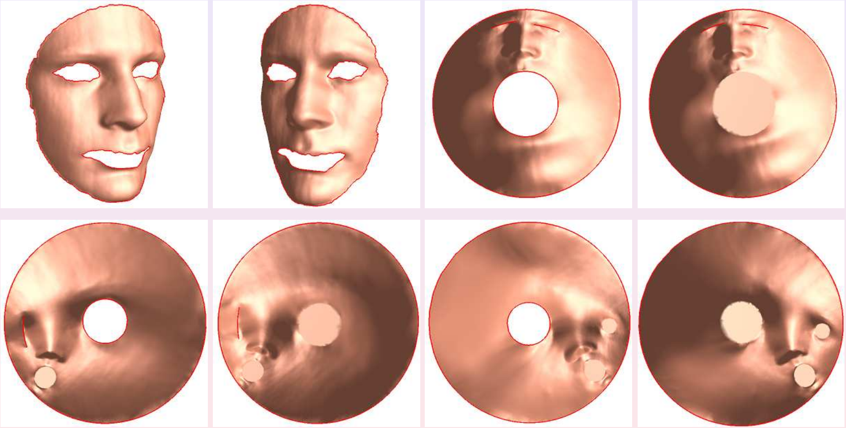


Figure: Fill interior holes.

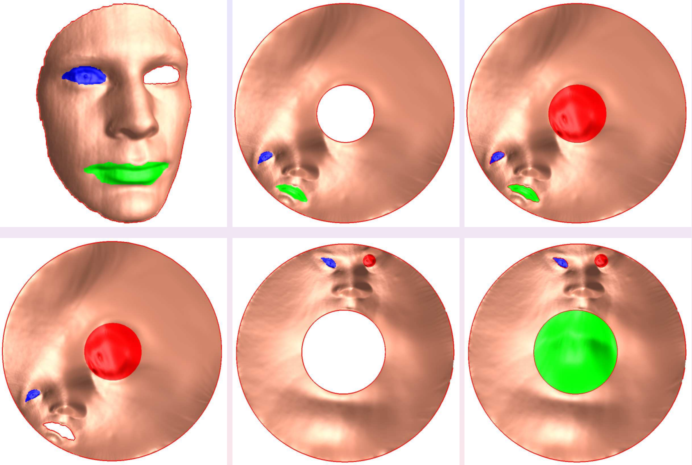


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - II

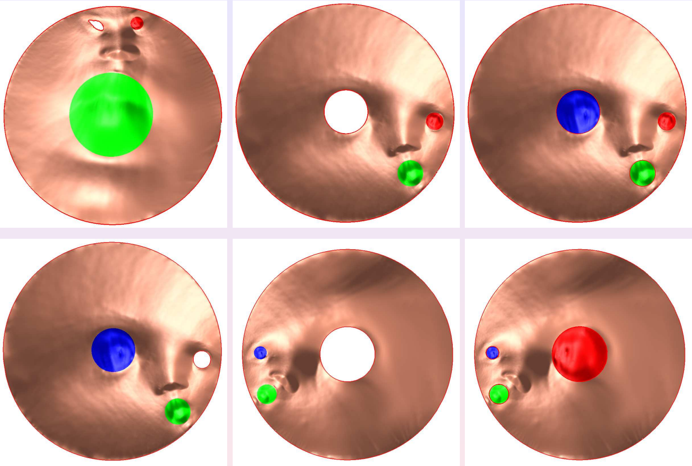


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - III

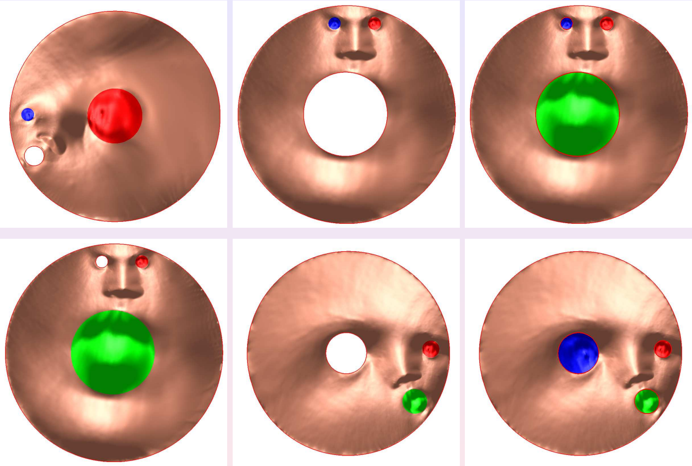


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Convergence Analysis

Theorem (Gu and Luo 2009)

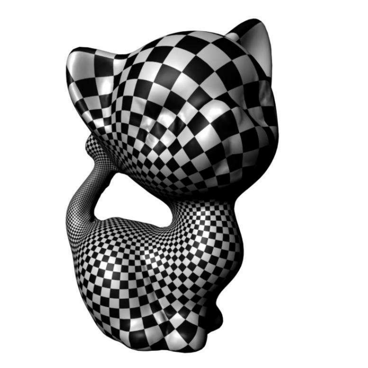
Suppose genus zero surface has n boundaries, then there exists constants $C_1 > 0$ and $0 < C_2 < 1$, for step k, for all $z \in \mathbb{C}$,

$$|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2[\frac{k}{n}]},$$

where f is the desired conformal mapping.

Topological Torus

Topological torus



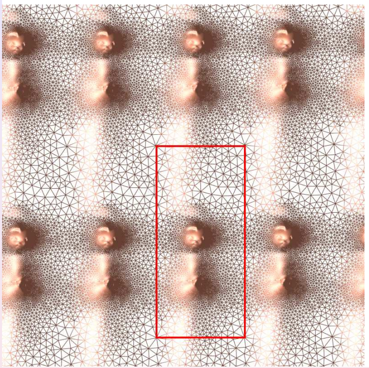


Figure: Genus one closed surface.

Topological Torus

1. We compute a basis for the fundamental group $\pi_1(S)$, $\{\gamma_1, \gamma_2\}$.
2. Compute the holomorophic 1-form basis ω_1, ω_2, such that $\int_{\gamma_i} \omega_j = \delta_{ij}$.
3. Slice the surface along γ_1, γ_2 to get a fundamental domain $\tilde{S}$,
4. The conformal mapping $\phi : \tilde{S} \to \mathbb{C}$ is given by

$$\phi(p) = \int_q^p \omega_1,$$

where q is the base point, the path from q to p in $\tilde{S}$ can be arbitrarily chosen.

Topological Torus

Suppose $a + ib = \int_{\gamma_2} \omega_1$, then $a + ib$ is the conformal module of the torus. The deck transformation group generators are

$$T_1(z) = z + 1, T_2(z) = z + a + ib.$$

By using all deck transformations to translate $\phi(\tilde{S})$, we can conformally map the universal covering space of S onto the whole complex plane $\mathbb{C}$, each fundamental domain is a parallelogram.

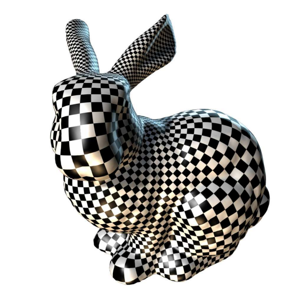
Computational Method - Surface Ricci Flow

Discrete Curvature Flow

Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

The Gaussian curvature is given by

$$K(u,v) = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda(u,v)}}\Delta\lambda(u,v),$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Conformal Metric Deformation

Definition

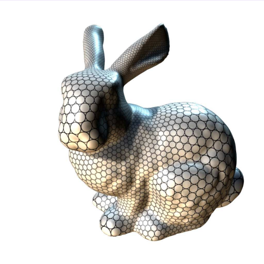
Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Yamabe Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

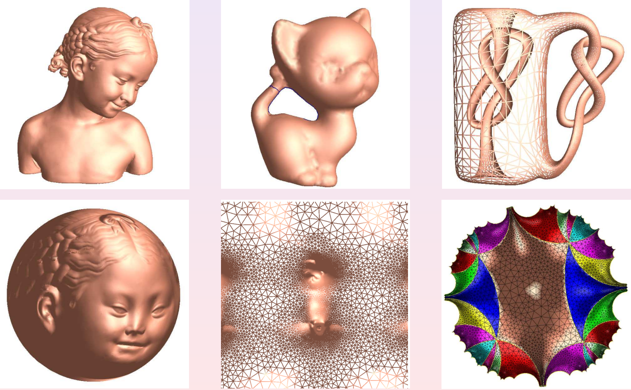
geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(-\partial_n\lambda + k_g).$$

Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical | Euclidean | Hyperbolic

Surface Ricci Flow

Definition (Hamilton's Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -2Kg_{ij}.$$

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant every where.

Key Idea

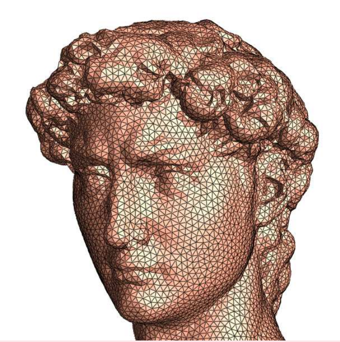
$K = -\Delta_{\mathbf{g}}\lambda$, Roughly speaking, $\frac{dK}{dt} = \Delta_{\mathbf{g}}\frac{d\lambda}{dt}$. Let $\frac{d\lambda}{dt} = -K$, then

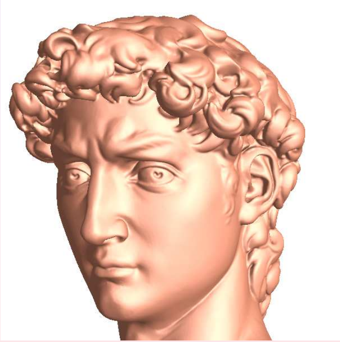
$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K + 2K^2$$

Heat equation!

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.



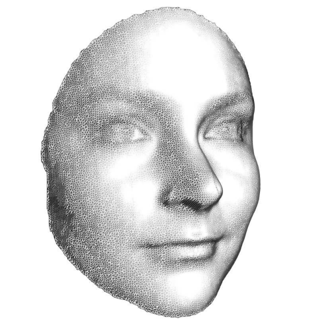


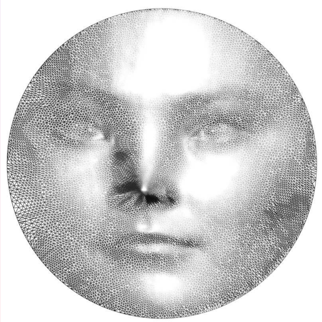
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

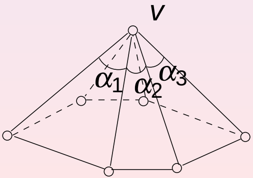
Definition (Discrete Curvature)

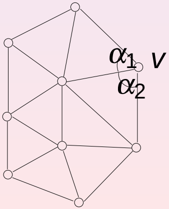
Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

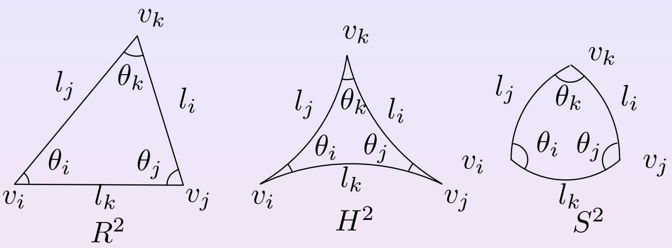
Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$





Discrete Metrics Determines the Curvatures



Angle and edge length relations: cosine laws $\mathbb{R}^2, \mathbb{H}^2, \mathbb{S}^2$

$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \tag{3}$$

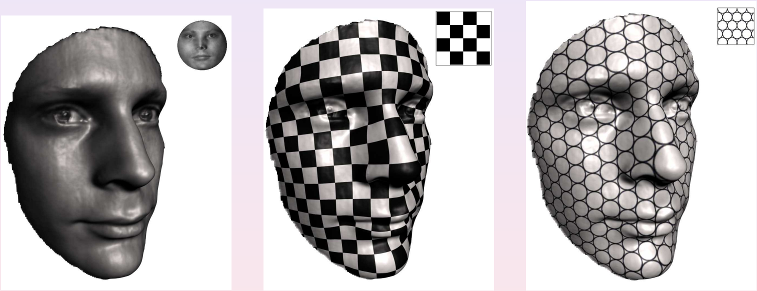
$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} \tag{4}$$

$$1 = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \tag{5}$$

Discrete Conformal Metric Deformation

Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

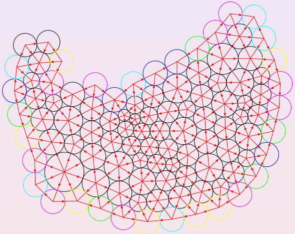


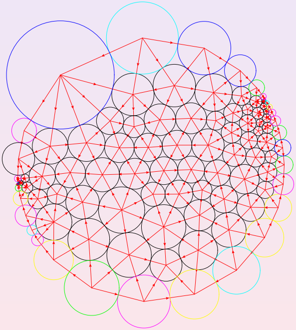
Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

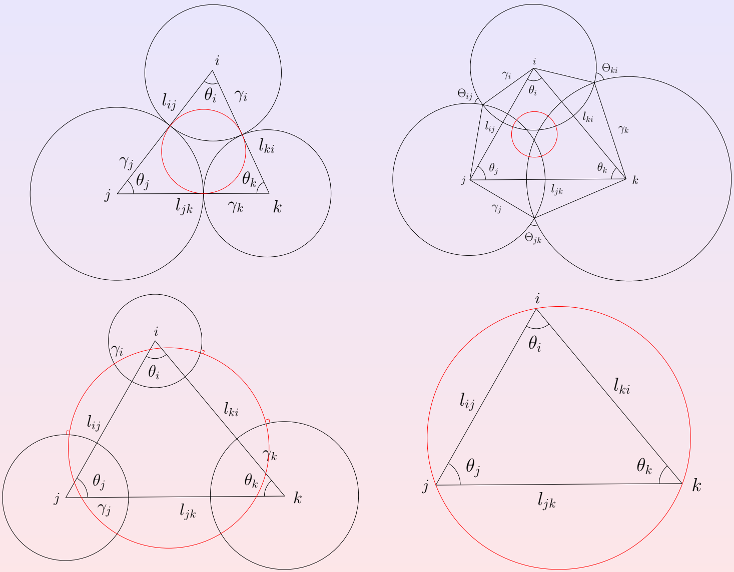
Circle Patterns

There are many local settings for circle patterns. The radius is variable, the intersection angles do not change.

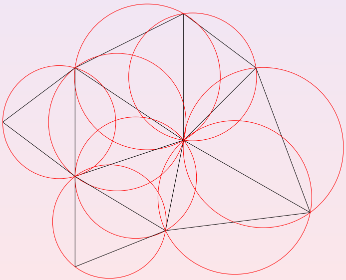


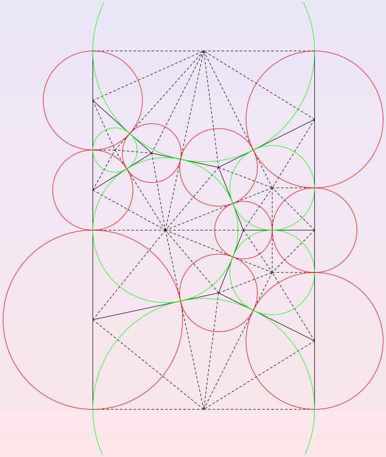


Circle Patterns



Circle Patterns





Discrete Approach

Circle Packing

- Thurston introduced circle packing metric for studying 3-manifolds in 1978.
- Sullivan and Rodin proved Thurston's circle packing conjecture in 1987.
- The first variational principle for CP metrics was founded by Colin de Verdiere (1991).
- Zheng-Xu He and O. Schramm proved the classical Riemannian mapping theorem in 1994.
- Chow and Luo built the connection between Ricci flow and circle packing in 2003.
- Springborn, Bobenko and Schroder's circle pattern in 2005.

Circle Packing Metric

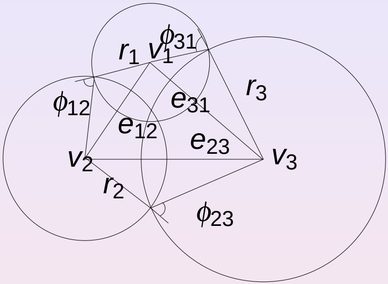
CP Metric

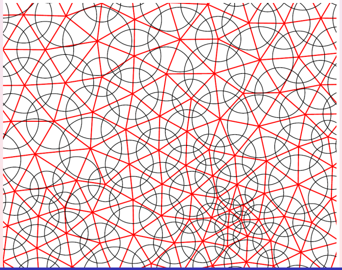
We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$





Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u} : V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation

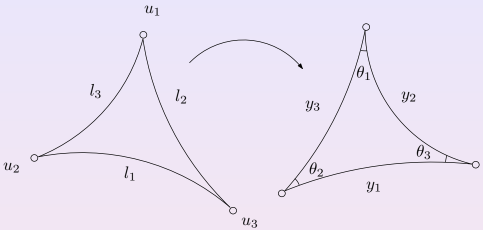
$$d\mathbf{K} = \Delta d\mathbf{u},$$

Δ is a discrete Lapalce-Beltrami operator.

