Manifold T-spline

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Geometric Modeling and Processing 2006

Erlangen Program - F. Klein 1872

Different geometries study the invariants under different transformation groups.

- Euclidean Geometry : Rigid motion on $\mathbb{R}^2$. Distances between arbitrary two points are the invariants.
- Affine Geometry: Affine transformations. Parallelism and barry centric coordinates are the invariants.
- Real Projective Geometry: Real projective transformations. Collinearity and cross ratios are the invariants.

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Algorithms vs. Geometries

Central Problem

- Can different geometries be defined on general surfaces?
- Can different planar algorithms be generalized to surface domains directly?

The answers are yes and yes. The major theoretic tool is the *Geometric Structure.*

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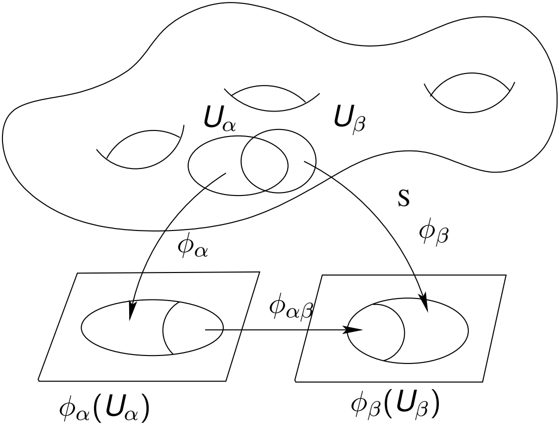
Main Ideas

Geometry Structure

A surface is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Manifold


U_α
U_β
S
ϕ_β
ϕ_α
$\phi_{\alpha\beta}$
$\phi_\alpha(U_\alpha)$
$\phi_\beta(U_\beta)$

Manifold

Definition (Manifold)

A manifold is a topological space Σ covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a chart of Σ, the set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of Σ. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

Transition maps satisfy cocycle condition, suppose $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then

$$\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}.$$

(X, G) structure

Definition ((X,G) Atlas)

Suppose X is a topological space, G is the transformation group of X. A manifold Σ with an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is an *(X, G) atlas* if

1. $\phi_\alpha(U_\alpha) \subset X$, for all charts (U_α, ϕ_α).
2. Transition maps $\phi_{\alpha\beta} \in G$.

Definition (Equivalent (X, G) atlases)

Two (X, G) atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of Σ are *equivalent*, if their union is still an (X, G) atlas of Σ.

Definition ((X,G) structure)

An *(X, G) structure* of a manifold Σ is an equivalent class of its (X, G) atlases.

(X,G) structure

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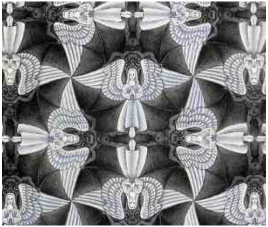
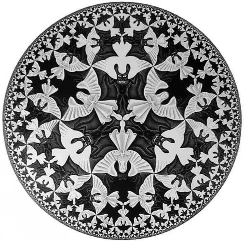
M.C.Esher's art works: Angels and Devils



