Surface and Volume Based Techniques for Shape Modeling and Analysis

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SIGGRAPH Asia 2013 Course

Diffeomorphic registration with Large deformation using Quasi-conformal maps

Collaborators

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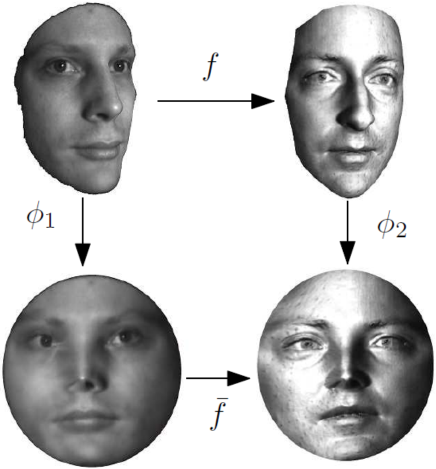
Registration Framework

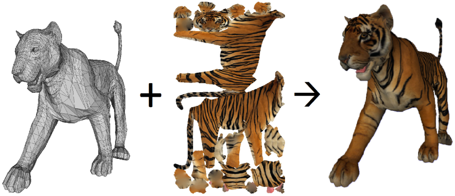
Framework



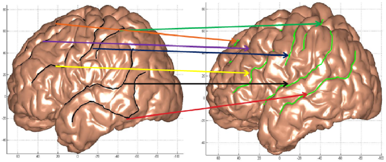
ϕ_1, ϕ_2 can be computed using Ricci flow, $\bar{f}$ quasi-conformal mapping.

Motivation

- Registration : Meaningful 1-1 correspondence between data (Images, surfaces)
- Importance : Measure of image / surface alignment
- Applications : Medical shape analysis, texture mapping, video compression, etc.



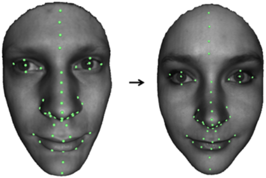
Texture mapping



Medical registration

Intensity-based registration : Matching intensity

Landmark-based registration : Matching feature points



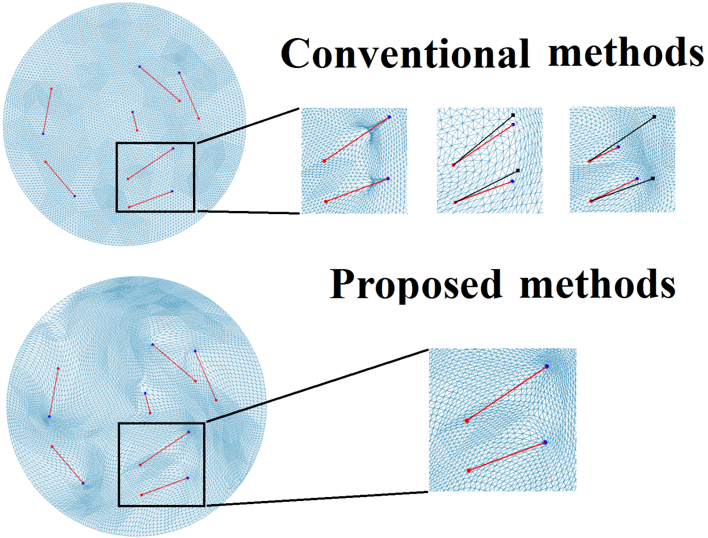
Landmark + intensity-based registration : Matching Both important information

Challenges of registration

- Bijectivity cannot be guaranteed under large deformation
- Bijectivity cannot be guaranteed when there are many landmark constraints
- Huge geometric distortion
- Slow

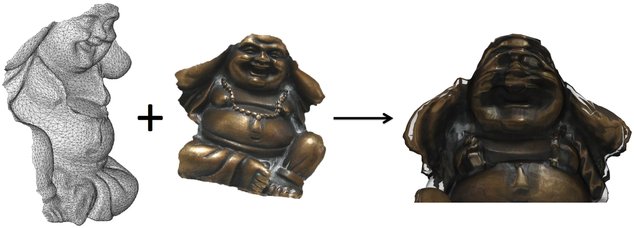
Contributions:

- Efficiently obtain bijective registration under large deformations and large number of landmark constraints.

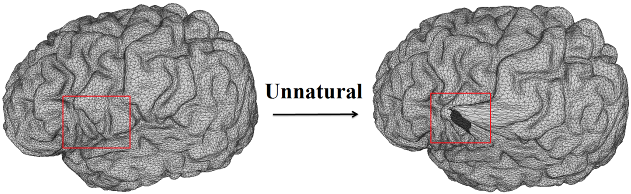

Conventional methods
Proposed methods

Importance of Bijectivity

Texture mapping :



Medical registration :



Mathematical background

Definition (Quasi-conformal mapping)

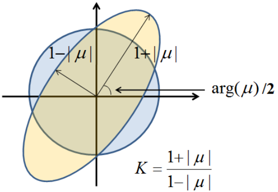
A function $f : \mathbb{C} \to \mathbb{C}$ is quasi-conformal if it satisfies the Beltrami equation:

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \tag{1}$$

for some complex valued function μ satisfying $\|\mu\|_\infty < 1$. μ is called the Beltrami coefficients.

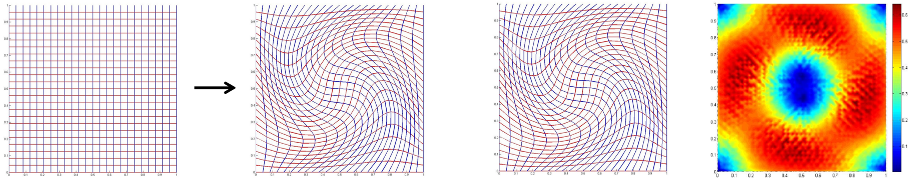
1-1 correspondence

Set of all Beltrami coeff. $|\mu| < 1$ $\Leftrightarrow$ Set of all quasi-conformal homeomorphisms (up to Mobiüs transf.)