- Feng Luo, *Combinatorial Yamabe Flow on Surfaces*, Commun.Contemp.Math., Vol.6 Num. 5, Pages 765-780, **2004**.
- Boris, Springborn and Peter Schröder and Ulrich Pinkall, *Conformal equivalence of triangle meshes*, ACM Trans. Graph., vol.27 Num. 3, pages 1-11, **2008**.

Discrete Conformal Factor for Yamabe Flow

Discrete conformal metric deformation:



conformal factor

$$
\begin{array}{rcll}
\frac{y_k}{2} & = & e^{u_i}\frac{l_k}{2}e^{u_j} & \mathbb{R}^2 \\
\sinh\frac{y_k}{2} & = & e^{u_i}\sinh\frac{l_k}{2}e^{u_j} & \mathbb{H}^2 \\
\sin\frac{y_k}{2} & = & e^{u_i}\sin\frac{l_k}{2}e^{u_j} & \mathbb{S}^2
\end{array}
$$

Properties: $\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$ and $d\mathbf{K} = \Delta d\mathbf{u}$.

Unified Framework of Discrete Curvature Flow

Unified framework for both Discrete Ricci flow and Yamabe flow

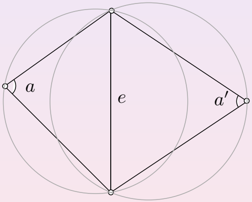
- Curvature flow
$$\frac{du}{dt} = \bar{K} - K,$$
- Energy
$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$
- Hessian of E denoted as Δ,
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

One Example: Discrete Yamabe Flow

Definition (Delaunay Triangulation)

Each PL metric d on (S,V) has a Delaunay triangulation T, such that for each edge e of T,

$$a + a' \leq \pi,$$

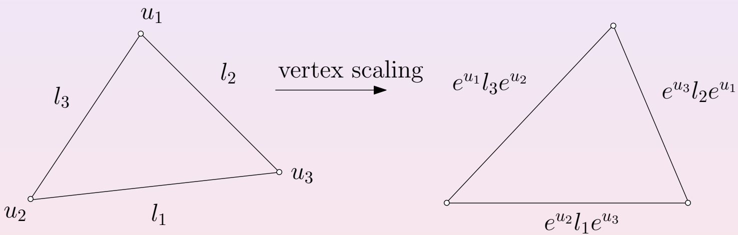


It is the dual of Voronoi decomposition of (S,V,d)

$$R(v_i) = \{x | d(x,v_i) \leq d(x,v_j) \text{ for all } v_j\}$$

Definition (Conformal change)

Conformal factor $u : V \to \mathbb{R}$. Discrete conformal change is vertex scaling.



proposed by physicists Rocek and Williams in 1984 in the Lorenzian setting. Luo discovered a variational principle associated to it in 2004.

Discrete Yamabe Flow

Definition (Discrete Yamabe Flow)

$$\frac{du(v_i)}{dt} = \bar{K}(v_i) - K(v_i)$$

Theorem (Luo)

The discrete Yamabe flow converge exponentially fast, $\exists c_1, c_2 > 0$, such that

$$|u_i(t) - u_i(\infty)| < c_1 e^{-c_2 t}, |K_i(t) - K_i(\infty)| < c_1 e^{-c_2 t},$$

Discrete Conformality

Definition (Discrete Conformal Equivalence)

PL metrics d, d' on (S, V) are discrete conformal,

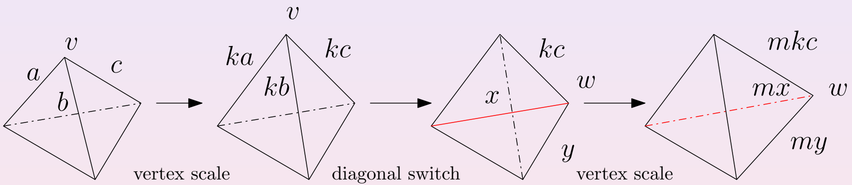
$$d \sim d'$$

if there is a sequence $d = d_1, d_2, \cdots, d_k = d'$ and $T_1, T_2, \cdots, T_k$ on (S, V), such that

1. T_i is Delaunay in d_i
2. if $T_i \neq T_{i+1}$, then $(S, d_i) \cong (S, d_{i+1})$ by an isometry homotopic to *id*
3. if $T_i = T_{i+1}$, $\exists u : V \to \mathbb{R}$, such that $\forall$ edge $e = [v_i, v_j]$,

$$l_{d_{i+1}}(e) = e^{u(v_i)} l_{d_i} e^{u(v_j)}$$

Discrete conformal metrics



Main Theorem

Theorem (Gu-Luo-Sun-Wu (2013))

$\forall$ PL metrics d on closed (S,V) and $\forall \bar{K}: V \to (-\infty, 2\pi)$, such that $\sum \bar{K}(v) = 2\pi\chi(S)$, $\exists$ a PL metric $\bar{d}$, unique up to scaling on (S,V), such that

1. *$\bar{d}$ is discrete conformal to d*
2. *The discrete curvature of $\bar{d}$ is $\bar{K}$.*

Furthermore, $\bar{d}$ can be found from d from a discrete curvature flow.

Remark

$\bar{K} = \frac{2\pi\chi(S)}{|V|}$, discrete uniformization.

Main Theorem

Theorem (Gu-Luo-Wu (2016))

Discrete conformal mapping produced by dynamic Yambe flow converges to the smooth conformal mapping.

Main Theorem

1. The uniqueness of the solution is obtained by the convexity of discrete surface Ricci energy and the convexity of the admissible conformal factor space (u-space).
2. The existence is given by the equivalence between PL metrics on (S,V) and the decorated hyperbolic metrics on (S,V) and the Ptolemy identity.

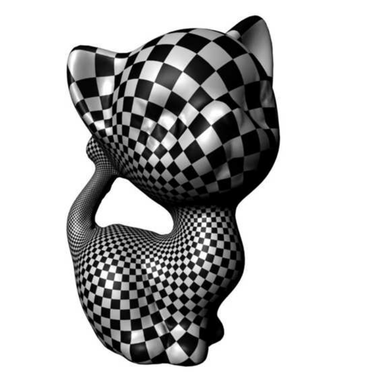
X. Gu, F. Luo, J. Sun, T. Wu, ”A discrete uniformization theorem for polyhedral surfaces”, arXiv:1309.4175.

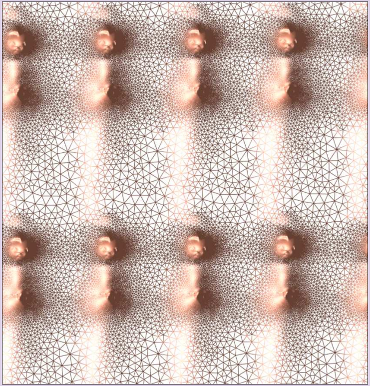
Algorithm

Input: a closed triangle mesh M, target curvature $\bar{K}$, step length δ, threshold ε
Output:a PL metric conformal to the original metric, realizing $\bar{K}$.

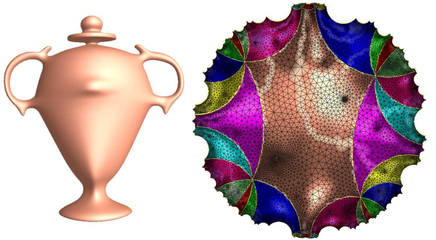
1. Initialize $u_i = 0, \forall v_i \in V$.
2. compute edge length, corner angle, discrete curvature K_i
3. update to Delaunay triangulation by edge swap
4. compute edge weight w_{ij}.
5. $\mathbf{u} += \delta \Delta^{-1}(\bar{\mathbf{K}} - \mathbf{K})$
6. normalize $\mathbf{u}$ such that the mean of u_i's is 0.
7. repeat step 2 through 6, until the max $|\bar{K}_i - K_i| < \varepsilon$.

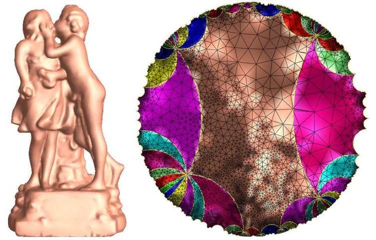
Genus One Example



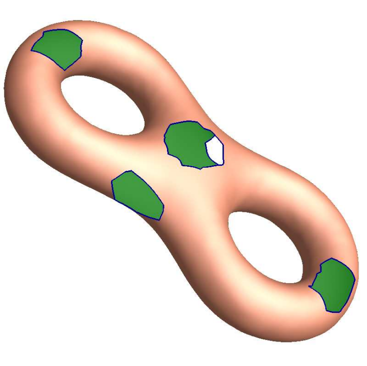


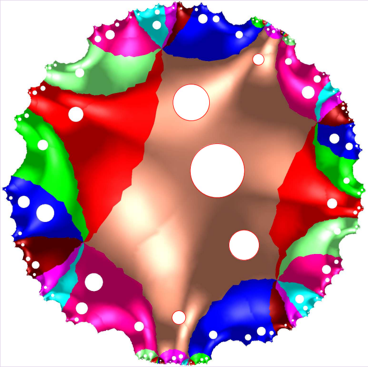
Computational results for genus 2 and genus 3 surfaces.





Hyperbolic Koebe's Method





Applications

Computer Graphics

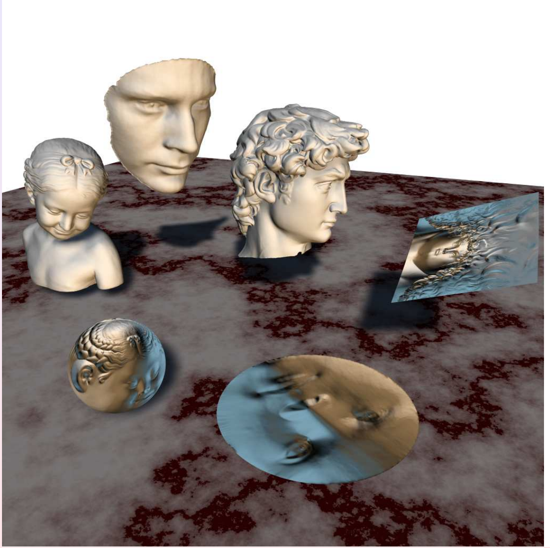
Computer Graphics Application

Graphics

- Surface Parameterization, texture mapping
- Texture synthesis, transfer
- Vector field design
- Shape space and retrieval.

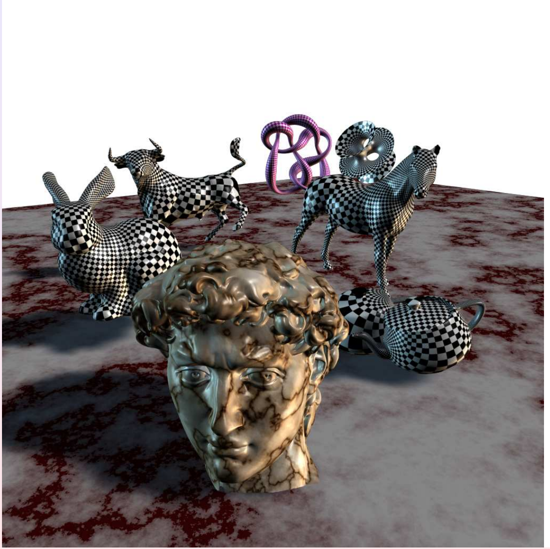
Surface Parameterization

Map the surfaces onto canonical parameter domains



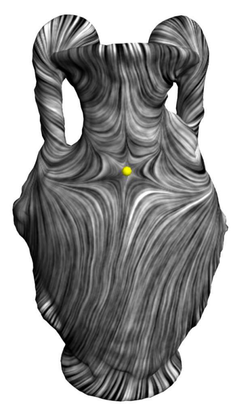
Surface Parameterization

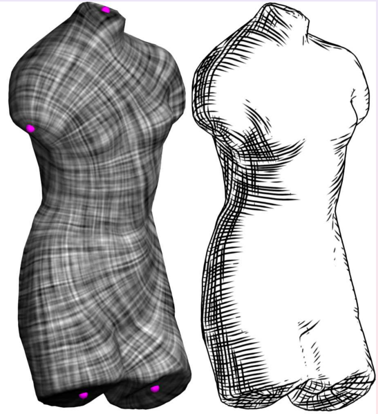
Applied for texture mapping.



n-Rosy Field Design

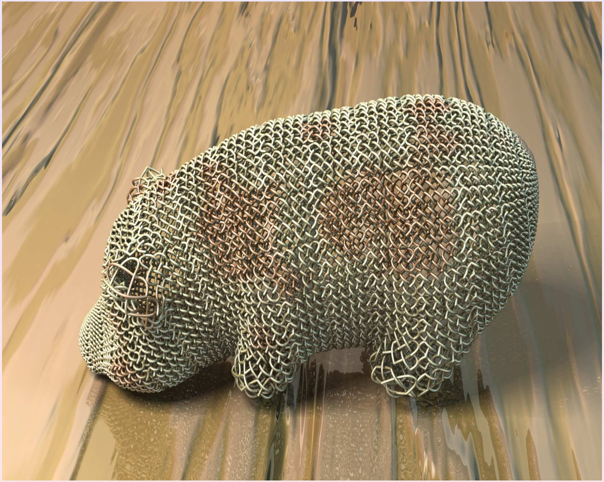
Design vector fields on surfaces with prescribed singularity positions and indices.





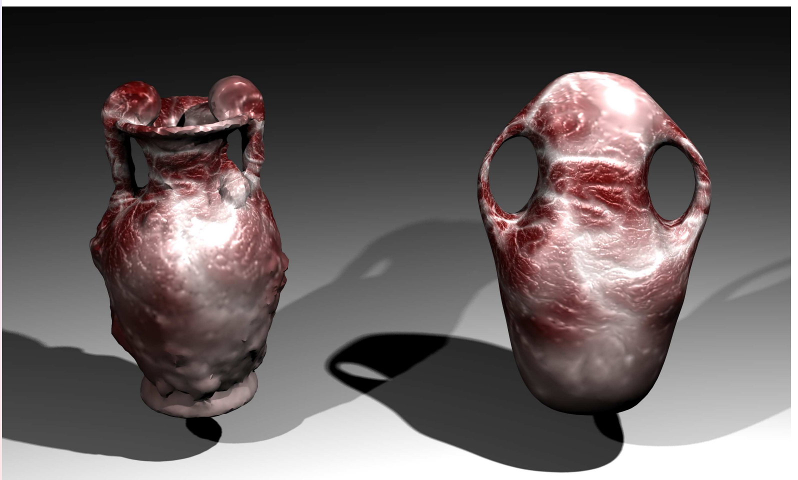
n-Rosy Field Design

Convert the surface to knot structure using smooth vector fields.



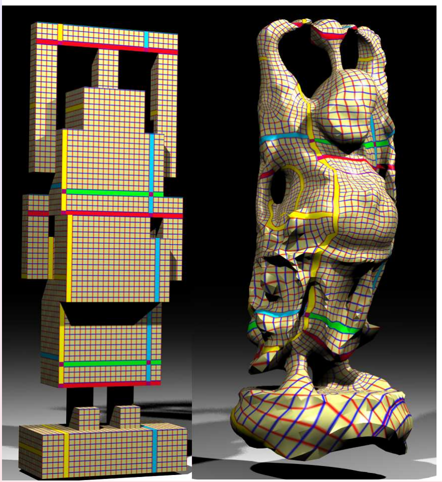
Texture Transfer

Transfer the texture between high genus surfaces.



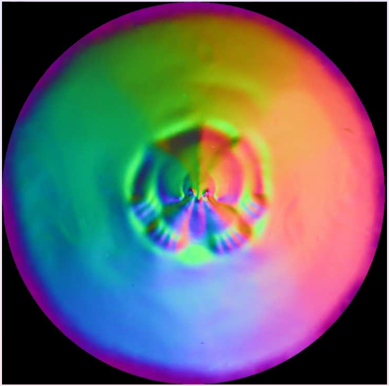
Polycube Map

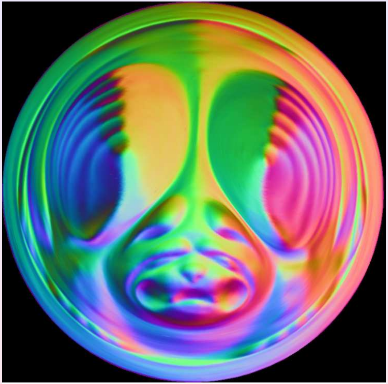
Compute polycube maps for high genus surfaces.