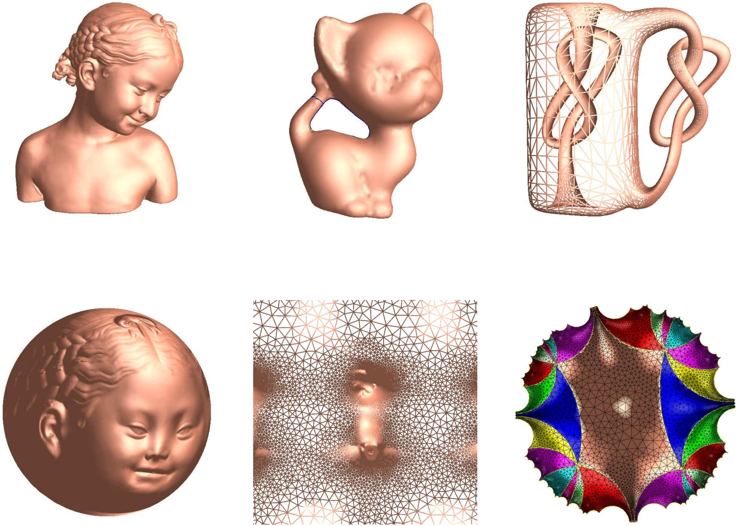
Regular divisin
of the plane

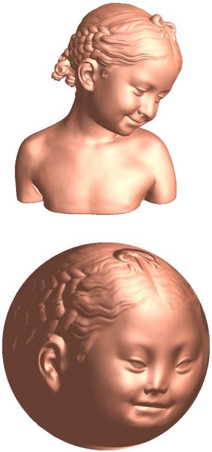
Sphere with Angels
and Devils



Circle limit IV
Heaven and Hell

Geometries defined on surfaces

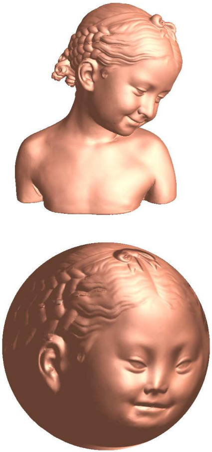




Spherical Structure

- X: Unit sphere $\mathbb{S}^2$.
- G: Rotation group.
- Surfaces: Genus zero closed surfaces; any open surfaces.
- Harmonic maps.

Common (X,G) structure



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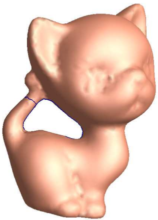

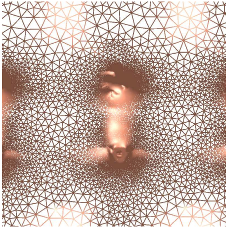

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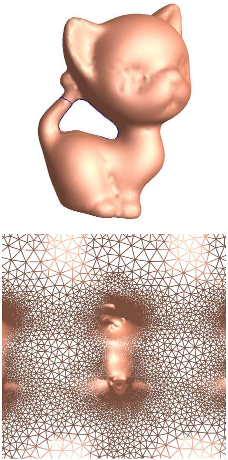
- Holomorphic 1-forms.



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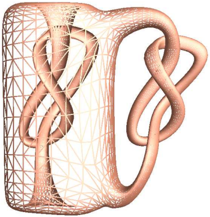
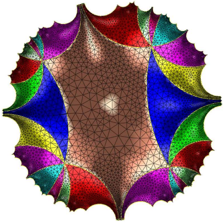
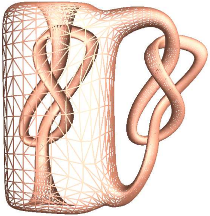


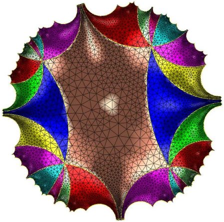


Hyperbolic Structure

- *X*: Hyperbolic plane $\mathbb{H}^2$.
- *G*: Möbius transformation group.
- Surfaces: with negative Euler number.
- Hyperbolic Ricci flow

Common (X,G) structure





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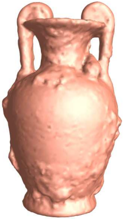

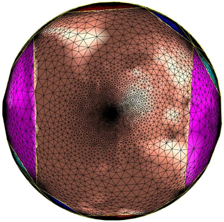

Real Projective Structure

- X: Real projective plane $\mathbb{RP}^2$.
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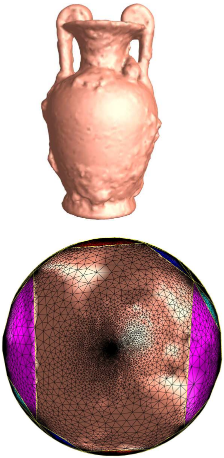
Common (X,G) structure