$\mu \to f$: Linear Beltrami Solver(LBS)

LBS – Reconstruct QC map f from the BC μ



Original diffeomorphism
(A)

Reconstructed diffeomorphism from Beltrami representation
(B)

Beltrami representation
(C)

Recall Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}$$

Let $f = u + iv$, where $i = \sqrt{-1}$. Then

$$\mu(f) = \frac{(u_x - v_y) + i(v_x + u_y)}{(u_x + v_y) + i(v_x - u_y)} \tag{2}$$

Substitute $\mu(f) = \rho + i\tau$ and

$$\alpha_1 = \frac{(\rho-1)^2+\tau^2}{1-\rho^2-\tau^2}; \quad \alpha_2 = -\frac{2\tau}{1-\rho^2-\tau^2}; \quad \alpha_3 = \frac{(1+\rho)^2+\tau^2}{1-\rho^2-\tau^2}$$

we have

$$\begin{cases} -v_y = \alpha_1 u_x + \alpha_2 u_y \\ v_x = \alpha_2 u_x + \alpha_3 u_y \end{cases} \tag{3}$$

Similarly,

$$\begin{cases} -u_y = \alpha_1 v_x + \alpha_2 v_y \\ u_x = \alpha_2 v_x + \alpha_3 v_y \end{cases} \tag{4}$$

Take divergence on both sides and let

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}$$

we have:

$$\nabla \cdot \left(A \begin{pmatrix} u_x \\ u_y \end{pmatrix} \right) = 0 \; ; \; \nabla \cdot \left(A \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right) = 0 \tag{5}$$

- Discretize → SPD linear systems → CG method
- Obtain x-coordinate, y-coordinate functions

Models for registration

In this talk, we introduce two models of registration using QC maps

- Landmark-based registration using QC maps: Obtain 1-1 correspondence based on feature landmarks.
- Landmark and intensity based registration using QC maps: Obtain 1-1 correspondence based on feature landmarks and intensity (such as image intensity or curvatures).

Model 1 : Landmark-based registration

Model : Landmark matching registration

Problem Solving :

$$\nu^* = \mathbf{argmin}_{(\nu)} \alpha \int |\nu|^p + \beta \int |\nabla \nu|^2 \tag{6}$$

subject to :

- $\nu^* = \mu(f)$;
- $\|\nu^*\|_\infty < 1$;
- $f(p_i) = q_i \rightarrow$ (Landmark constraints).

Idea

- BC ν controls smoothness & bijectivity & conformality distortion;
- Existence of minimizer ν^* is theoretically guaranteed (Lam-Lui, 2013).

Strategy :

Transform the problem :

$$(\nu^*, f^*) = \textbf{argmin}_{(\nu,f)} \alpha \int |\nu|^2 + \beta \int |\nabla \nu|^2 + \gamma_n \int |\nu - \mu(f)|^2 \qquad (7)$$

subject to :

- $||\nu||_\infty < 1$;
- $f(p_i) = q_i \rightarrow$ (Landmark constraints).

Idea :

$\Rightarrow$ Introduce auxiliary variable f

$\Rightarrow$ Alternate optimization over f and ν to decouple the minimization

Algorithm

$$\textbf{argmin}_{\nu,f} \int \alpha|\nu|^2 + \beta|\nabla\nu|^2 + +\gamma \int |\nu - \mu(f)|^2 \tag{8}$$

$$\Downarrow$$

A

Minimize (fixing f_n):

$$\mu_{n+1} = \textbf{argmin}_{\nu} \int \alpha|\nu|^2 + \beta|\nabla\nu|^2 + \int |\nu - \mu(f_n)|^2 \tag{9}$$

B

Minimize (fixing μ_{n+1}):

$$f_{n+1} = \textbf{argmin}_{f} \int |\mu_{n+1} - \mu(f)|^2 \tag{10}$$

Alternatively update μ_n and f_n using (A) and (B).

Summary of algorithm :

- Start with initial BC, ν_0 (In practice, set $\nu_0 = 0$).
- Obtain $\mu_0 = \mathbf{argmin}_\nu \alpha \int 2\nu + \beta \int |\nabla \nu|^2 + \gamma \int (\nu - \nu_0)^2$
- Use LBS to reconstruct f_0 from μ_0 satisfies landmark and boundary constraints.
- Iterate until $|\mu_{n+1} - \mu_n| \leq \epsilon$ to get the resultant map $f = f_n$.

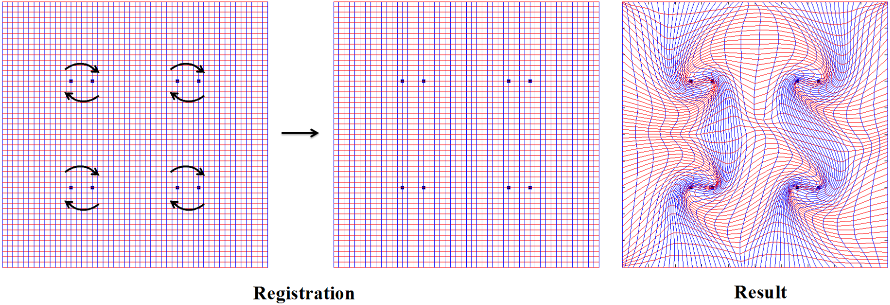
Minimization :

$$\begin{aligned}
&\mathbf{argmin}_{\nu}\alpha \int |\nu|^2 + \beta \int |\nabla \nu|^2 + \gamma \int (\nu - \nu_0)^2 \\
&\Rightarrow \alpha \int 2\nu + \beta \int \triangle \nu + \gamma \int 2(\nu - \nu_0) = 0 \\
&\Rightarrow \nu = 2\gamma(\beta\triangle + 2\alpha I + 2\gamma I)^{-1}\nu_0 \\
&\Rightarrow \text{Solve a linear system}
\end{aligned} \tag{11}$$

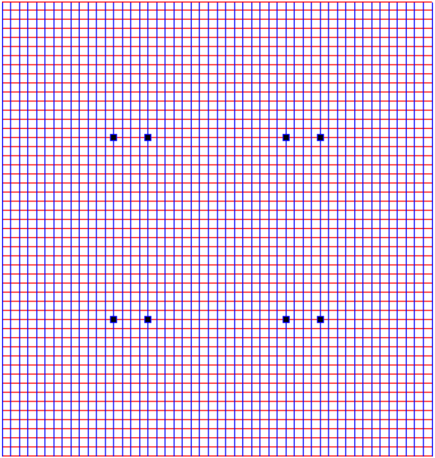
Experimental result: Landmark matching registration

Landmark matching example(1)