Normal Map

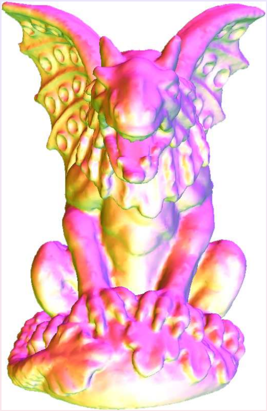
Normal Map

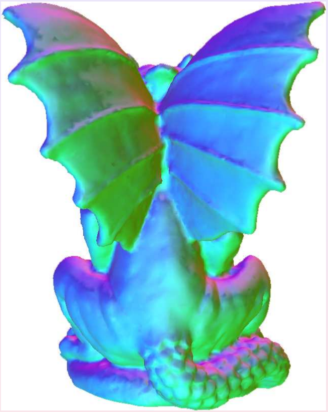




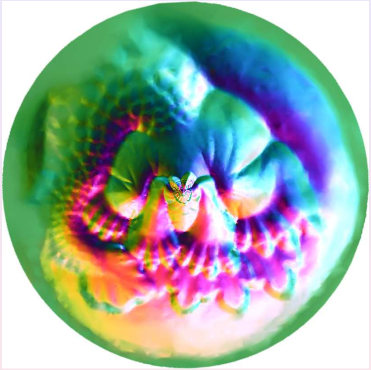
- Quasi-conformal geometry controls angle-distortion;
- Optimal mass transportation map controls area-distortion;

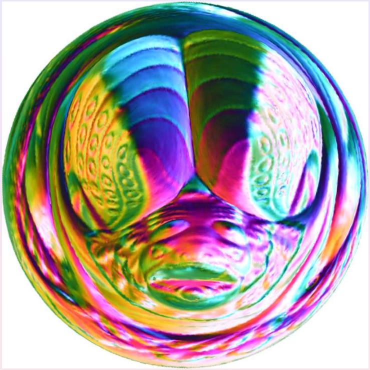
Normal Map





Normal Map





Volumetric Parameterization

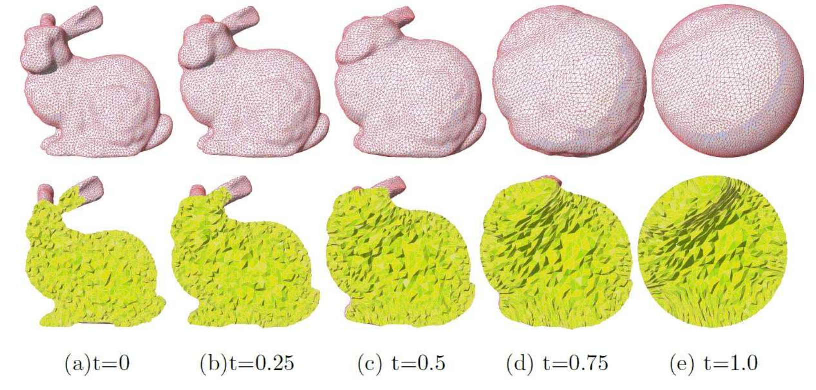


Figure: Volumetric morphing using our method.

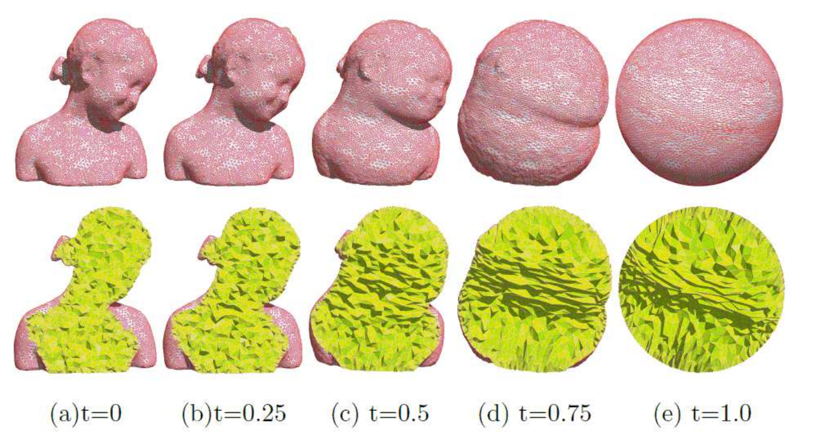


Figure: Volumetric morphing using our method.

Computer Vision

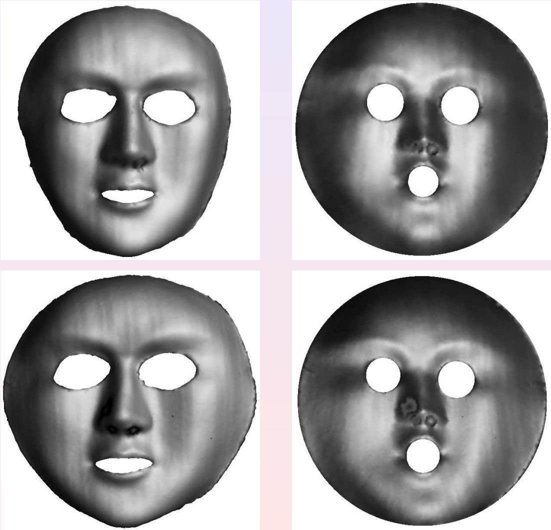
Computer Vision Application

Vision

- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Tracking.

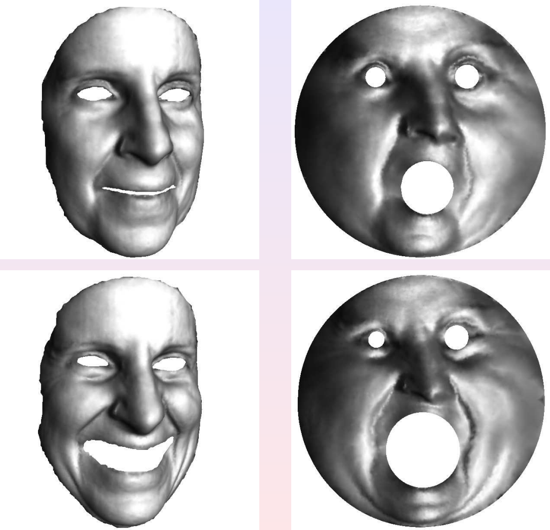
Surface Matching

Isometric deformation is conformal. The mask is bent without stretching.



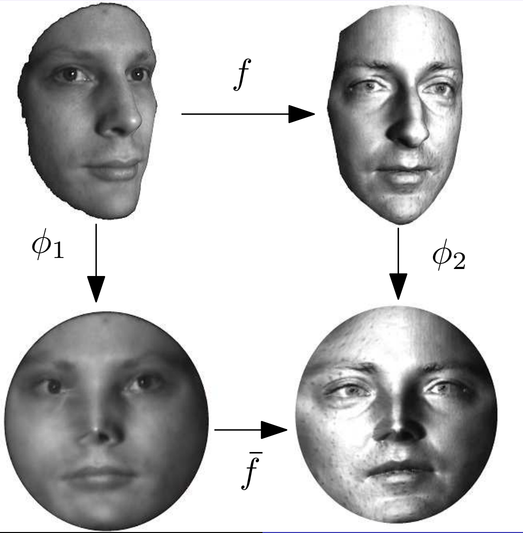
Surface Matching

Facial expression change is not-conformal.

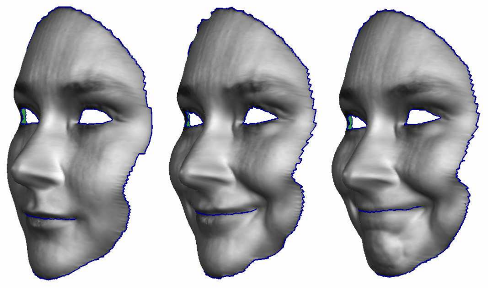


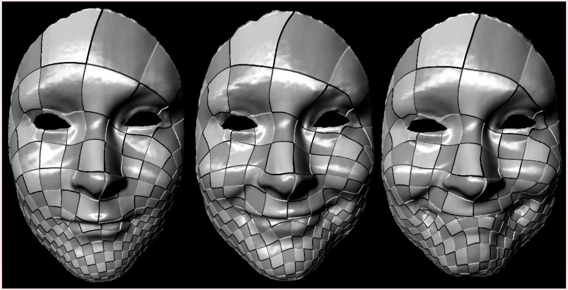
Surface Matching

3D surface matching is converted to image matching by using conformal mappings.

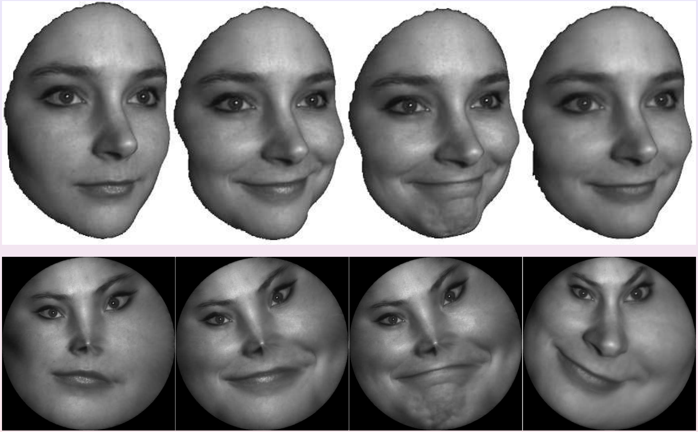


Face Surfaces with Different Expressions are Matched

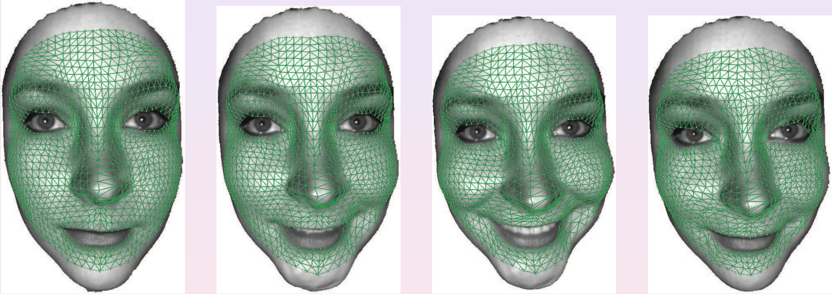




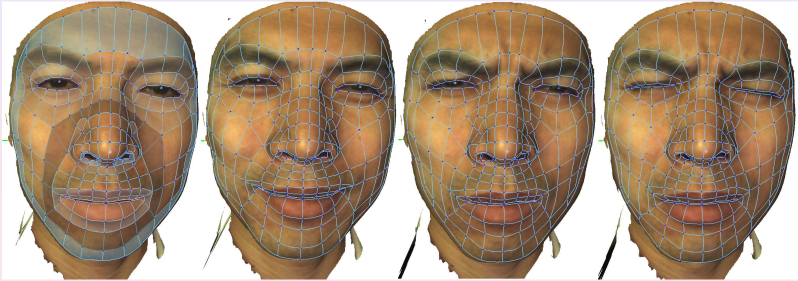
Face Expression Tracking



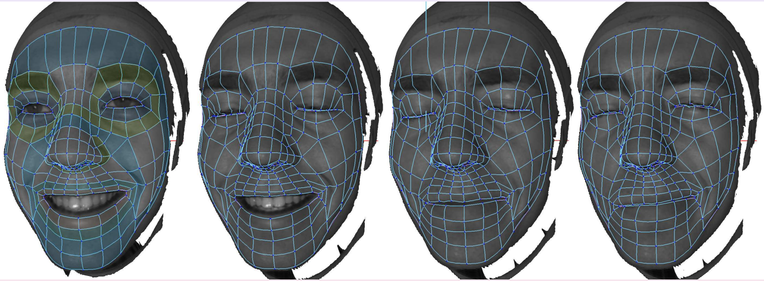
Face Expression Tracking



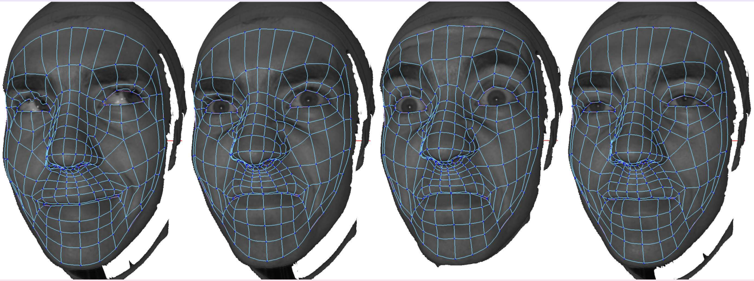
Single Mesh for facial expression transfer.



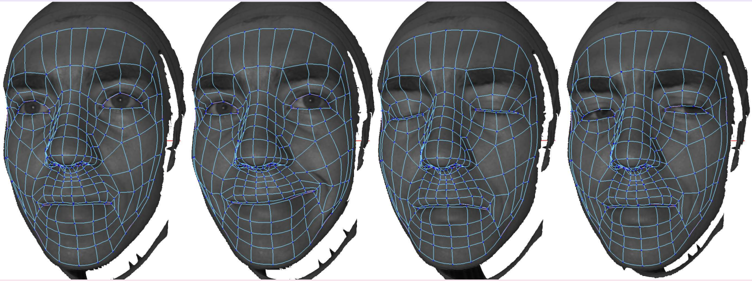
Face Expression Tracking



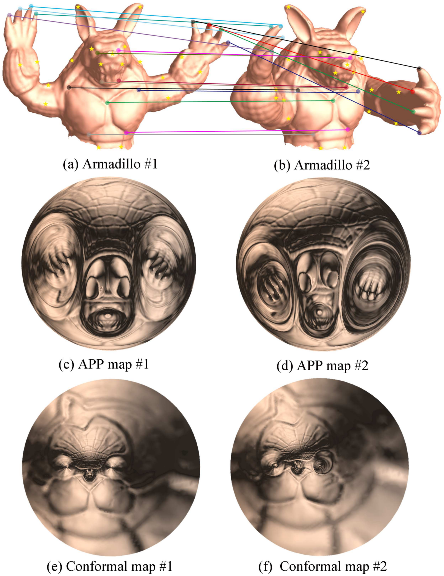
Face Expression Tracking



Face Expression Tracking



Surface Registration



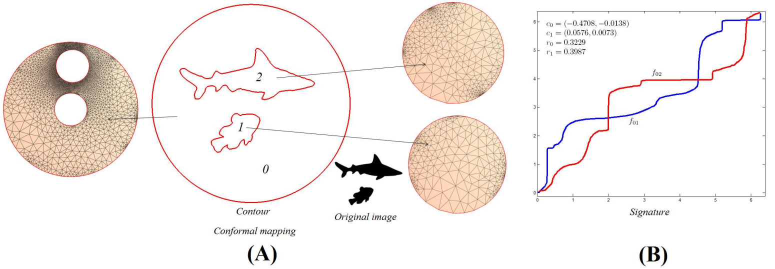
(a) Armadillo #1 (b) Armadillo #2

(c) APP map #1 (d) APP map #2

(e) Conformal map #1 (f) Conformal map #2

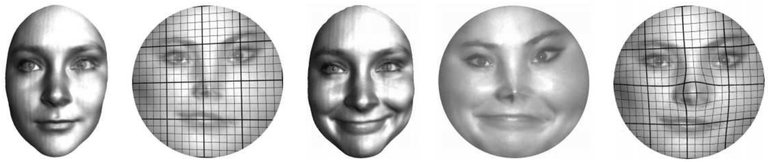
2D Shape Space-Conformal Welding

$$\{2D\ Contours\} \cong \frac{\{Diffeomorphism\ on\ S^1\} \cup \{Conformal\ Module\}}{\{Mobius\ Transformation\}}$$



Wasserstein Distance

Given a metric surface $(S,\mathbf{g})$, a Riemann mapping $\varphi : (S,\mathbf{g}) \to \mathbb{D}^2$, the conformal factor $e^{2\lambda}$ gives a probability measure on the disk. The shape distance is given by the Wasserstein distance.