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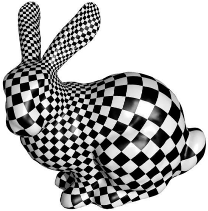
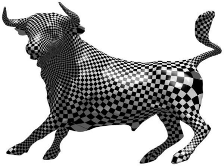
Pseudo (X,G) structure

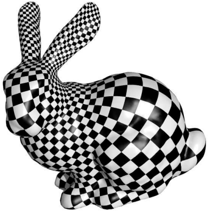


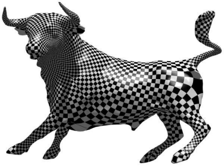


Conformal Structure

- *X*: Complex plane $\mathbb{C}$.
- *G*: Biholomorphic maps.
- Surfaces: any surface.
- Holomorphic 1-forms

Pseudo (X,G) structure





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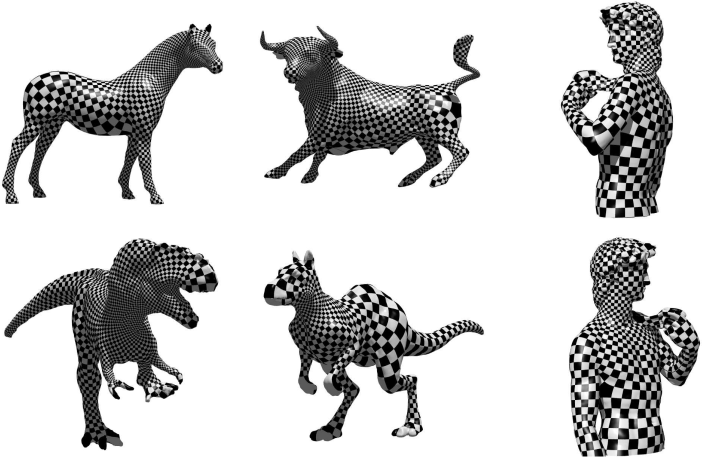


(X,G) Invariant Algorithms

Definition ((X,G) invariant Algorithm)

Suppose X is a topological space, G is the transformation group on X. A geometric operator Ω defined on X is (X, G) invariant, if and only if

$$\Omega \circ g = g \circ \Omega, \forall g \in G.$$

Examples:

- Convex Hull: Projective invariant.
- Voronoi Diagram: Rigid motion invariant.
- Polar form : Affine invariant.

(X,G) structure

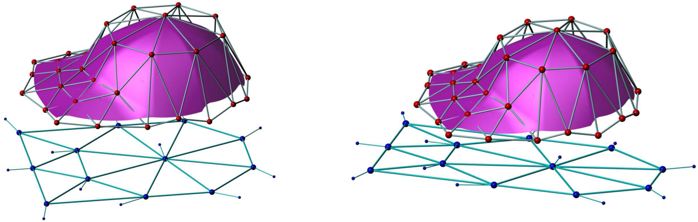
Theorem

Suppose a manifold with an (X, G) structure, then any (X, G) invariant algorithms can be generalized on the manifold.

Corollary (Manifold Splines - Gu,He,Qin 2005)

Spline schemes based on polar forms can be defined on a manifold, if and only if the manifold has an affine structure.

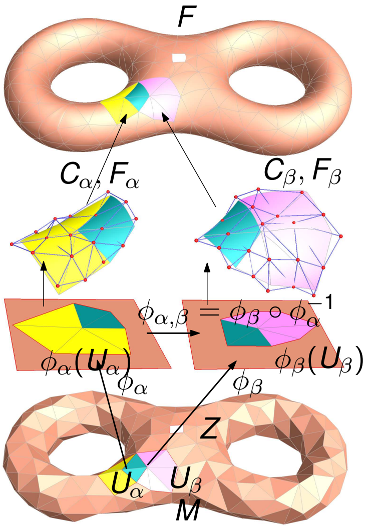
Planar Splines



Parametric Affine Invariant

The spline is invariants under the affine transformations of the knots and the parameters.