Matching eight points

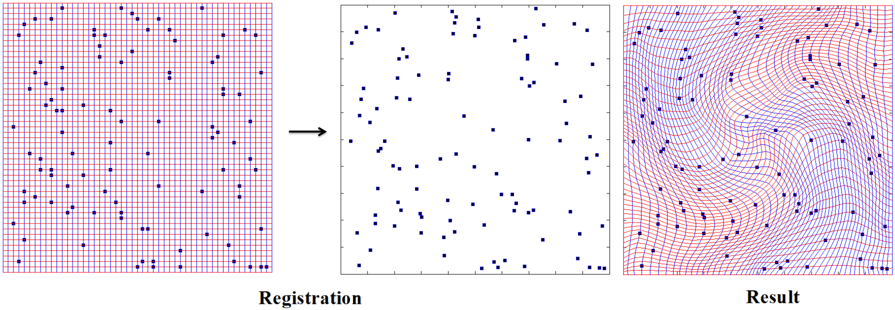


Animation

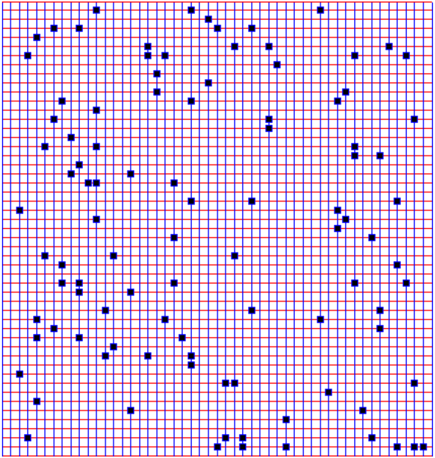


Landmark matching example(2)

Matching 100 points



Animation



Model : Extremal problem

Problem Solving :

$$\nu^* = \textbf{argmin}_{(\nu)}\{||\nu||_\infty\} \tag{12}$$

subject to :

- $\nu^* = \mu(f)$;
- $\|\nu^*\|_\infty < 1$;
- $f(p_i) = q_i \rightarrow$ (Landmark constraints).

Idea

- Replace the energy functional by L^∞-norm of the Beltrami coefficient;
- The problem is equivalent to finding the extremal or Teichmüller map.

Definition (Extremal mapping)

Let $f : S_1 \in \mathbb{C} \to S_1 \in \mathbb{C}$ be a quasi-conformal mapping between S_1 and S_2. f is said to be an extremal mapping if for any quasi-conformal mapping $h : S_1 \to S_2$ isotopic to f relative to the boundary,

$$\|\mu(f)\|_\infty \leq \|\mu(h)\|_\infty \tag{13}$$

It is uniquely extremal if the inequality is strict.

Extremal mapping = Quasi-conformal map with least conformality distortion

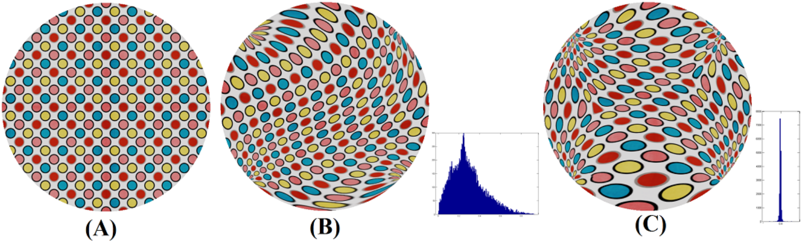
Teichmüller extremal mapping (T-map)

Definition (Teichmüller mapping)

Let $f : S_1 \in \mathbb{C} \to S_1 \in \mathbb{C}$ be a quasi-conformal mapping between S_1 and S_2. f is said to be an Teichmüller mapping associated with $\varphi : S_1 \to \mathbb{C}$ if its associated Beltrami coefficient is of the form:

$$\mu(f) = k\frac{\bar{\varphi}}{|\varphi|} \tag{14}$$

for some constant $k < 1$ and $\varphi \neq 0$.



Definition (Extremal mapping)

Let $f : S_1 \in \mathbb{C} \to S_1 \in \mathbb{C}$ be a quasi-conformal mapping between S_1 and S_2. f is said to be an extremal mapping if for any quasi-conformal mapping $h : S_1 \to S_2$ isotopic to f relative to the boundary,

$$\|\mu(f)\|_\infty \leq \|\mu(h)\|_\infty \tag{15}$$

It is uniquely extremal if the inequality is strict.

Extremal mapping = Quasi-conformal map with least conformality distortion

Relation:

Under suitable boundary cond.

($h' \neq 0$ **and** $|h''| < C$)

T-map = Unique Extremal map

Properties of T-maps:

- Minimal conformal distortion
- Uniqueness
- Bijectivity

Extended model : T-maps

Computing T-map

Problem Solving:

Find $f : S_1 \to S_2$ such that

$$\frac{\partial f}{\partial \bar{z}} = k \frac{\bar{\varphi}}{|\varphi|} \frac{\partial f}{\partial z} \tag{16}$$

and satisfies

- $0 \leq k < 1$ and $\varphi : S_1 \to \mathbb{C}$ holomorphic;
- $f|_{\partial S_1} : \partial S_1 \to \partial S_2 = g$ (Boundary condition);
- $f(p_i) = q_i$, $i = 1, ..., N$ (Landmark constraints).

Mathematical formulation of finding T-map

$$(\nu, f) = \mathbf{argmin}_{\nu: S_1 \to \mathbb{C}}\{||\nu||_\infty\} \quad (17)$$

subject to:

- $\nu = \mu(f)$;
- $\nu(f) = k\frac{\bar{\varphi}}{|\varphi|}$ where $0 \leq k < 1$ and $\varphi : S_1 \to \mathbb{C}$ holomorphic;
- $f|_{\partial S_1} = g$ (boundary condition);
- $f(p_i) = q_i$, $i = 1, ..., N$ (Landmark constraints).

Idea:

- Find a path: Initial BC $\nu_0 \to$ unique BC ν of Teichmüller type.
- Two crucial properties of a extremal T-map:
  - (P1) $\mu = k\frac{\bar{\varphi}}{|\varphi|}$, $0 \leq k < 1$, $\varphi : S_1 \to \mathbb{C}$ holomorphic.
  - (P2) $\mu = argmin_{f: S_1 \to S_2}\{\|\mu(f)\|_\infty\}$, where f satisfies the given boundary and landmark constraints.