(a) (b) (c) (d) (e)

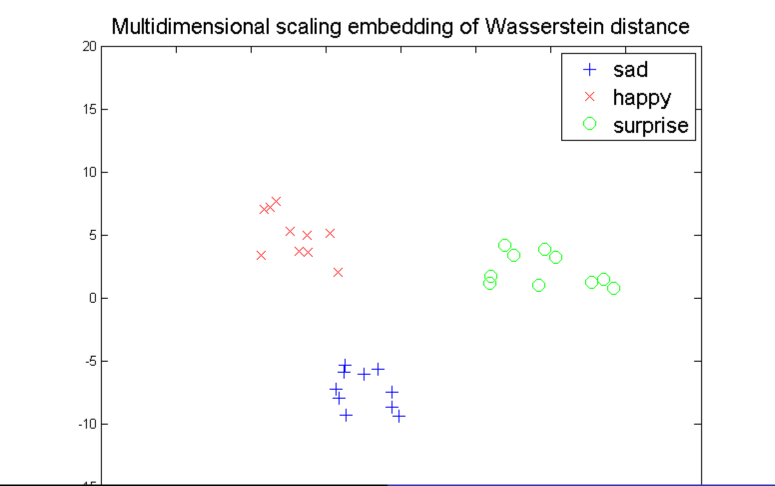
Fig. 9: **The computation of Wasserstein distance**

Expression Classification

Fig. 10: **Face surfaces for expression clustering. The first row is "sad", the second row is "happy" and the third row is "surprise".**

Expression Classification

Compute the Wasserstein distances among all the facial surfaces, isometrically embed on the plane using MDS method, perform clustering.



Curvature Sensitive Remeshing

Algorithm Pipeline

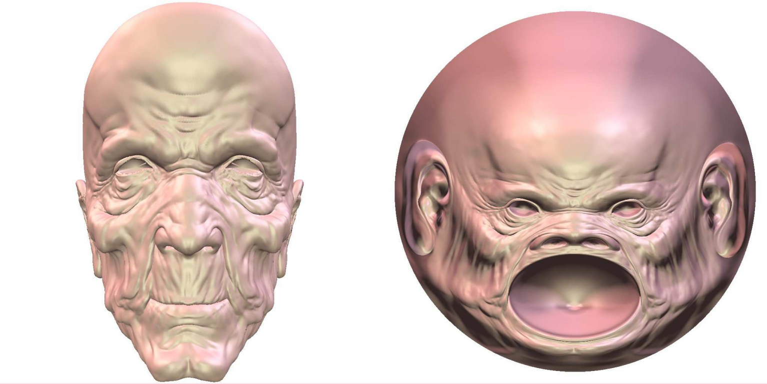
1. Compute the conformal parameterization of the input surface $\varphi : (S, \mathbf{g}) \to (\mathbb{D}, dzd\bar{z})$,
2. Compute the optimal mass transportation map $\psi : (\mathbb{D}, dxdy) \to (\mathbb{D}, \mu)$, where μ is the combination of the surface area element $e^{2\lambda}dxdy$ and the absolute value of the Gaussian curvature measure $|K(x,y)|dxdy$,
3. Uniformly sample on the preimage of the OMT map $\psi^{-1}(\mathbb{D})$,
4. Pull back the samples to the conformal parameter domain $\varphi(S)$, compute the Delaunay triangulation T,
5. Pull back the triangulation T to the original surface S, which induces the remeshing of S.

Curvature Sensitive Parameterization(CSP)

Original face mesh

Conformal parameteraiztion

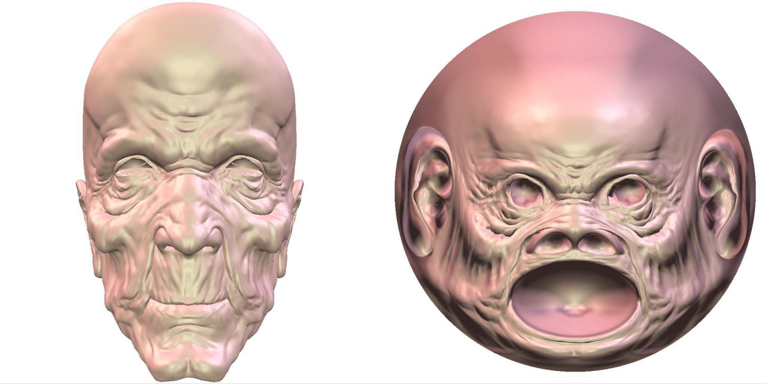
Curvature Sensitive Parameterization(CSP)



Original face mesh

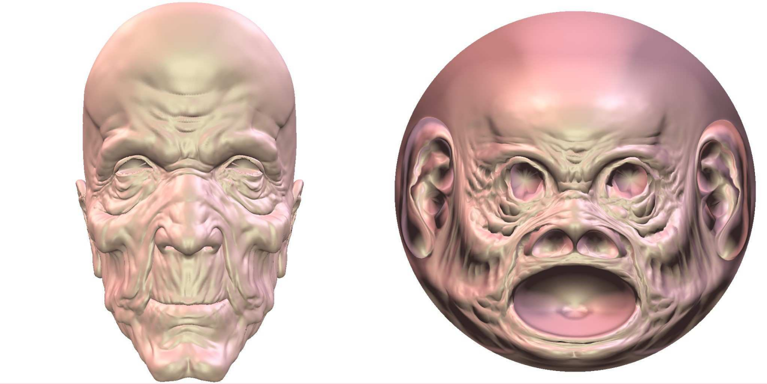
Area preserving parameteraiztion

Curvature Sensitive Parameterization(CSP)



Original face mesh CSP: area + curvature$\times$0.1

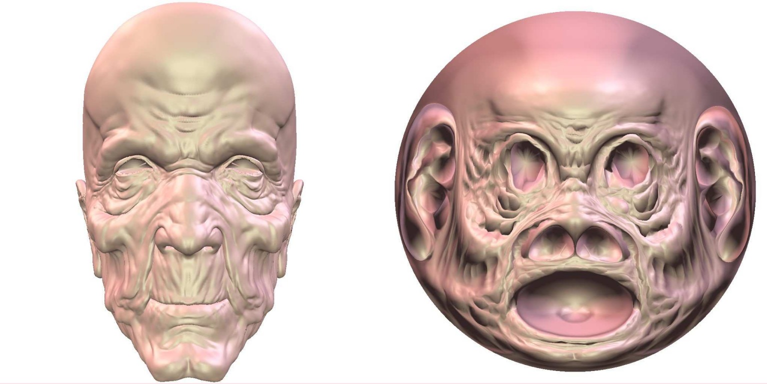
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$0.2

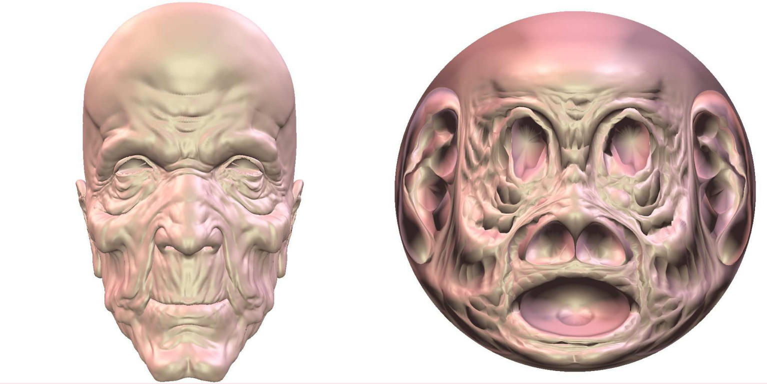
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$0.4

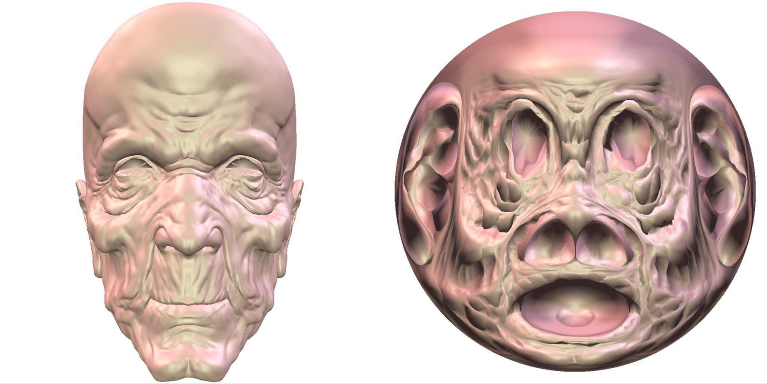
Curvature Sensitive Parameterization(CSP)



Original face mesh

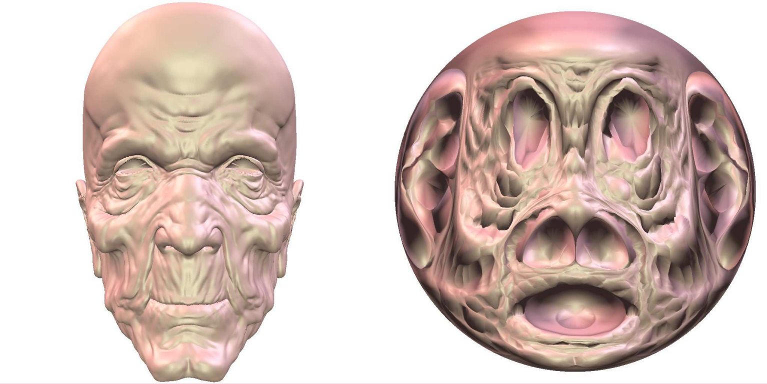
CSP: area + curvature$\times$0.8

Curvature Sensitive Parameterization(CSP)



Original face mesh   CSP: area + curvature $\times$1.0

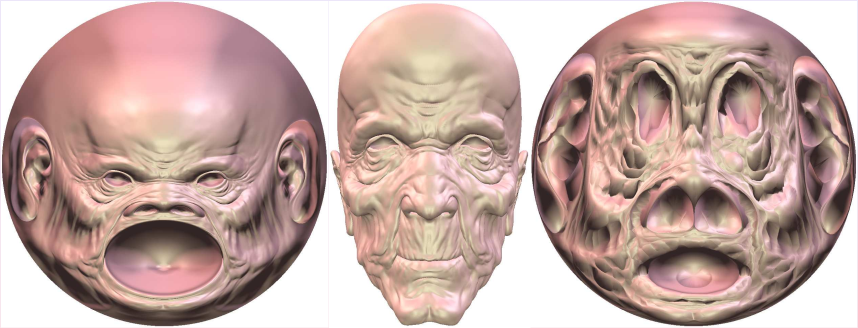
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature $\times$ 2.0

Compare Multi-scale remeshing between APP and CSP



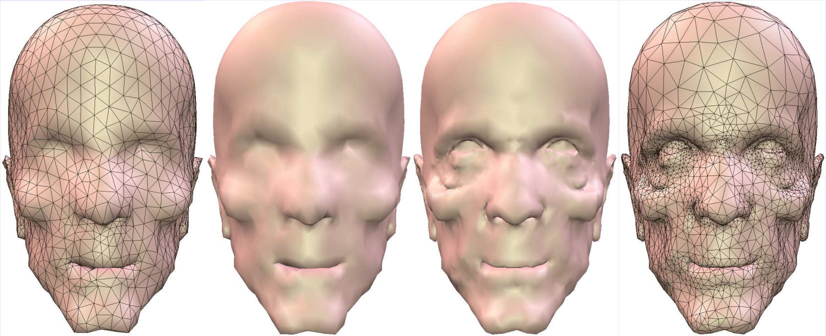
(a) APP

(b) original mesh(140K)

(c) CSP

Compare Multi-scale remeshing between APP and CSP

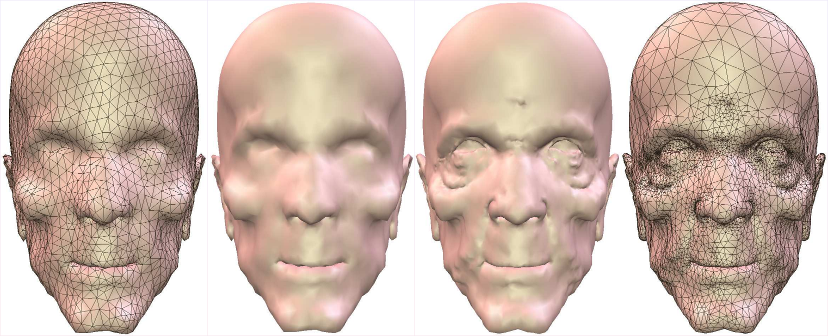
Remeshing: 1K vertices



(a) APP wireframe (b) APP smooth (c) CSP smooth (d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 2K vertices



(a) APP wireframe

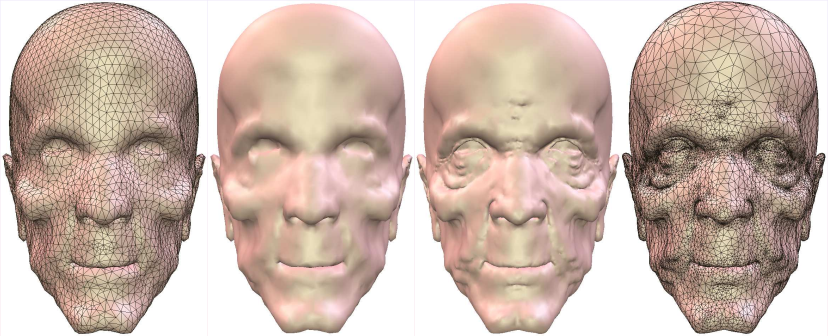
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 4K vertices



(a) APP wireframe

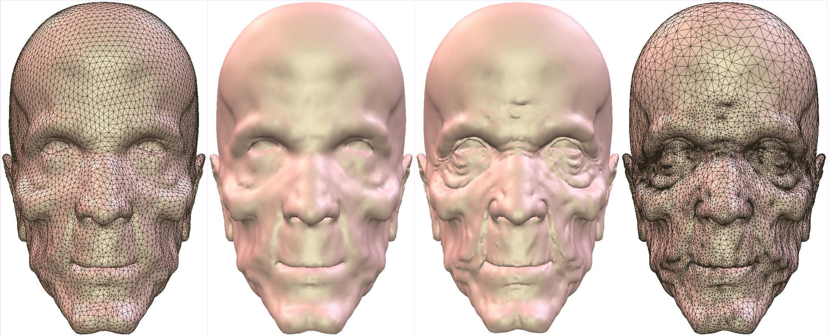
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 8K vertices



(a) APP wireframe

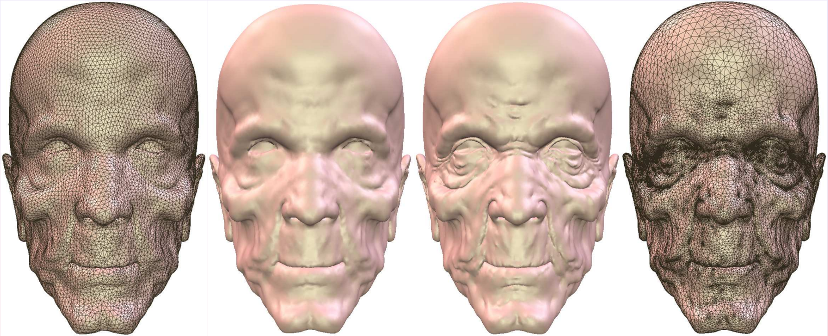
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

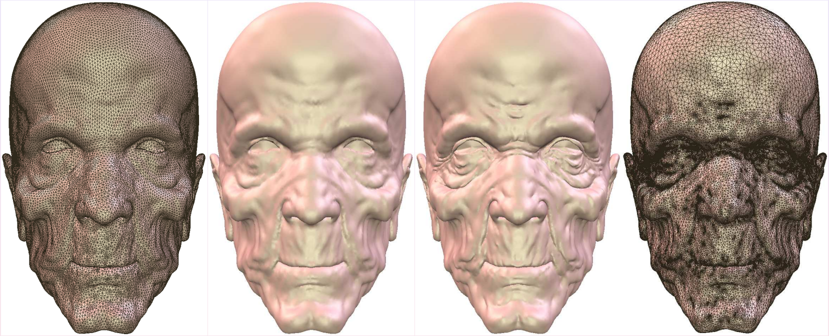
Remeshing: 16K vertices



(a) APP wireframe (b) APP smooth (c) CSP smooth (d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 32K vertices



(a) APP wireframe

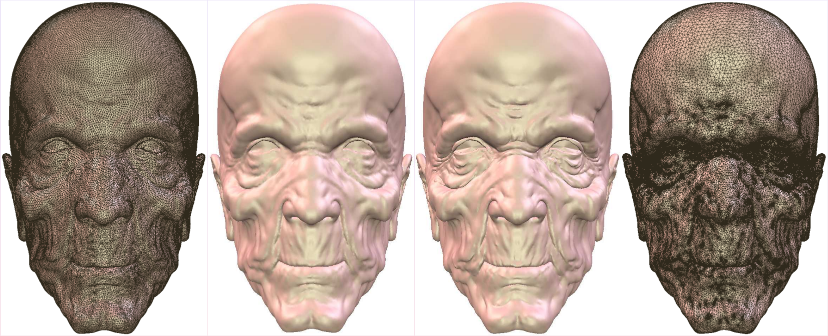
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 64K vertices



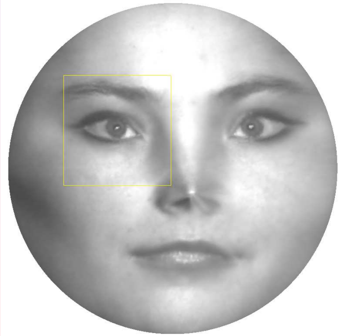
(a) APP wireframe (b) APP smooth (c) CSP smooth (d) CSP wireframe

Meshing

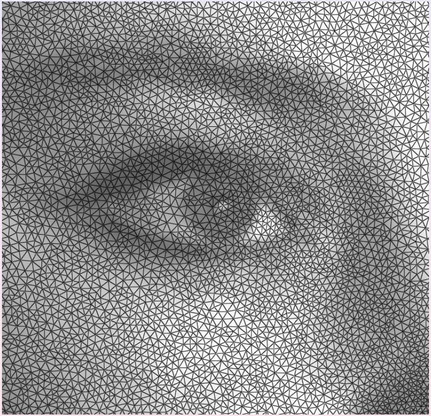
Theorem

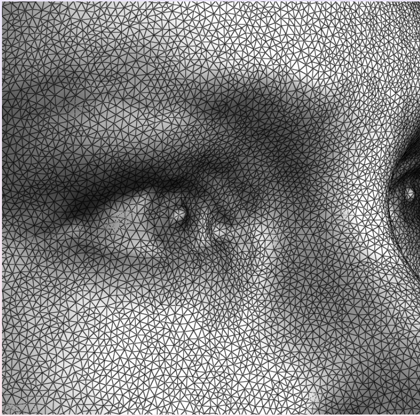
Suppose S is a surface with a Riemannian metric. Then there exist meshing method which ensures the convergence of curvatures.