Manifold SPlines



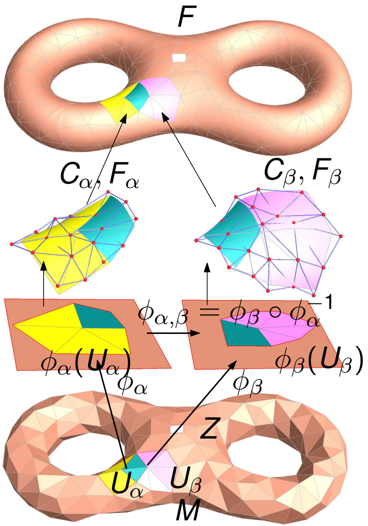
Idea: Geometry Structure

A mesh is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Global Parameterization

Find atlas with special transition functions.

Manifold SPlines



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Topological Obstructions

Theorem (Benzécri 1959)

If a closed surface admits an affine structure, it has zero Euler class.

Real projective structure

- Real projective structure is general, it exists for all surfaces.
- Real projective structure is simple, all transitions are linear rational functions.
- Real projective structure is suitable for designing manifold spline schemes.

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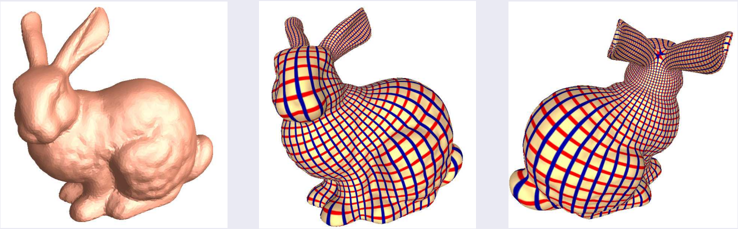
Conformal Structure

Global Tensor Product Structure



Definition (Holomorphic 1-form)

Suppose Σ is a Riemann surface, $\{z_\alpha\}$ is a local complex parameter, a holomorphic 1-form ω has a local representation as

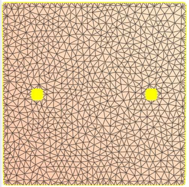
$$\omega = f(z_\alpha) dz_\alpha,$$

where $f(z_\alpha)$ is a holomorphic function.

Locally, ω is the derivative of a holomorphic function. Globally, it is not.

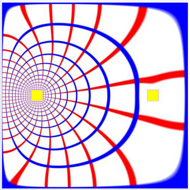
Holomorphic 1-forms

Original Surface



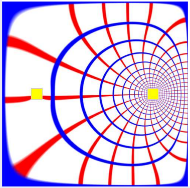
Holomorphic 1-forms

One basis holomorphic 1-form



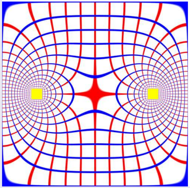
Holomorphic 1-forms

Another one basis holomorphic 1-form



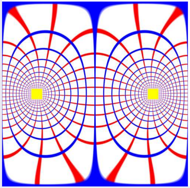
Holomorphic 1-forms

Summation of ω_1 and ω_2



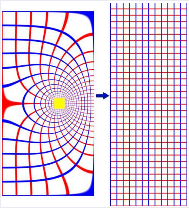
Holomorphic 1-forms

Difference between ω_1 and ω_2



Holomorphic 1-forms

Holomorphic 1-form induces a conformal parameterization.