QC-iteration

- Start with an initial map f_0 whose BC is $\nu_0 := \mu(f_0)$.
- Project ν_0 to a BC of constant norm and perform Laplace smoothing on $\arg(\nu_0) \to \mu_1$. (Property 1)
- **LBS** reconstructs f_1 from μ_1. satisfies boundary and landmark constraints.
- Iterate until f_n converges to the unique extremal T-map. (Property 2)

Projection onto Teichmüller type

Consider $\mu = k \arg(\nu_0)$,

Aim :

- $\nu_0 \to \mu := k \arg(\nu_0)$
- k decreases in each iteration

A natural choice:

$$k = \mathcal{A}(\nu_0) := \frac{\int_{S_1} |\nu_0| dS_1}{\text{Area}(S_1)} \tag{18}$$

- When $\nu_0 \neq$ Teichmüller type $\to k < \|\nu_0\|_\infty$
- When $\nu_0 =$ Teichmüller type $\to k$ unchanged.

Holomorphic property of φ

Aim : Optimal ν : $\arg(\nu) = \frac{\bar{\varphi}}{|\varphi|}$

Consider: $\varphi = |\varphi| e^{i\theta}$ and $\nu = |\nu| e^{i\theta}$

$$\log \varphi = \log|\varphi| + i\theta \Rightarrow \theta \text{ harmonic} \Rightarrow \triangle\theta = 0$$

$\Rightarrow$ Laplace smoothing on θ in each iteration

$$\mathfrak{L}(\nu_0)(T) = |\nu_0(T)| \left(\sum_{T_i \in \text{Nbhd}(T)} \frac{\arg(\nu_0(T_i))}{|\text{Nbhd}(T)|} \right) \tag{19}$$

Optimal state:

θ harmonic
$\rightarrow \zeta$ harmonic conjugate of θ
$\rightarrow \varphi = e^{\zeta - i\theta}$.

Convergence of the QC-iteration

Theorem (Convergence of the QC iteration, Lui-Gu-Yau, 2013)

The QC iteration gives a convergent sequence of pairs (f_n, ν_n), *where* ν_n *is the Beltrami coefficient of* f_n, *whose limit point is* (f^*, ν^*). *Here,* ν^* *is the unique admissible Beltrami coefficient of Teichmüller type associated with the extremal Teichmüller map.*

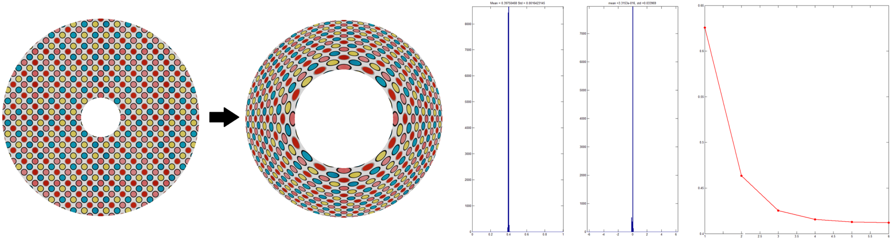
K.C. Lam, X.F. Gu, S.T. Yau, L.M. Lui, "Teichmuller mapping (T-Map) and its applications to landmark matching registrations", SIAM Journal on Imaging Sciences (2013)

Experimental result: T-Maps

T-maps example(1)

Analytic example : Ring with inner and outer radii r_0 and r_1 to ring with inner and outer radii r'_0 and r'_1.

Corresponding maximal dilation $K = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty} = \frac{\ln\left(r'_1/r'_0\right)}{\ln(r_1/r_0)}$

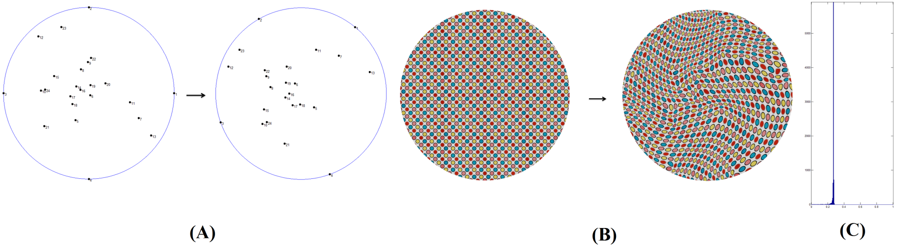


(A) (B) (C) (D)

Take $r_0 = r'_0 = 1$ and $r_1 = 0.2$, $r'_1 = 0.5$, error of $K = 0.0013219$.

T-maps example(2)

Example 2 : 4-point boundary constraints + 24 interior landmark constraints enforced.