Key idea: Delaunay triangulations on uniformization domains. Angles are bounded, areas are bounded.



Meshing





Theorem

Let M be a compact Riemannian surface embedded in $\mathbb{E}^3$ with the induced Euclidean metric, T the triangulation generated by Delaunay refinement on conformal uniformization domain, with circumradius bound ε. If B is the relative interior of a union of triangles of T, then

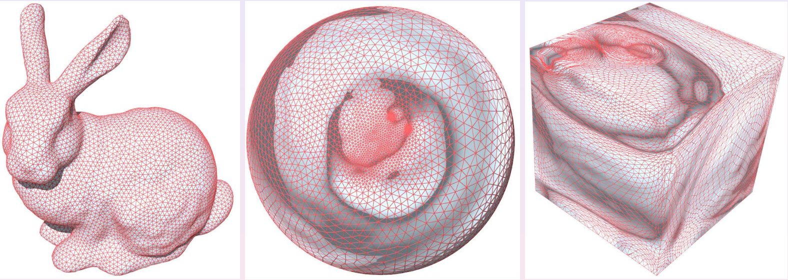
$$\begin{aligned} |\phi_T^G(B) - \phi_M^G(\pi(B))| &\leq K\varepsilon \\ |\phi_T^H(B) - \phi_M^H(\pi(B))| &\leq K\varepsilon \end{aligned}$$

where $\pi : T \to M$ is the closest point projection, ϕ^H, ϕ^G are the mean and Gaussian curvature measures, where

$$K = O(area(B)) + O(length(\partial B)).$$

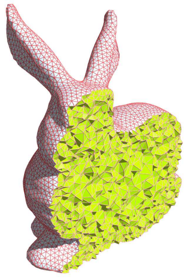
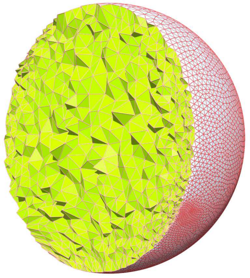
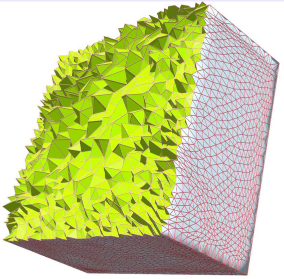
Hexahedral Meshing

Genus Zero Case



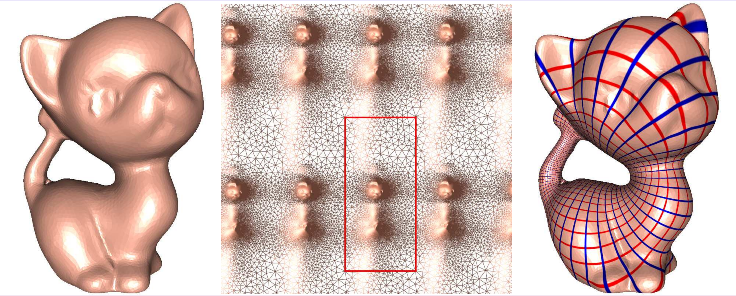
(a) Stanford bunny (b) Spherical mapping (c) Cube mapping

Genus Zero Case

(d) Solid bunny(e) Solid ball mapping(f) Solid cube mapping

Genus One Case



(a) Kitten surface (b) Flat torus (c) Quad-mesh

Figure: A genus one closed surface can be conformally and periodically mapped onto the plane, each fundamental domain is a parallelogram. The subidvision of the parallelogram induces a quad-mesh of the surface.

Genus One Case

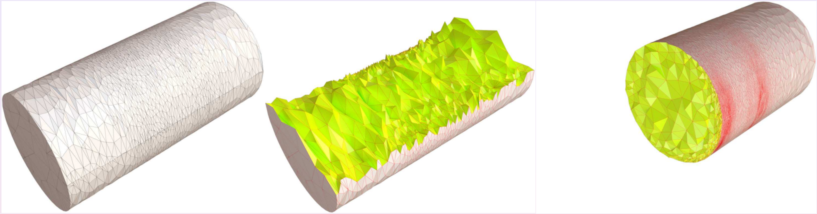


Figure: The interior of the kitten surface is mapped onto a canonical solid cylinder.

Algorithmic Pipeline

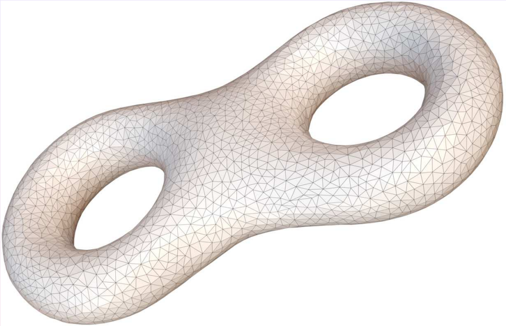


Figure: Input Surface $\partial\Omega$.

Algorithmic Pipeline

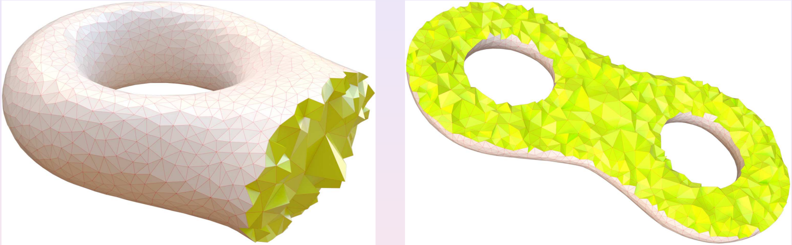


Figure: Tetrahedral meshing Ω.

Algorithmic Pipeline

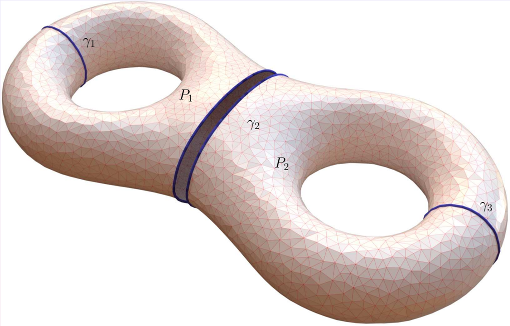


Figure: Admissible curve system $\{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3}\}$, pants decomposition $\{P_1, P_2, \cdots, P_{2g-2}\}$.

Algorithmic Pipeline

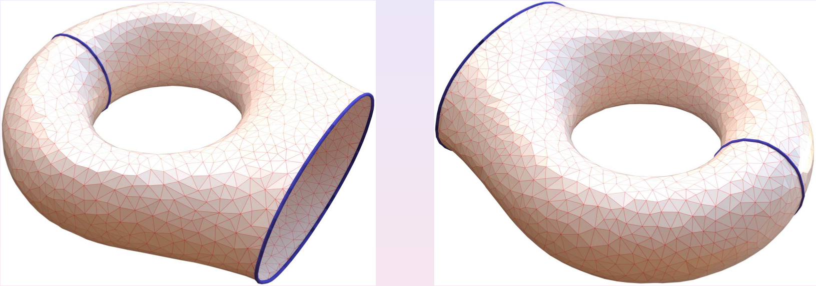


Figure: Pairs of pants $\{P_1, P_2, \cdots, P_{2g-2}\}$.

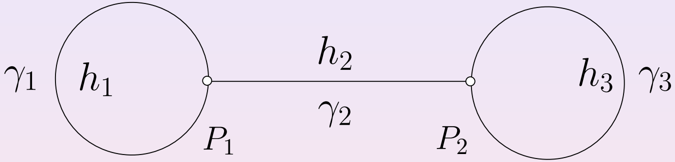


Figure: Pants decomposition graph.

Strebel Differentials

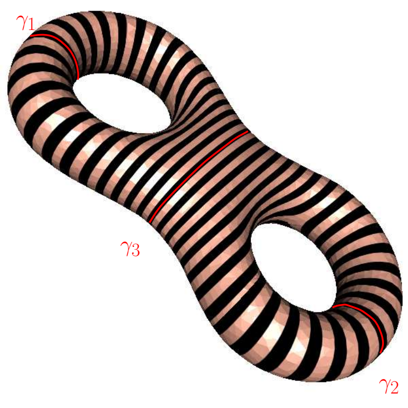


Figure: Foliation, Holomorphic quadratic differential.

Algorithmic Pipeline

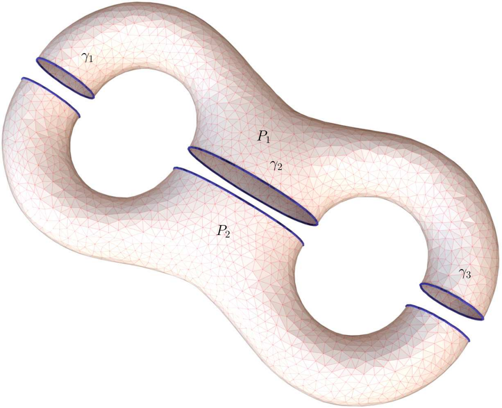


Figure: Admissible curve system, pants decomposition.

Algorithmic Pipeline

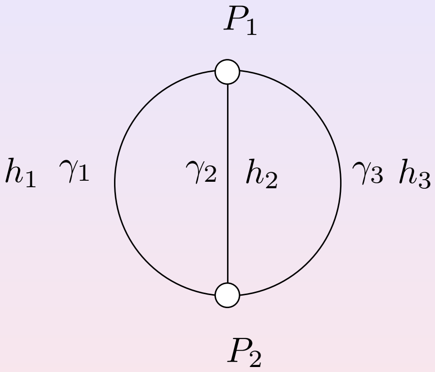


Figure: Pants decomposition graph.

Algorithmic Pipeline

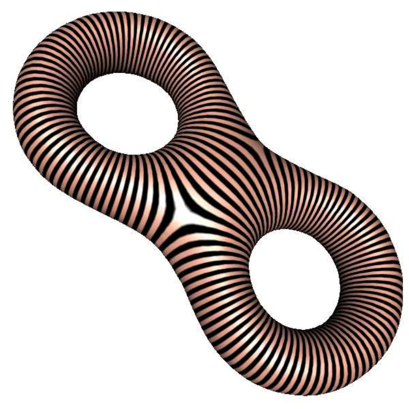


Figure: Foliation, Holomorphic quadratic differential.

Algorithmic Pipeline

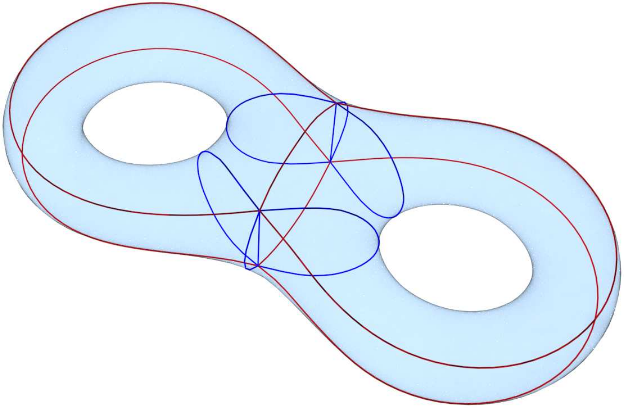


Figure: Critical horizontal trajectories and vertical trajectories.

Algorithmic Pipeline

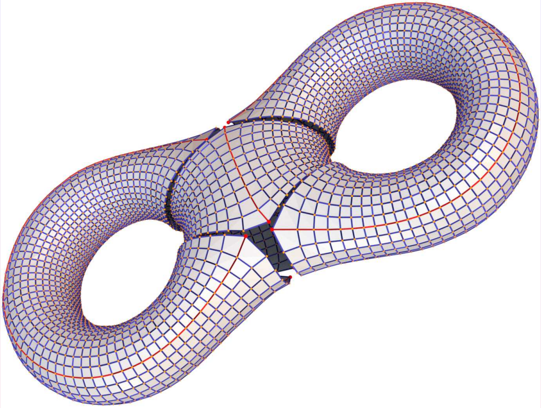


Figure: Quad-Mesh $\mathscr{Q}$ and the cylindrical decomposition.

Algorithmic Pipeline

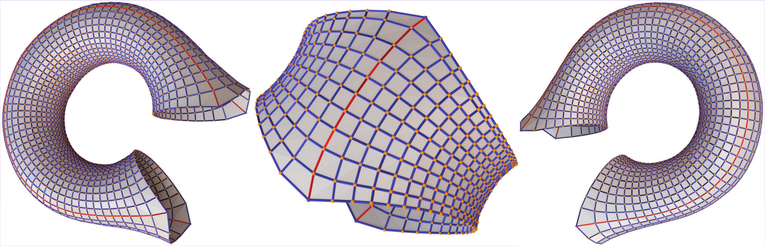


Figure: Cylindrical decomposition.

Algorithmic Pipeline

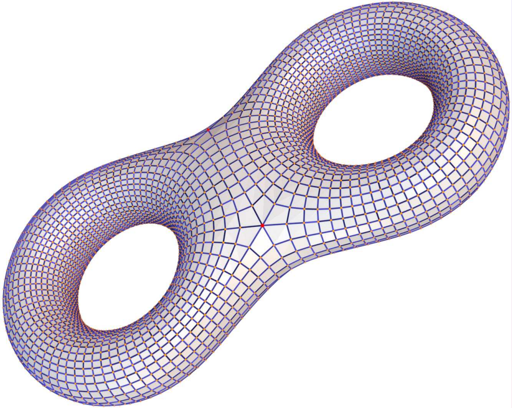
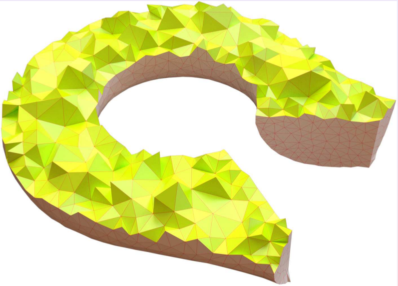


Figure: Colorable quadrilateral mesh, all vertex valences are even.

Algorithmic Pipeline



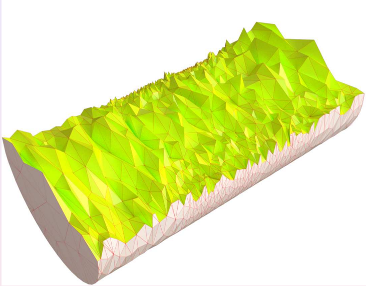
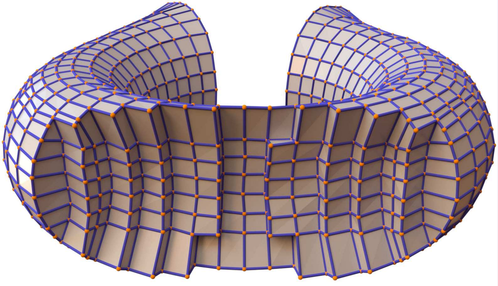


Figure: Left solid cylinder, maps to the canonical solid cylinder.

Algorithmic Pipeline



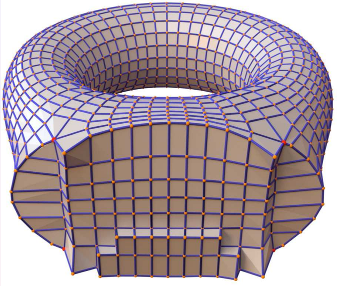


Figure: Hexahedral meshing of solid cylinders.