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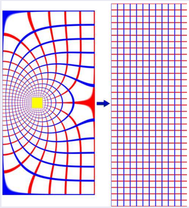
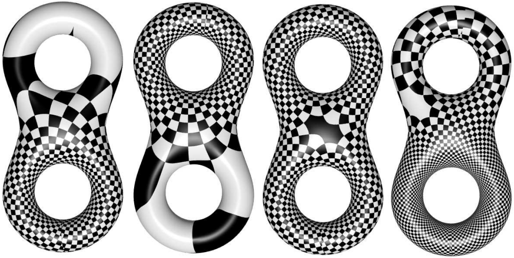


Holomorphic 1-forms

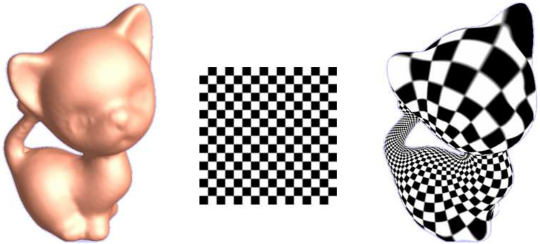
Theorem (Holomorphic 1-forms)

All holomorphic 1-forms form a linear space $\Omega(\Sigma)$ *which is isomorphic to the first cohomology group* $H^1(\Sigma, \mathbb{R})$*.*



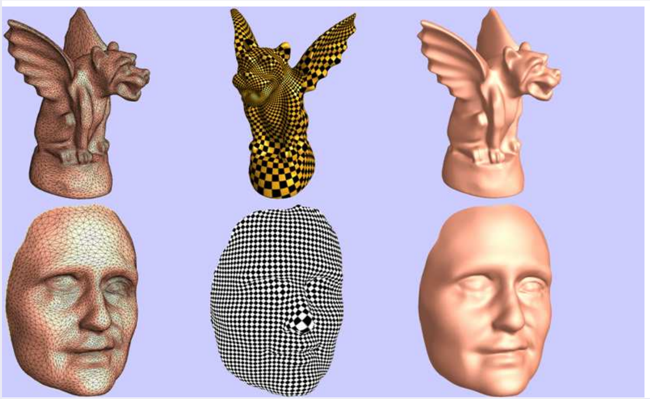
Theorem

Holomorphic 1-form induces affine structure By integrating a holomorphic 1-form, local coordinate charts can be established. The charts covers the surface without the singularises, the transition maps are translations.



Manifold Splines SPM2005

Manifold Powell-Sabin Spline



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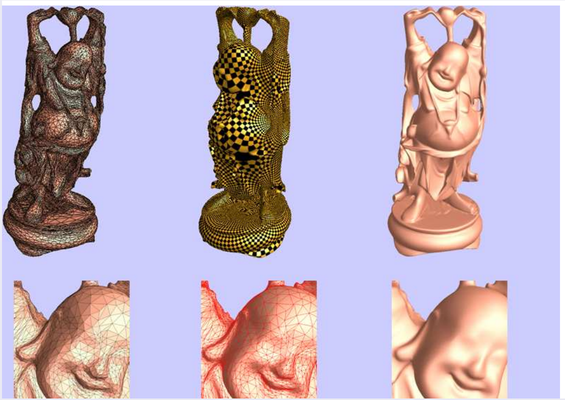
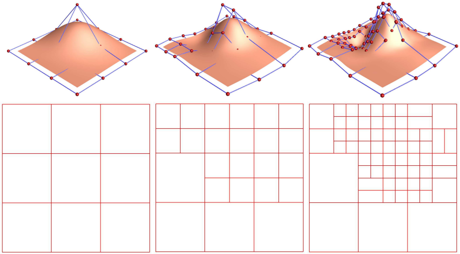
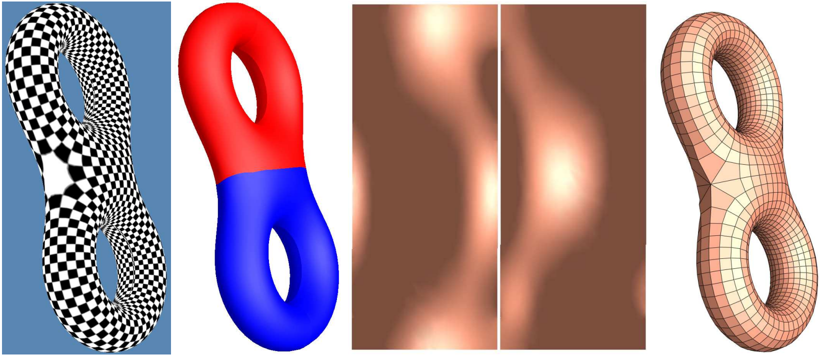