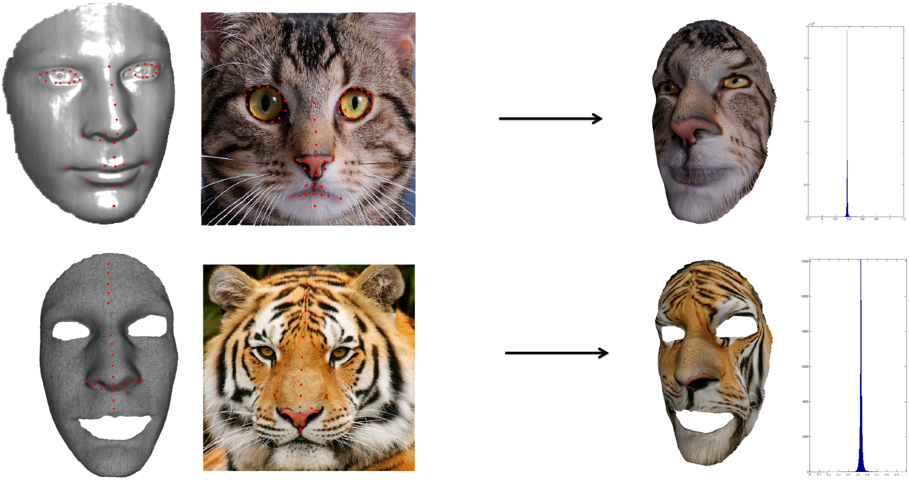


- (A) 24 landmark constraints
- (B) T-map
- (C) Histogram : $|\mu|$ of T-map

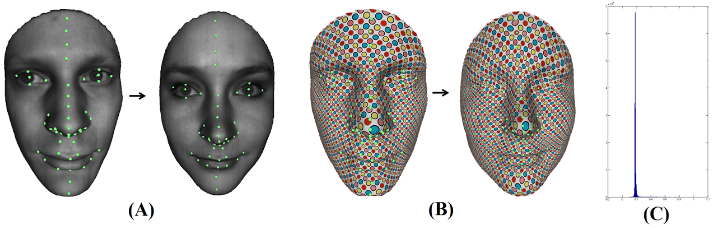
Applications

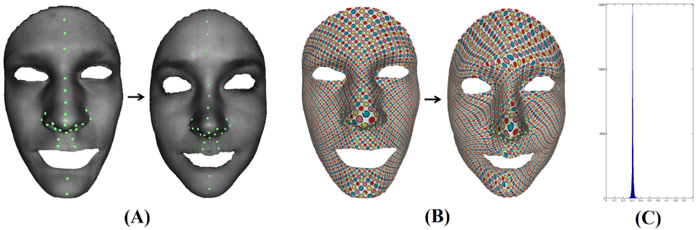
Constrained texture mapping

Constrained texture mapping

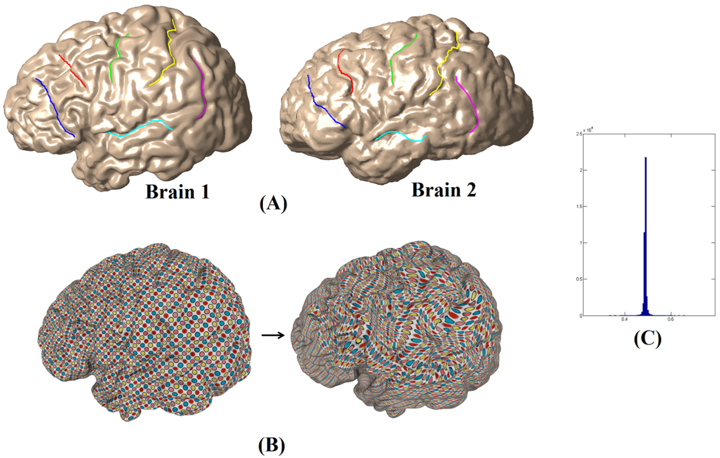


Face registration





Medical registration : Brain



Model 2 : Landmark + intensity registration

Model : Landmark + intensity registration

Problem Solving :

$$(\nu^*, f^*) = \mathbf{argmin}_{(\nu,f)} \alpha \int |\nu|^p + \beta \int |\nabla \nu|^2 + \gamma \int (I_1 - I_2(f))^2 \tag{20}$$

subject to :

- $\nu^* = \mu(f^*)$;
- $\|\nu^*\|_\infty < 1$;
- $f^*(p_i) = q_i \;\; \rightarrow$ (Landmark constraints).

Idea

- BC μ controls smoothness & bijectivity & conformality distortion;
- $\int (I_1 - I_2(f))^2$ controls the sum of squared difference (SSD) of the intensity difference in the two data.

Algorithm

Use penalty method to minimize :

$$(\nu^*, \mu^*) = \mathbf{argmin}_{\mu,\nu} \alpha \int |\nu|^2 + \beta |\triangle \nu|^2 + \frac{1}{\sigma^2} |\nu - \mu|^2 + \gamma \int (I_1 - I_2(f^\mu))^2 \tag{21}$$

subject to :

- $\|\nu^*\|_\infty < 1$;
- $f^{\mu^*}(p_i) = q_i \rightarrow$ (Landmark constraints).

Idea :

⇒ Introduce auxiliary variable μ

⇒ Alternate optimization over μ and ν to decouple the minimization

Algorithm

$$\textbf{argmin}_{\mu,\nu} \int \alpha|\nu|^2 + \beta|\nabla\nu|^2 + \frac{1}{\sigma^2}|\nu-\mu|^2 + \int \gamma(I_1 - I_2(f^\mu))^2 \quad (22)$$

$$\Downarrow$$

A

Minimize :

$$\mu_{n+1} = \textbf{argmin}_{\mu}\gamma \int (I_1 - I_2(f^\mu))^2 + \frac{1}{\sigma_n^2}|\mu - \nu_n|^2 \quad (23)$$

B

Minimize :

$$\nu_{n+1} = \textbf{argmin}_{\nu} \int \alpha|\nu|^2 + \beta|\nabla\nu|^2 + \frac{1}{\sigma_n^2}|\nu - \mu_{n+1}|^2 \quad (24)$$

Alternatively update μ_n and ν_n using (A) and (B).

Minimization A

Strategy : Gradient descent

Note that :

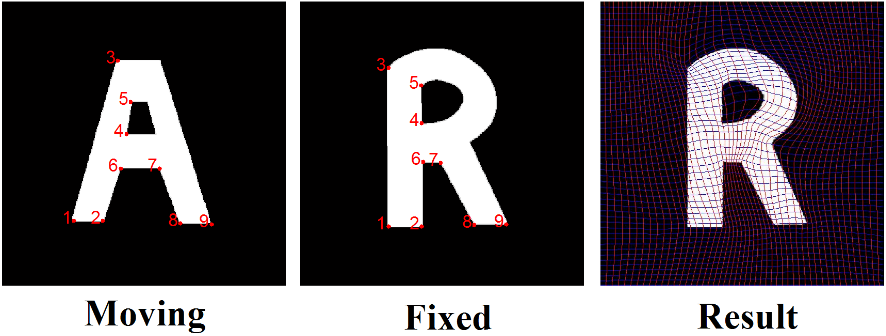
$$
\begin{aligned}
\frac{\partial(f+df)}{\partial\bar{z}} &= (\mu+d\mu)\frac{\partial(f+df)}{\partial z} \\
\Rightarrow \frac{\partial f}{\partial\bar{z}}+\frac{\partial df}{\partial\bar{z}} &= \mu\frac{\partial f}{\partial z}+d\mu\frac{\partial df}{\partial z}+\mu\frac{\partial df}{\partial z}+d\mu\frac{\partial df}{\partial z} \\
\Rightarrow d\mu &= \frac{\frac{\partial df}{\partial\bar{z}}-\mu\frac{\partial df}{\partial z}}{\frac{\partial f}{\partial z}}
\end{aligned}
\tag{25}
$$

Gradient descent direction of BC from $\int(I_1 - I_2(f^{\mu}))^2$ can be obtained.

Gradient descent direction of BC from $|\mu - \nu_n|^2$ is $2(\mu - \nu_n)|$.