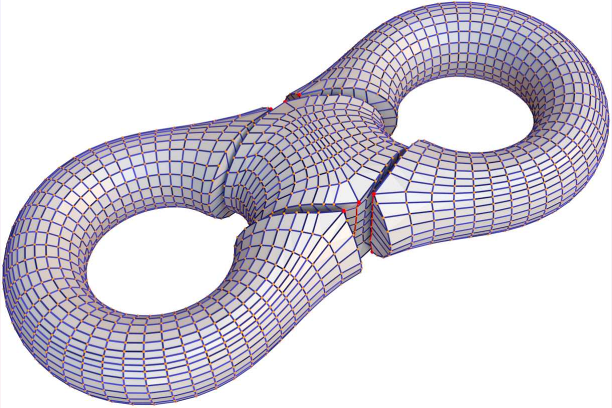


Figure: Hexahedral mesh $\mathscr{H}$ of the interior volume Ω.

Algorithmic Pipeline

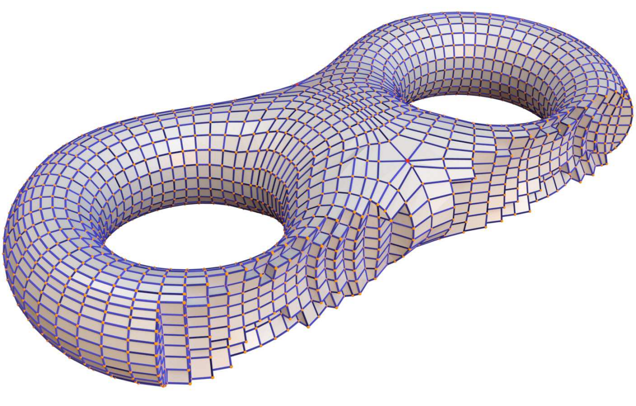


Figure: Hexahedral mesh $\mathscr{H}$ of the interior volume Ω.

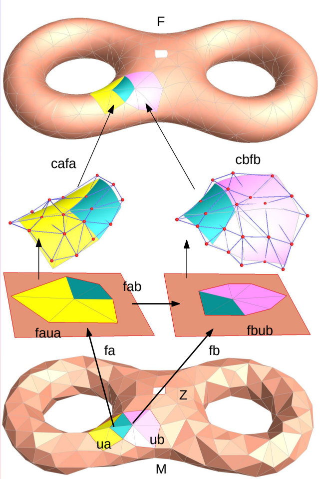
Manifold Spline

- Convert scanned polygonal surfaces to smooth spline surfaces.
- Conventional spline scheme is based on affine geometry. This requires us to define affine geometry on arbitrary surfaces.
- This can be achieved by designing a metric, which is flat everywhere except at several singularities (extraordinary points).
- The position and indices of extraordinary points can be fully controlled.

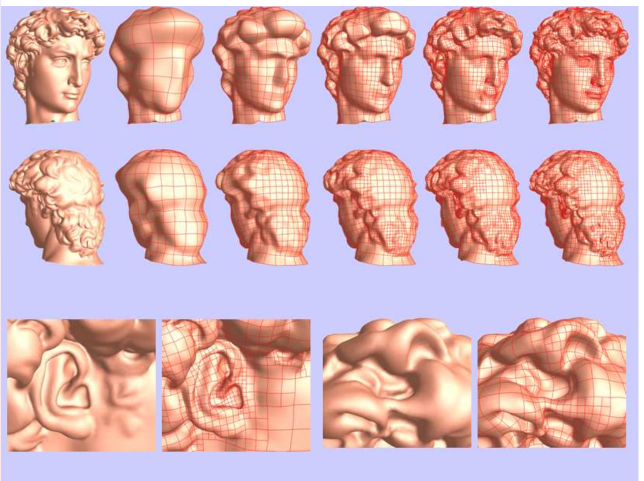
Extraordinary Points

- Fully control the number, the index and the position of extraordinary points.
- For surfaces with boundaries, splines without extraordinary point can be constructed.
- For closed surfaces, splines with only one singularity can be constructed.

Manifold Spline

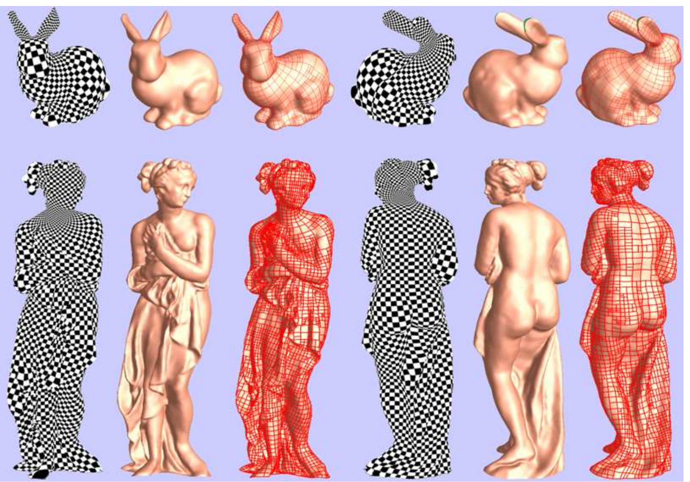


Converting a polygonal mesh to TSplines with multiple resolutions.



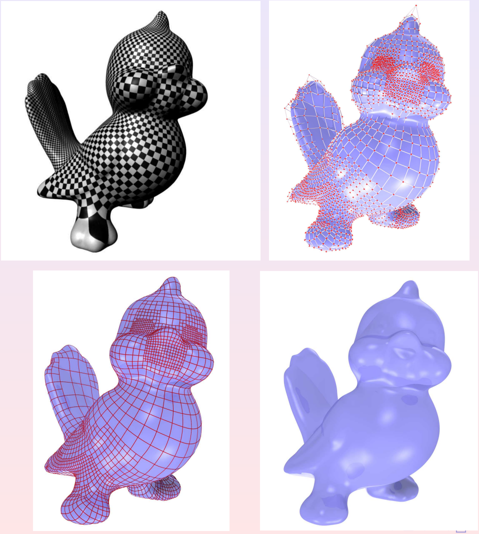
Manifold Spline

Converting scanned data to spline surfaces.



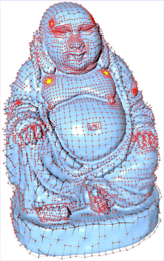
Manifold Spline

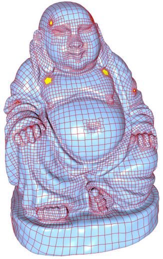
Converting scanned data to spline surfaces, the control points, knot structure are shown.

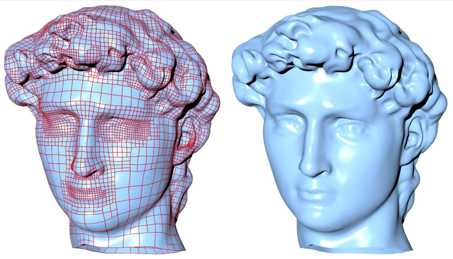


Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.

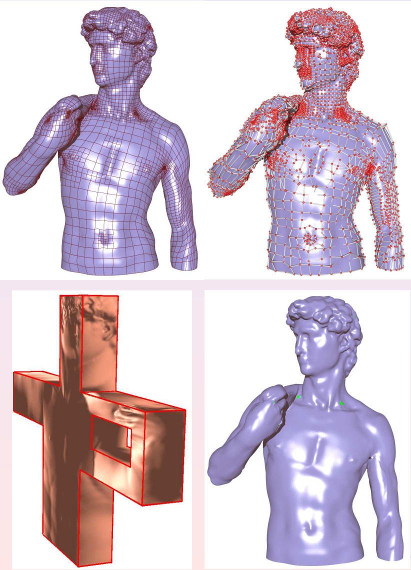




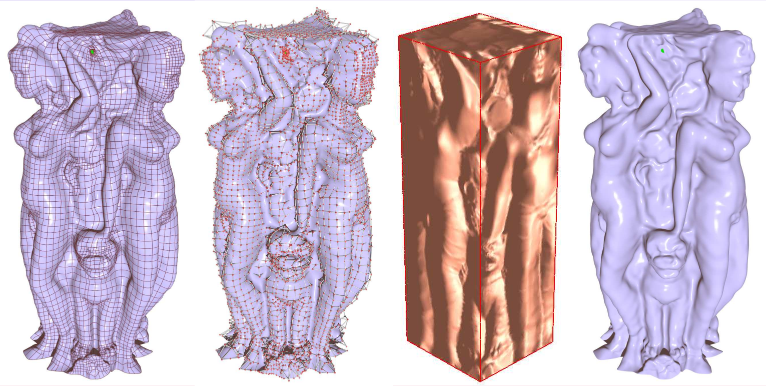


Manifold Spline

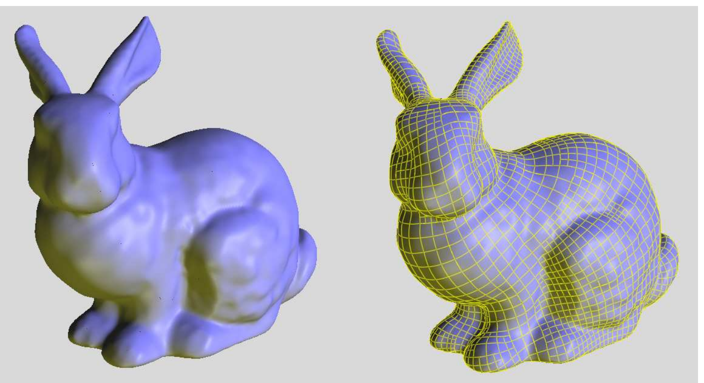
Polygonal mesh to spline, control net and the knot structure.



Manifold Spline

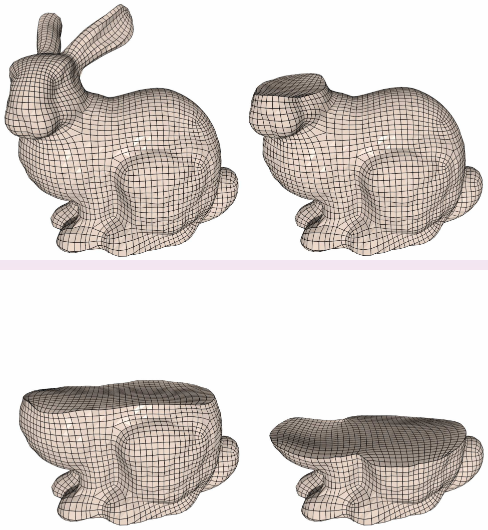


Manifold Spline



Manifold Spline

volumetric spline.



Visualization

(a) Front view (b) Angle-preserving (c) Area-preserving (d) Back view

Angle-perserving parameterization vs. area-preserving parameterization

(a) 2x (b) 3x (c) 4x (d) 6x

Importance driven parameterization. The Buddha's head region is magnified by different factors

Wireless Sensor Network

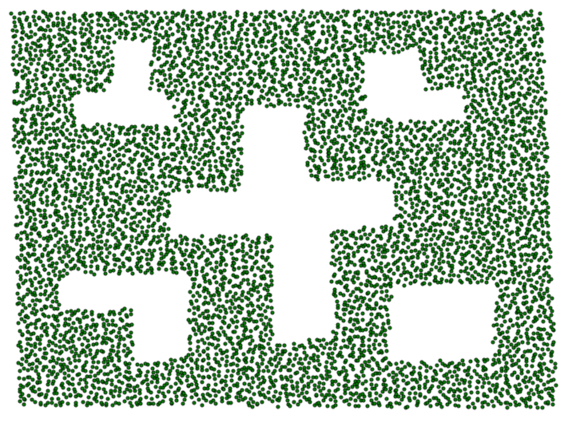
Wireless Sensor Network Application

Wireless Sensor Network

- Detecting global topology.
- Routing protocol.
- Load balancing.
- Isometric embedding.

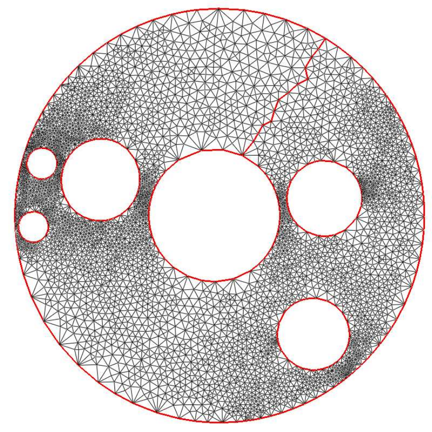
Greedy Routing

Given sensors on the ground, because of the concavity of the boundaries, greedy routing doesn't work.



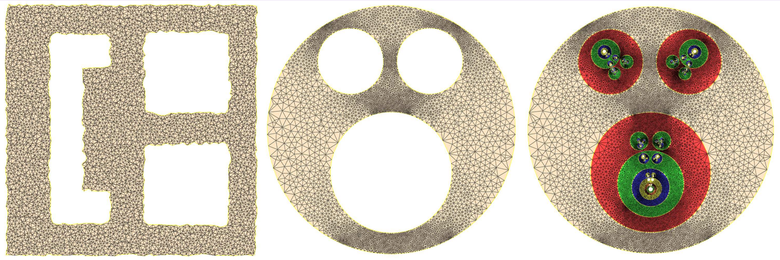
Greedy Routing

Map the network to a circle domain, all boundaries are circles, greedy routing works.

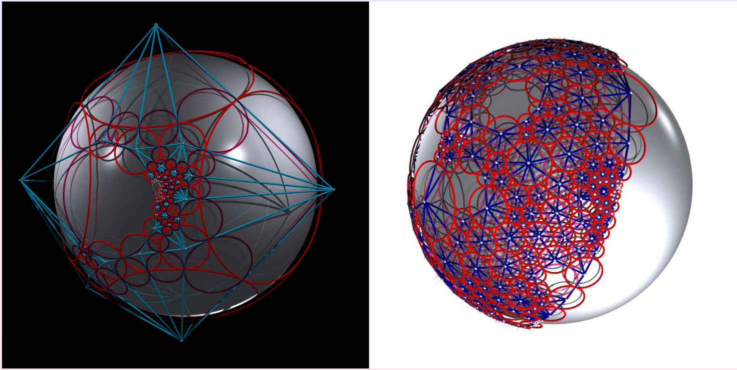


Load Balancing

Schoktty Group - Circular Reflection



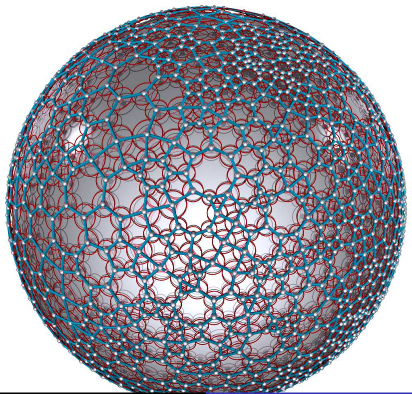
Optimal Planar Graph Embedding.



Graph Embedding

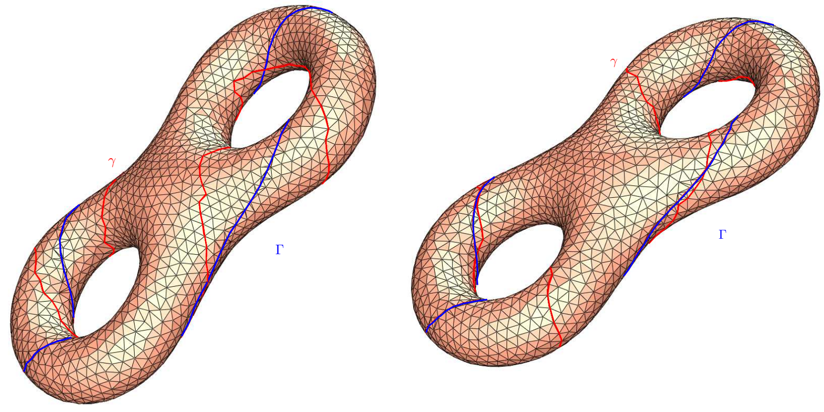
Thurston-Andreev Theorem

A planar graph can be embedded on the unit sphere, such that the face circles are orthogonal to vertex circles; the circles at the vertices of an edge are tangent to each other. Such kind of embedding differ by a Möbius transformation.