T-spline

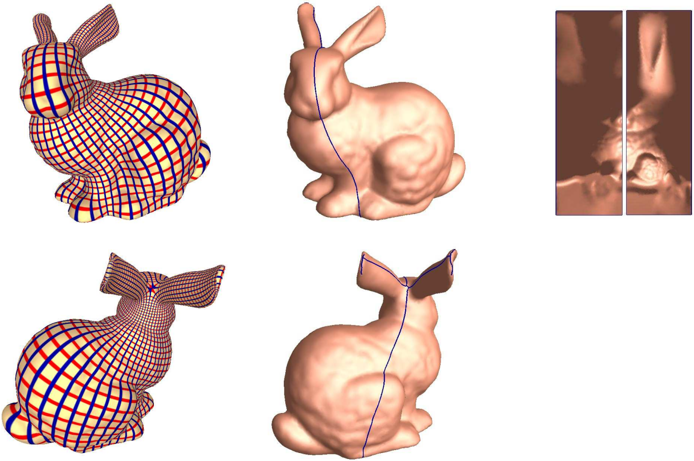
- T-spline is the superset of tensor-product B-spline and industry-standard NURBS
- Allow T-junction in parametric domain and control net
- Natural hierarchical structure
- Much more flexible than NURBS!



Critical Graph



Critical Graph and Local Charts



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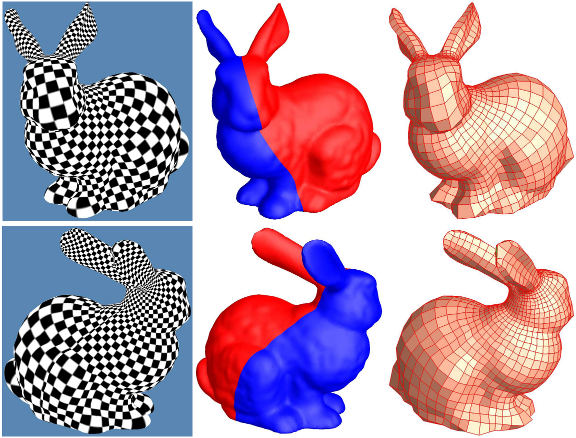


Given the domain manifold M with conformal structure $\phi : M \to \mathbb{R}^2$, the manifold T-spline can be formulated as follows:

$$\mathbf{F}(\mathbf{u}) = \sum_{i=1}^{n} \mathbf{C}_i B_i(\phi(\mathbf{u})), \ \mathbf{u} \in M, \tag{1}$$

where B_is are basis functions and $\mathbf{C}_i = (x_i, y_i, z_i, w_i)$ are control points in $\mathbb{P}^4$ whose weights are w_i, and whose Cartesian coordinates are $\frac{1}{w_i}(x_i, y_i, z_i)$. The cartesian coordinates of points on the surface are given by

$$\frac{\sum_{i=1}^{n}(x_i, y_i, z_i) B_i(\phi(\mathbf{u}))}{\sum_{i=1}^{n} w_i B_i(\phi(\mathbf{u}))}. \tag{2}$$

Given a parameter $\mathbf{u} \in M$, the evaluation can be carried out on arbitrary charts covering $\mathbf{u}$.