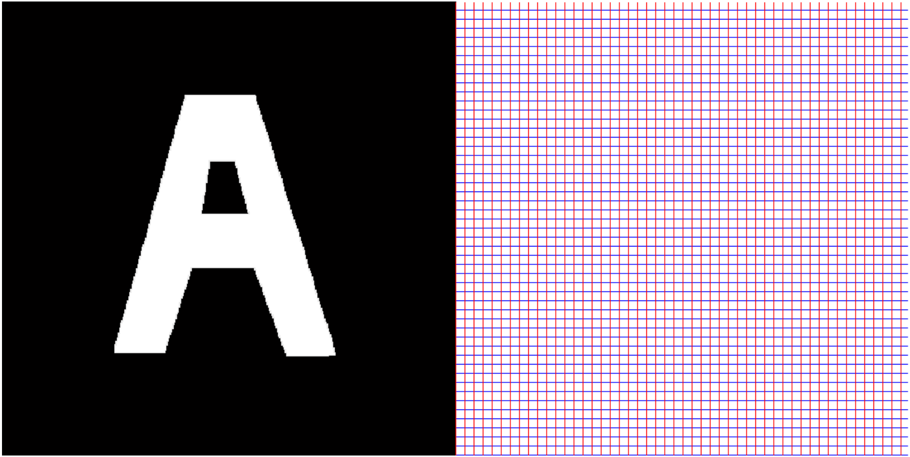
Synthetic example(1)

Alphabet "A" to alphabet "R"



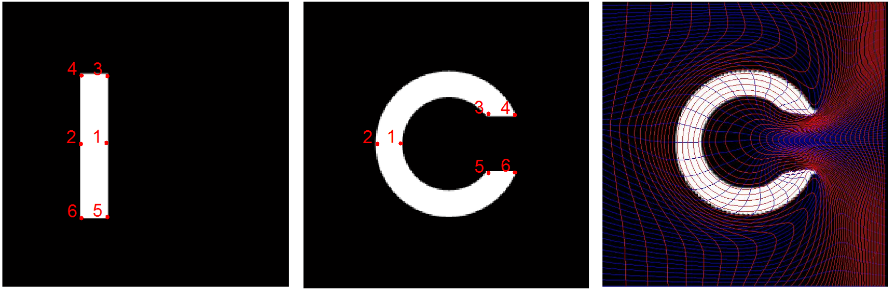
Synthetic example(1)

Animation



Synthetic example(2)

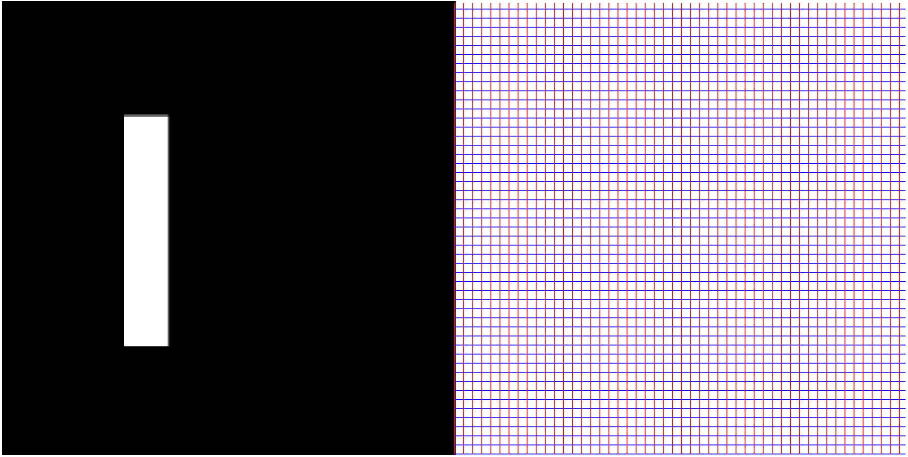
Alphabet "I" to alphabet "C"



Moving **Fixed** **Result**

Synthetic example(2)

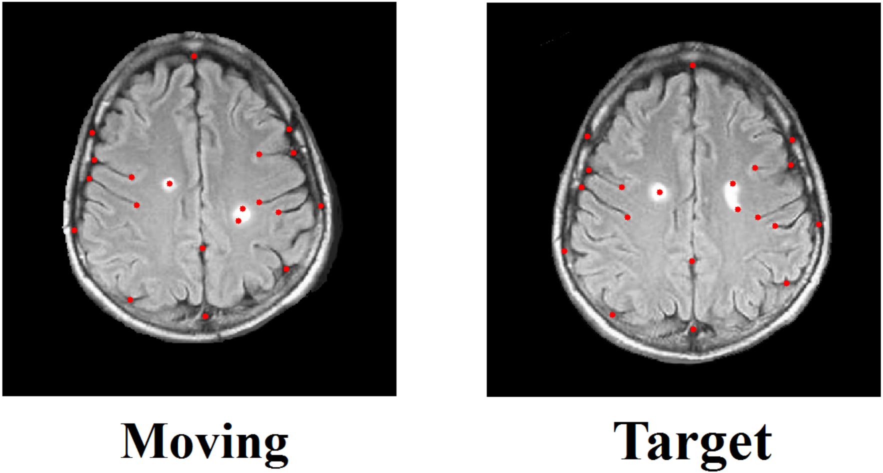
Animation



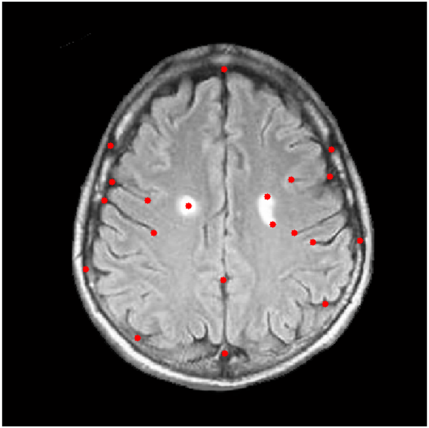
Applications

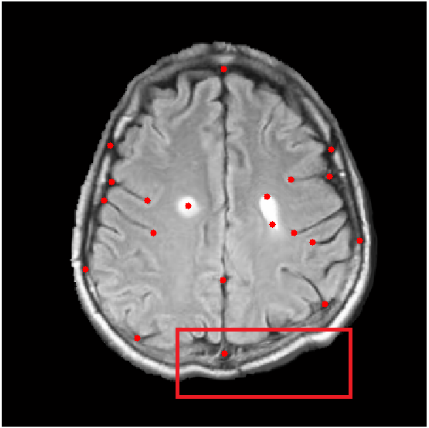
Image registration : Brain MRI

Registration problem :