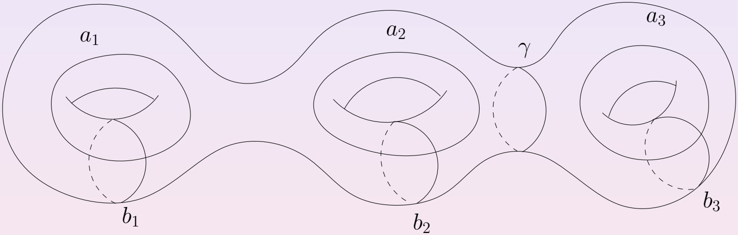


Canonical Homotopy Class Representative

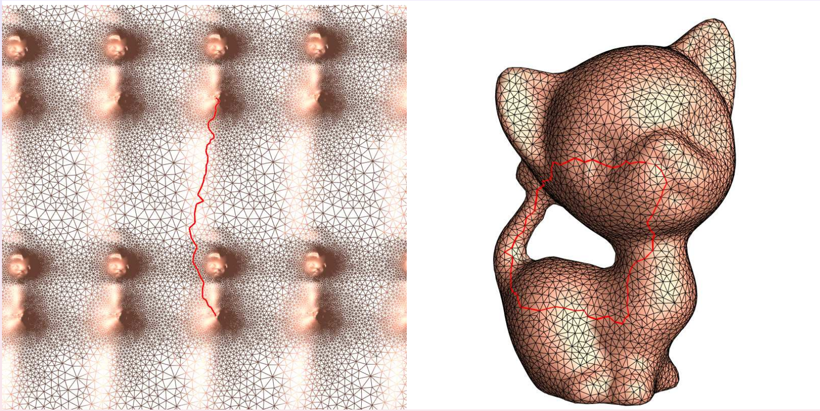
Under hyperbolic metric, each homotopy class has a unique geodesic, which is the representative of the homotopy class.



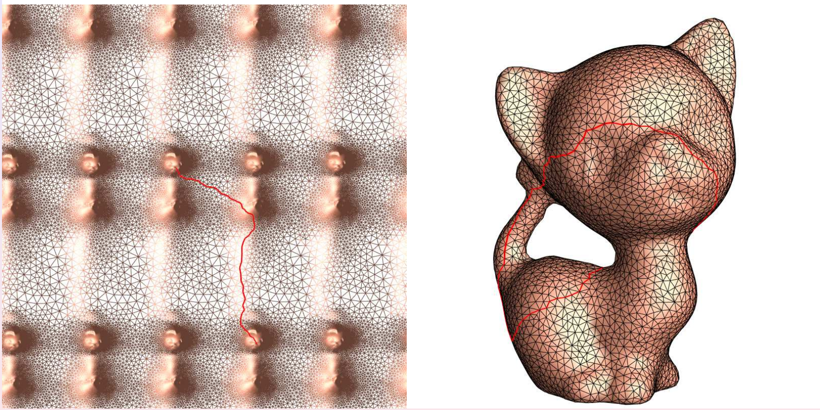
Shortest word Problem (NP Hard):



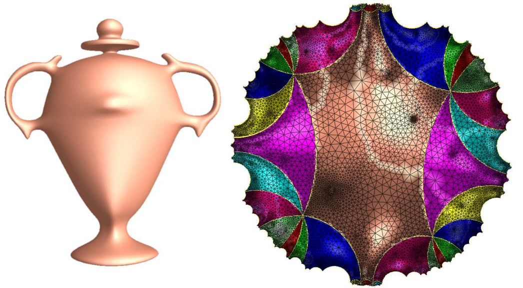
$$\gamma = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = (a_3 b_3 a_3^{-1} b_3^{-1})^{-1}$$

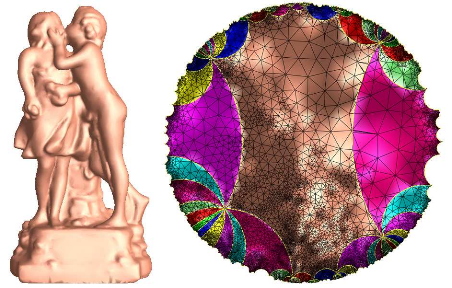


Loop Lifting



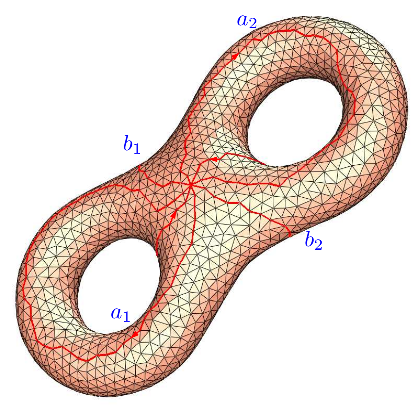
Hyperbolic Ricci Flow

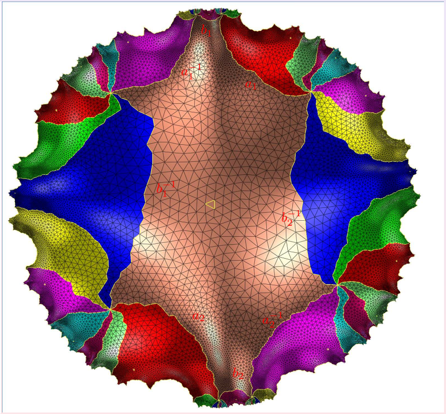




Hyperbolic Yamabe Flow

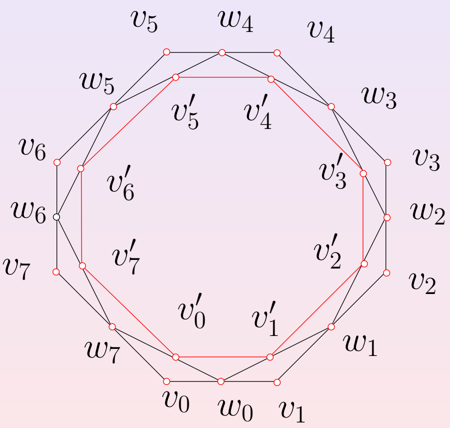
Lifting a loop from base surface to the universal covering space.





Birkoff Curve Shorting

Birkoff curve shortening deforms a loop to a geodesic.



Birkoff Curve Shorting

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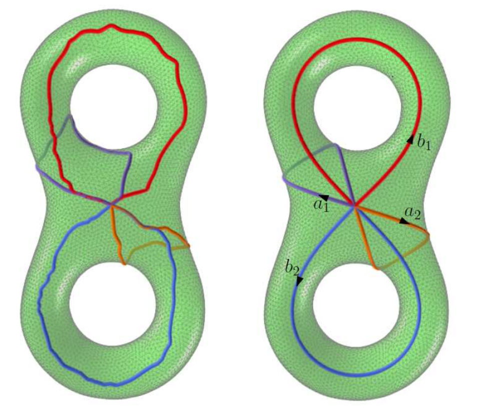


Computing Shortest Word

1. Compute the uniformization metric using Ricci flow.
2. Compute the geodesic loop by Birkoff curve shortening.
3. Lift the geodesic loop to the universal covering space.
4. Trace the lifted loop to compute the word.

Medical Imaging Application

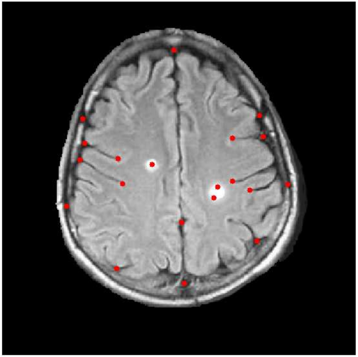
Medical Imaging

Quantitatively measure and analyze the surface shapes, to detect potential abnormality and illness.

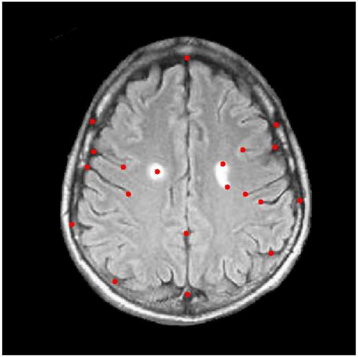
- Shape reconstruction from medical images.
- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Shape retrieval.

Image registration : Brain MRI

Registration problem :

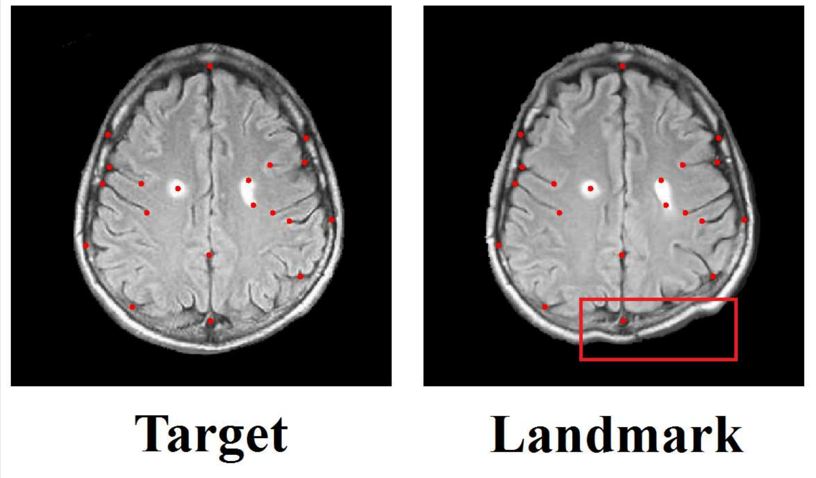


Moving

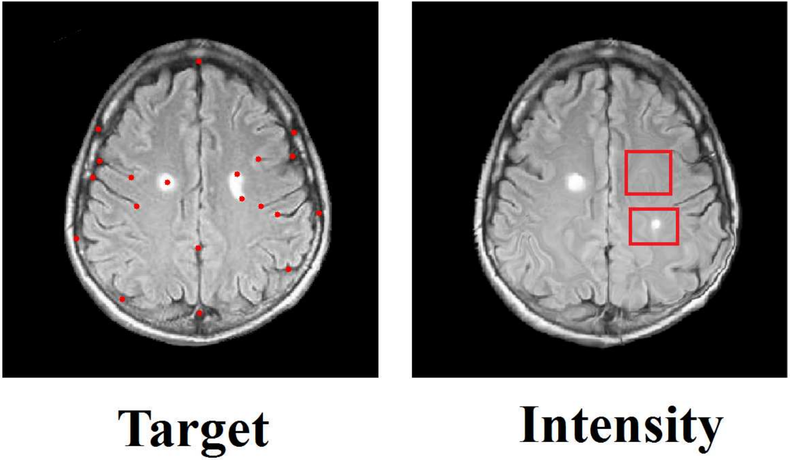


Target

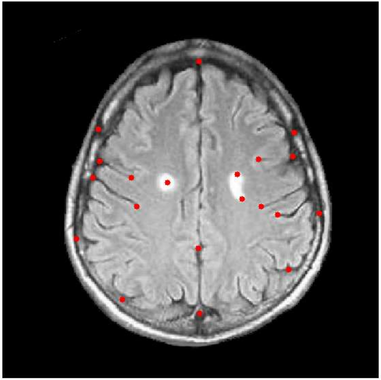
Landmark only :



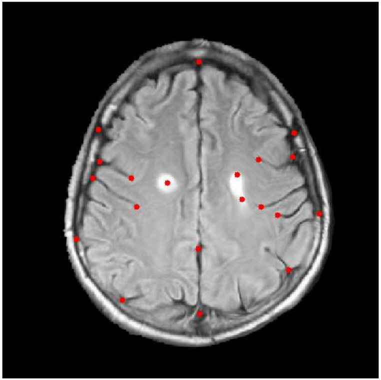
Intensity only :



Landmark + intensity only :

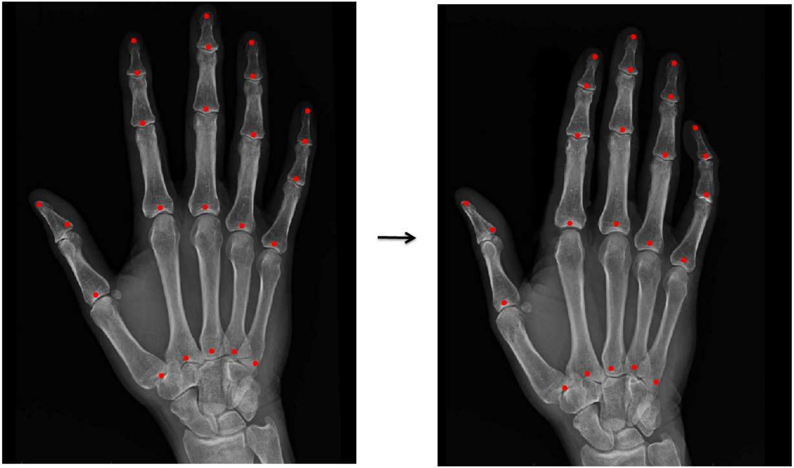


Target

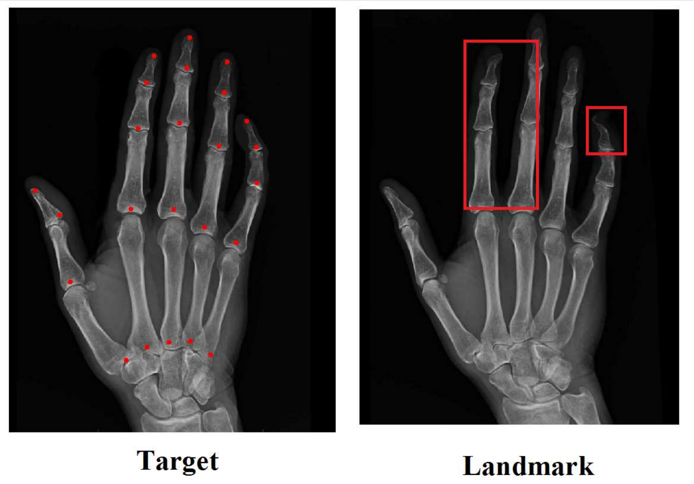


Landmark + intensity

Image registration : X-ray bone

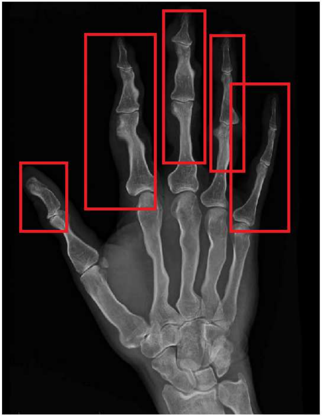


Landmark only :



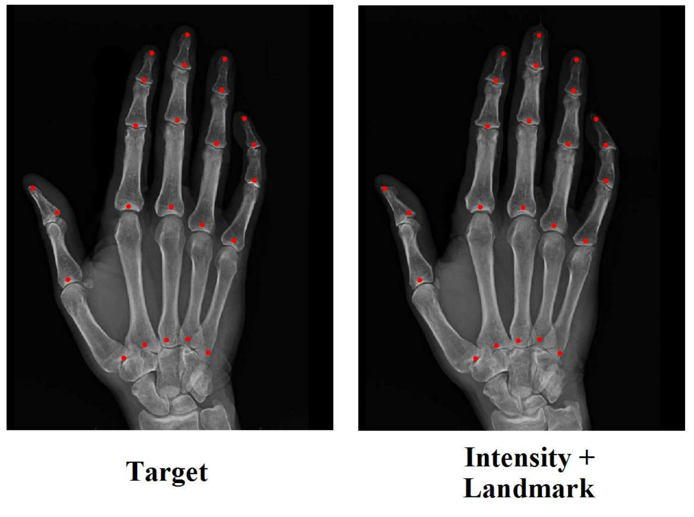
Intensity only :

Target

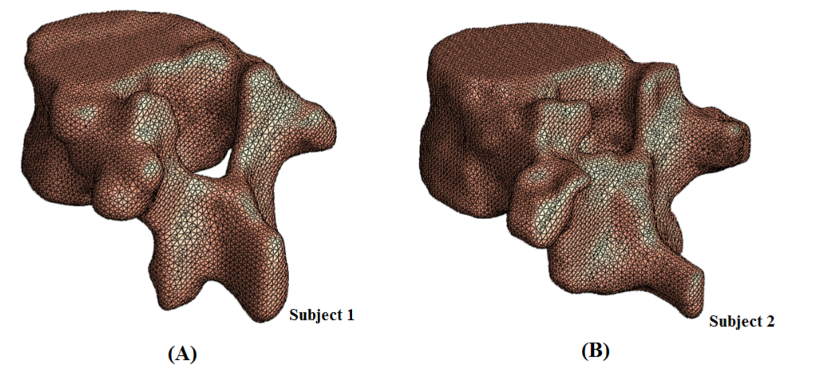


Intensity

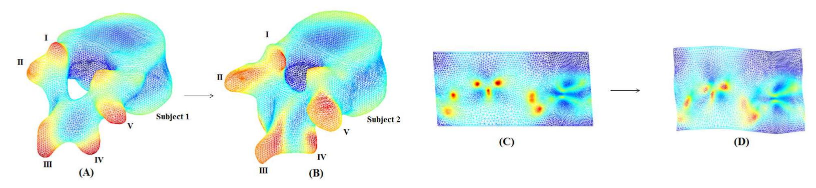
Landmark + Intensity :

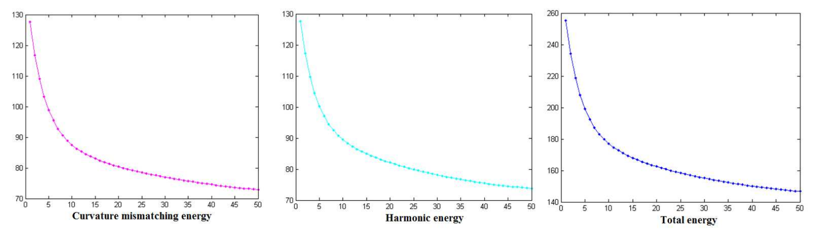


Medical registration : Vertebral bone

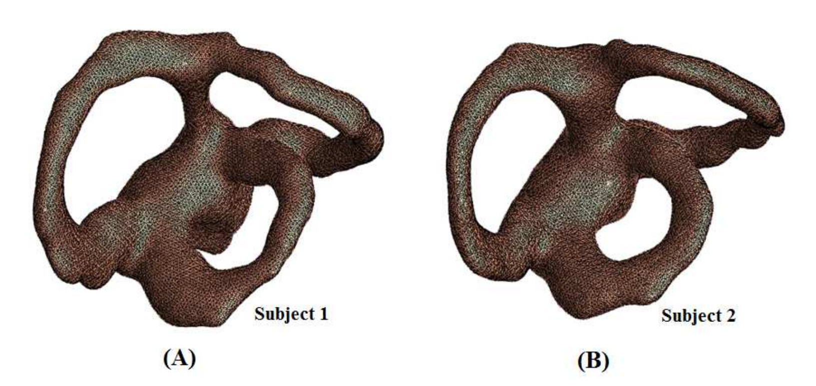


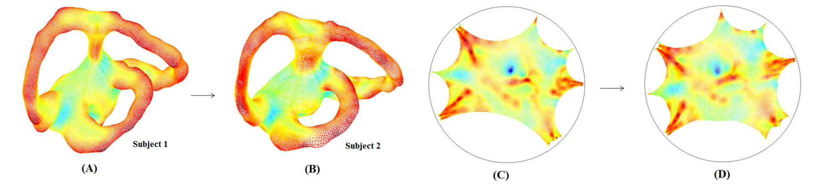
Registration result:

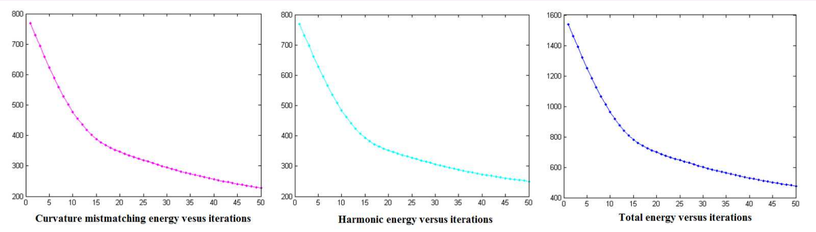




Medical registration : Vestibular system



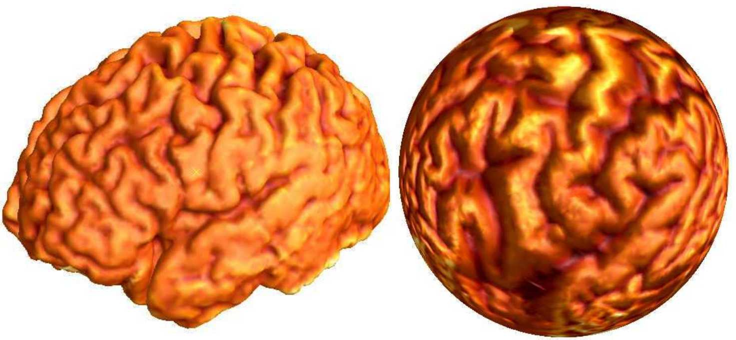

Subject 1
Subject 2
(A)
(B)
(C)
(D)


Curvature mistmatching energy vesus iterations
Harmonic energy versus iterations
Total energy versus iterations

Conformal Brain Mapping

Brain Cortex Surface

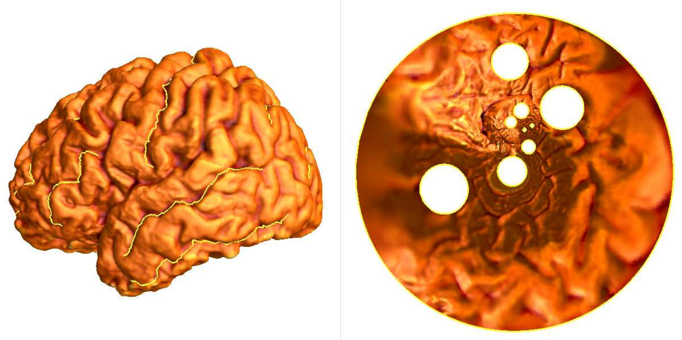
Conformal Brain Mapping for registration, matching, comparison.



Conformal Brain Mapping

Using conformal module to analyze shape abnormalities.

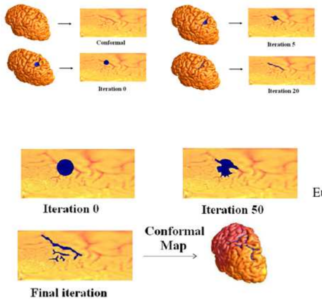
Brain Cortex Surface



Automatic sulcal landmark Tracking

- With the conformal structure, PDE on Riemann surfaces can be easily solved.
- Chan-Vese segmentation model is generalized to Riemann surfaces to detect sulcal landmarks on the cortical surfaces automatically

Extraction of high mean curvature region by Chan-Vese segmentation



$$F(c_1, c_2, \psi) = \int_S (I_f - c_1)^2 H(\psi) dS + \int_S (I_f - c_2)^2 (1 - H(\psi)) dS + \nu \int_S |\nabla_S H(\psi)| dS$$

$$c_1 = \frac{\int_D I_f \circ \phi(x,y) H(\psi \circ \phi(x,y)) \lambda(x,y) dxdy}{\int_D H(\psi \circ \phi(x,y)) \lambda(x,y) dxdy}$$

$$c_2 = \frac{\int_D I_f \circ \phi(x,y)(1 - H(\psi \circ \phi(x,y))) \lambda(x,y) dxdy}{\int_D (1 - H(\psi \circ \phi(x,y))) \lambda(x,y) dxdy}$$

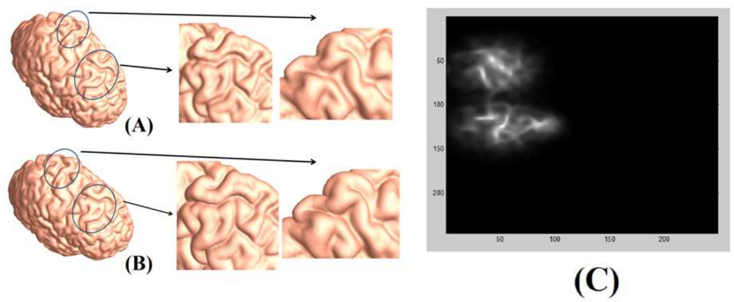
Euler Lagrange equation is:

$$\frac{\partial \psi}{\partial t} = \lambda \delta(\psi)[\,\nu div_S(\frac{\nabla_S \psi}{||\nabla_S \psi||_S}) - (I_f - c_1)^2 - (I_f - c_2)^2] \text{ or}$$

$$\frac{\partial \psi \circ \phi}{\partial t} = \lambda \delta(\psi \circ \phi)[\nu \frac{1}{\lambda} div(\sqrt{\lambda} \frac{\nabla \psi \circ \phi}{|\nabla \psi \circ \phi|}) - (I_f \circ \phi - c_1)^2 - (I_f \circ \phi - c_2)^2]$$

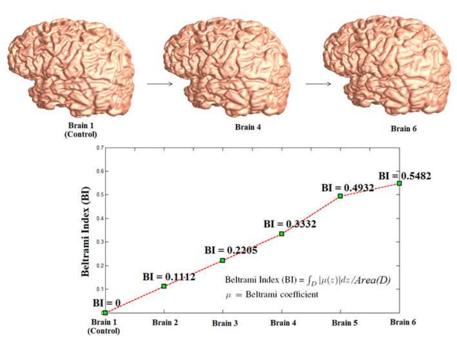
Abnormality detection on brain surfaces

The Beltrami coefficient of the deformation map detects the abnormal deformation on the brain.

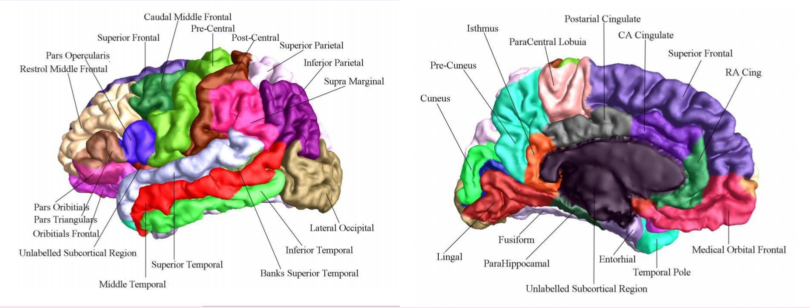


Abnormality detection on brain surfaces

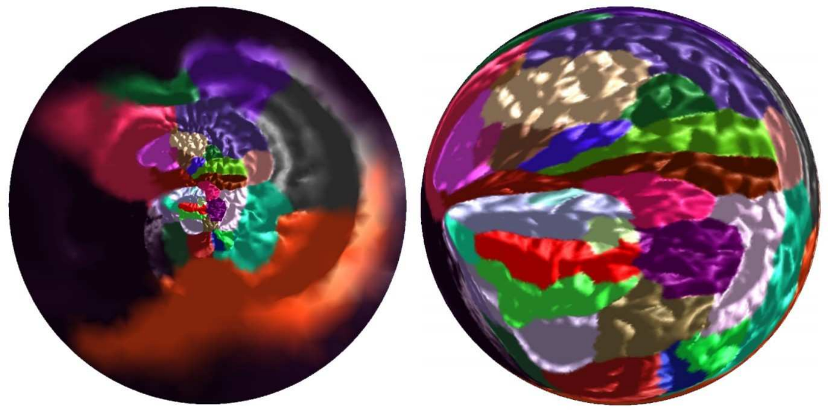
The brain is undergoing gyri thickening (commonly observed in Williams Syndrome) The Beltrami index can effectively measure the gyrification pattern of the brain surface for disease analysis.



Alzheimer Study

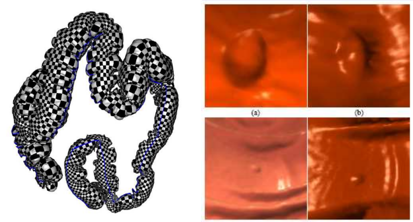


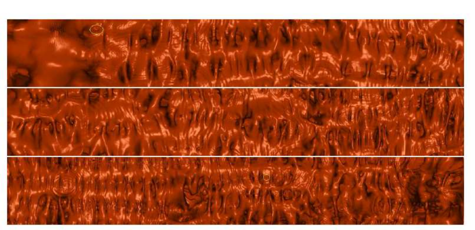
Alzheimer Study



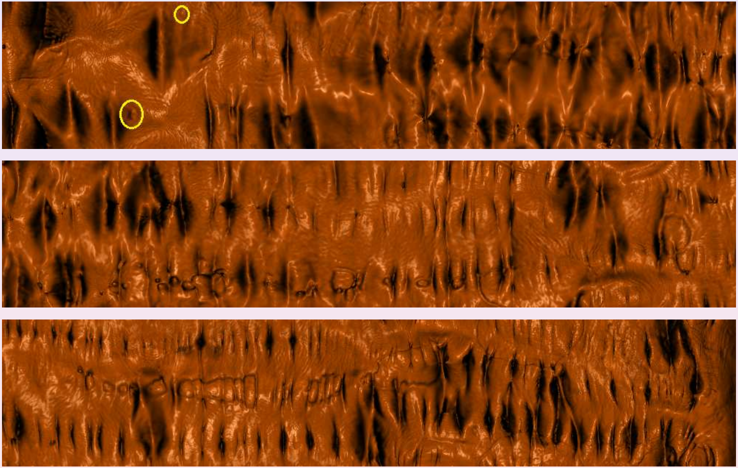
Virtual Colonoscopy

Colon cancer is the 4th killer for American males. Virtual colonosocpy aims at finding polyps, the precursor of cancers. Conformal flattening will unfold the whole surface.





Colon Flattening

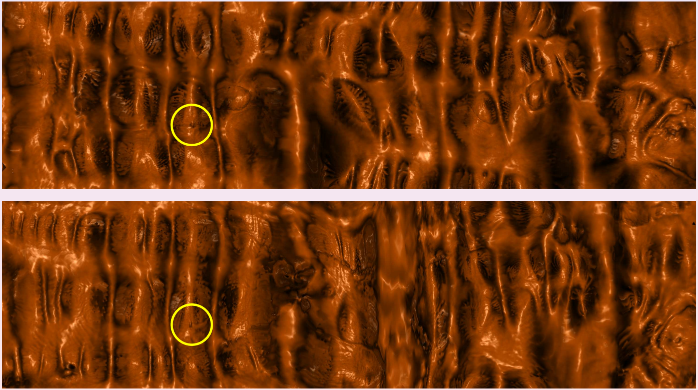


Virtual Colonoscopy

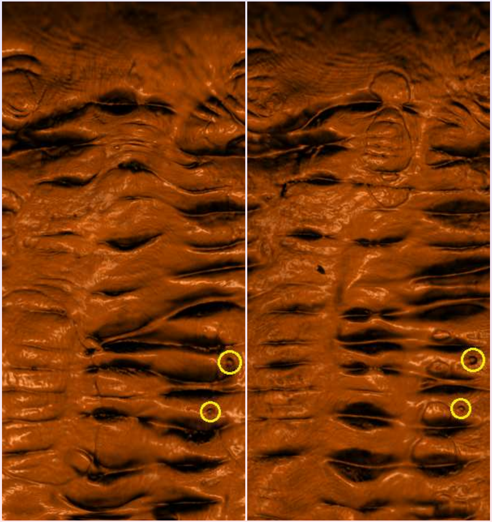
Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.

Virtual Colonoscopy

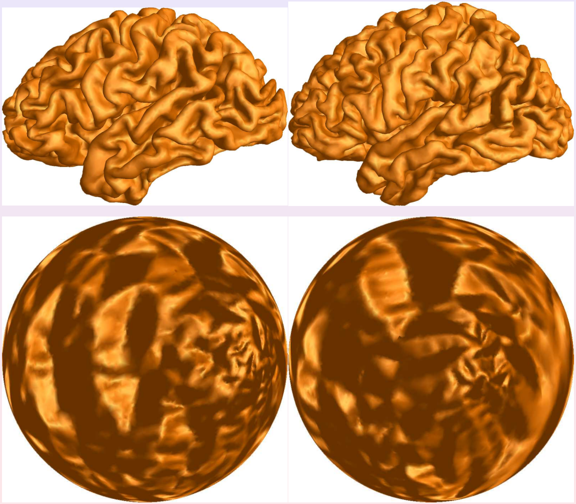
Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.



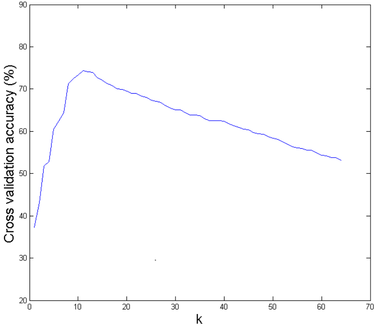
Colon Registration

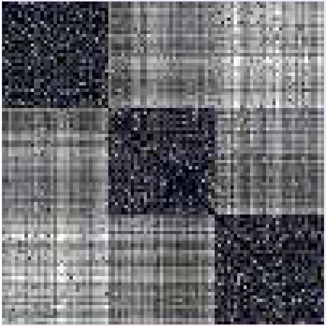


IQ from shape



IQ from shape





3D Fabrication

3D Fabrication

3D Fabrication

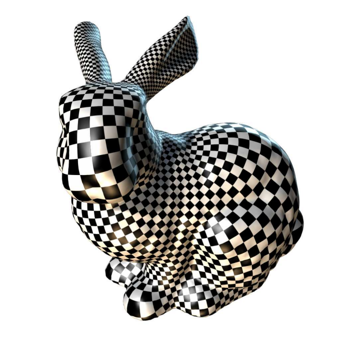
Figure: Fabrication by plain weaving.

Summary

- Conformal structure is more flexible than Riemannian metric
- Conformal structure is more rigid than topology
- Conformal geometry can be used for a broad range of engineering applications.

Thanks

For more information, please email to gu@cs.stonybrook.edu.



Thank you!