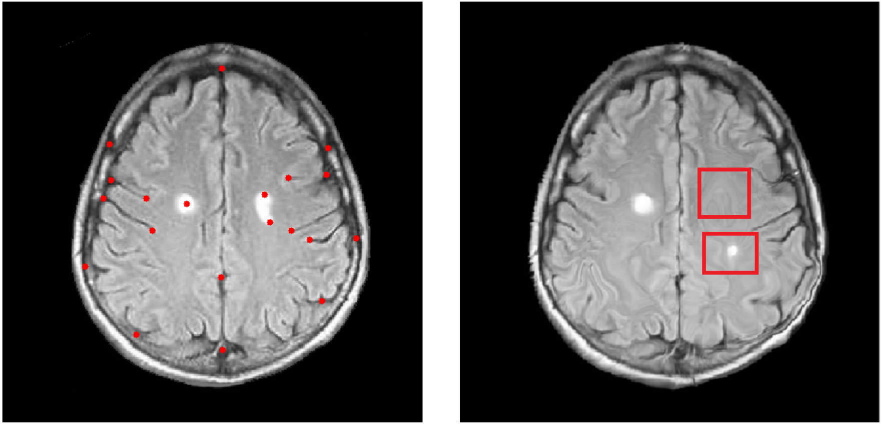
Landmark only :





Target **Landmark**

Intensity only :



Target

Intensity

Landmark + intensity only :

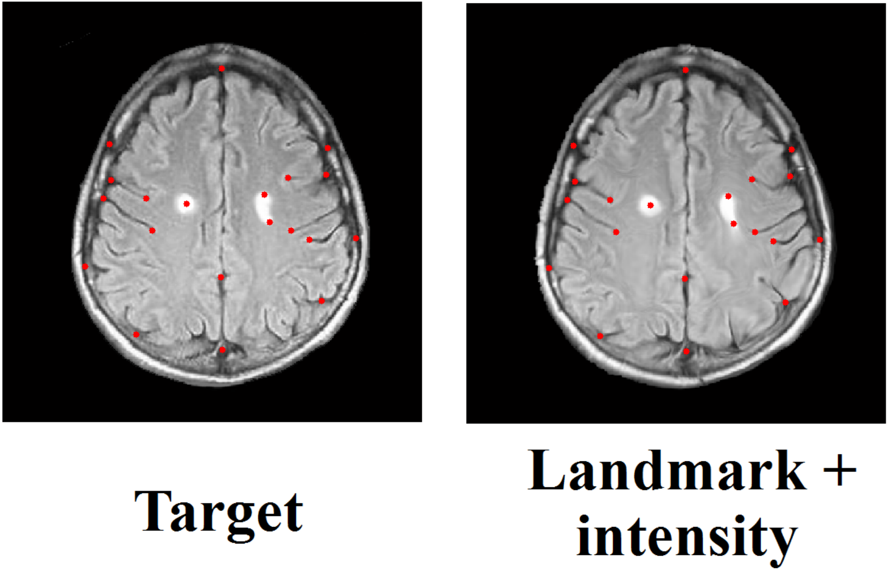
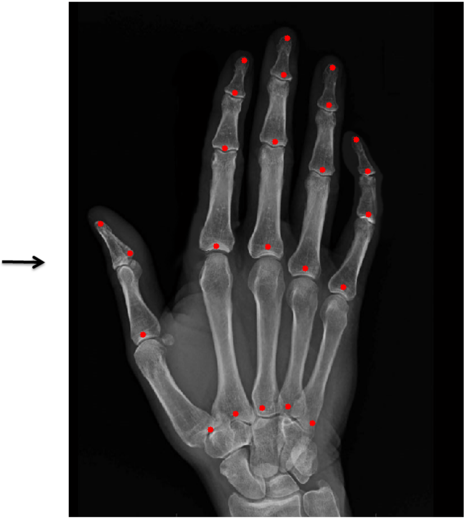


Image registration : X-ray bone



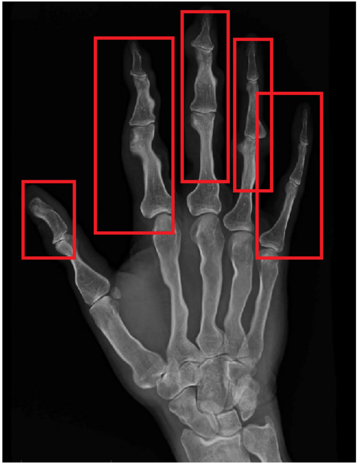
Landmark only :


Target


Landmark

Intensity only :

Target



Intensity

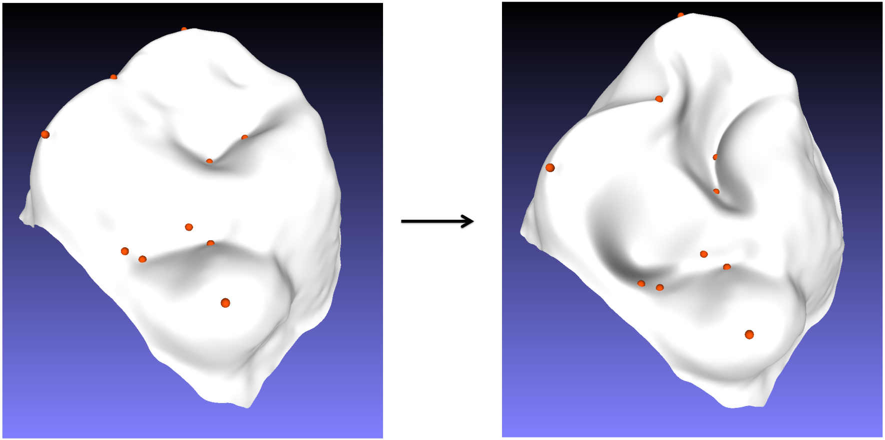
Landmark + Intensity :

Target

Intensity + Landmark

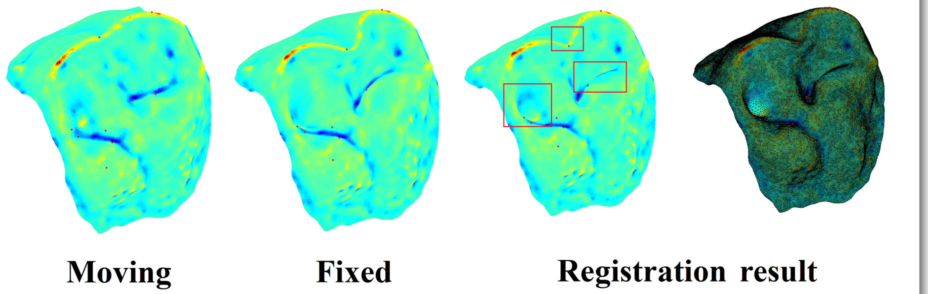
Surface registration : Teeth

Teeth registration

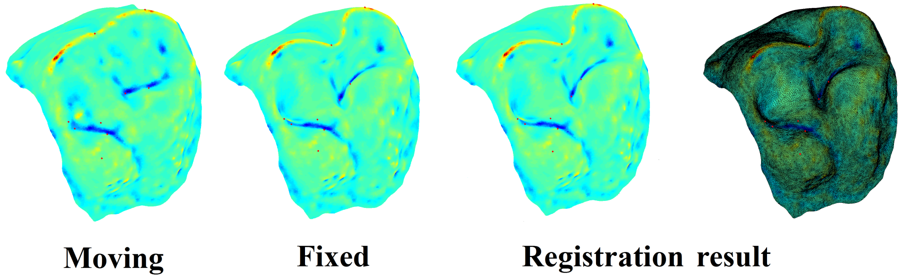


Teeth 1 → Teeth 2

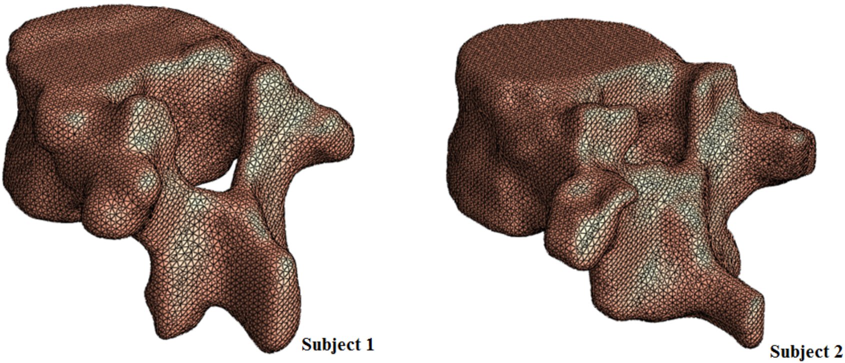
Landmark matching only



Landmark + intensity matching

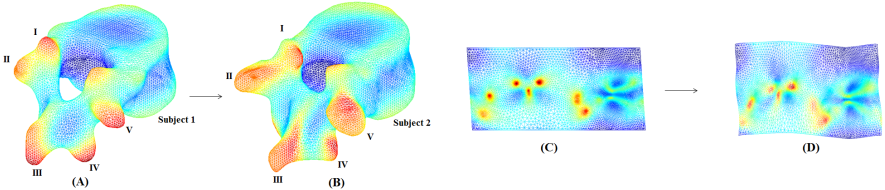


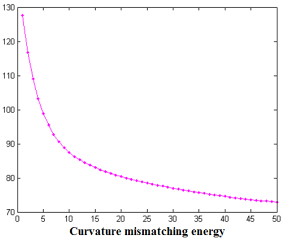
Medical registration : Vertebral bone

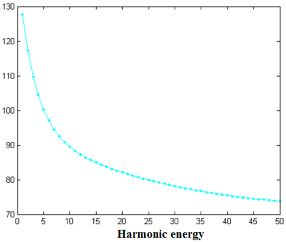


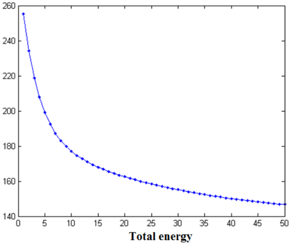
(A) (B)

Registration result:

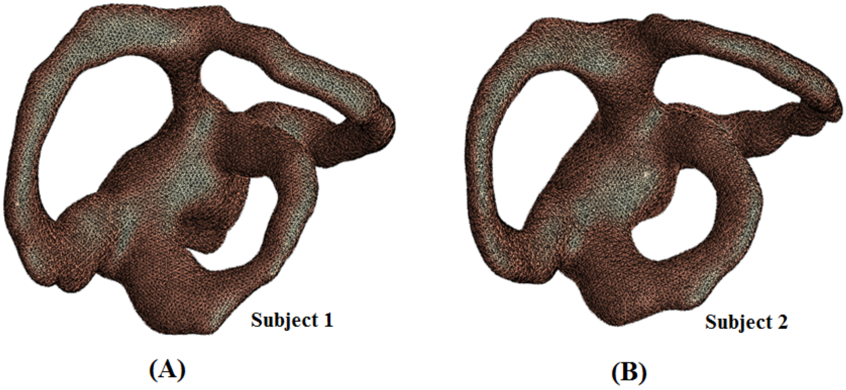


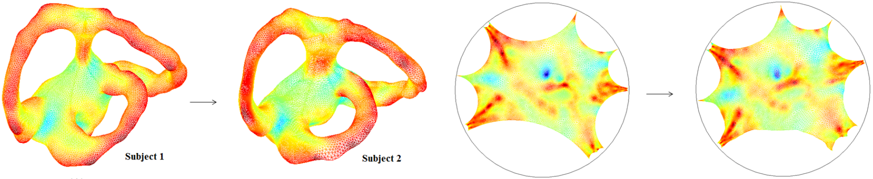




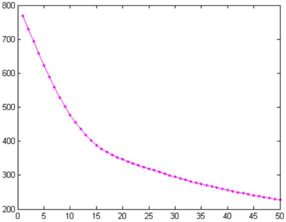


Medical registration : Vestibular system

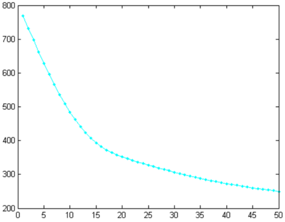




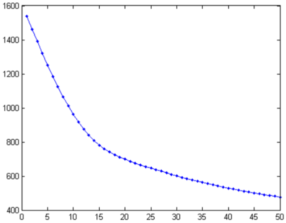
(A) (B) (C) (D)



Curvature mistmatching energy vesus iterations



Harmonic energy versus iterations



Total energy versus iterations

Conclusion

- Propose Landmark registration algorithm.
- Extend to obtain T-maps and show further application.
- Propose Landmark + intensity registration algorithm.
- Extension to high-genus surface registration and show further application.

Thank You