Hierarchical Surface Reconstruction

Minimize a linear combination of interpolation and fairness functionals,

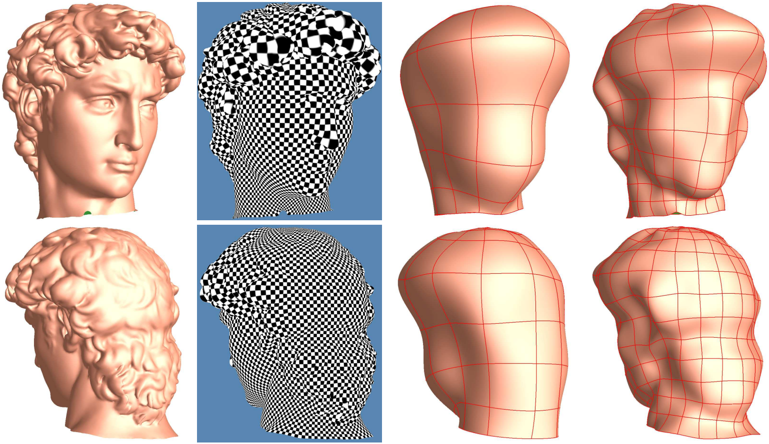
$$\min E = E_{dist} + \lambda E_{fair}, \qquad (3)$$

where

$$E_{dist} = \sum_{i=1}^{m} \|\mathbf{F}(\mathbf{u}_i) - \mathbf{p}_i\|^2$$

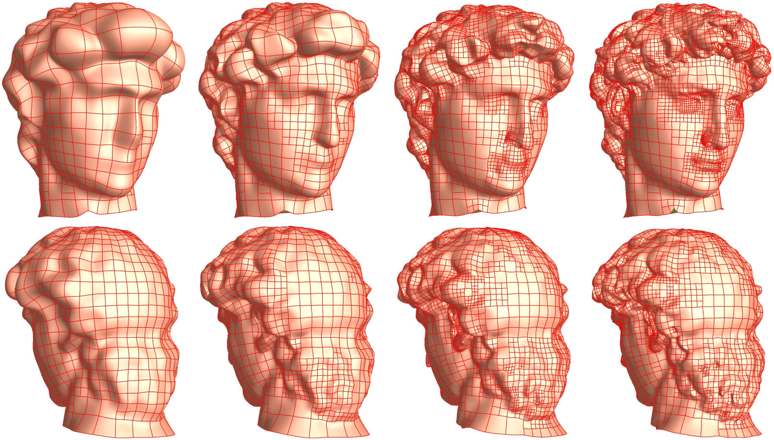
and E_{fair} in (3) is a smoothing term.

Hierarchical Surface Reconstruction



P, $N_v = 200K$ | Conformal structure | $N_c^1 = 105$, $L_\infty^1 = 9.6\%$ | $N_c^2 = 295$, $L_\infty^2 = 5.7\%$

Hierarchical Surface Reconstruction



$N_c^3 = 950$ $L_\infty^3 = 3.8\%$

$N_c^4 = 2130$ $L_\infty^4 = 2.4\%$

$N_c^5 = 5087$ $L_\infty^5 = 1.3\%$

$N_c^6 = 7706$ $L_\infty^6 = 0.74\%$

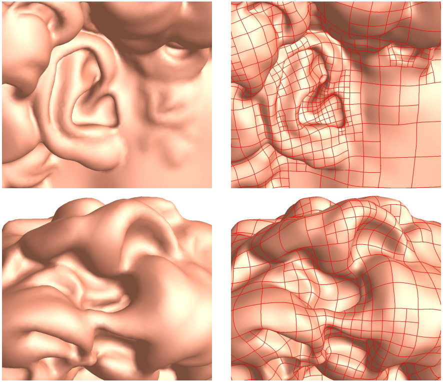
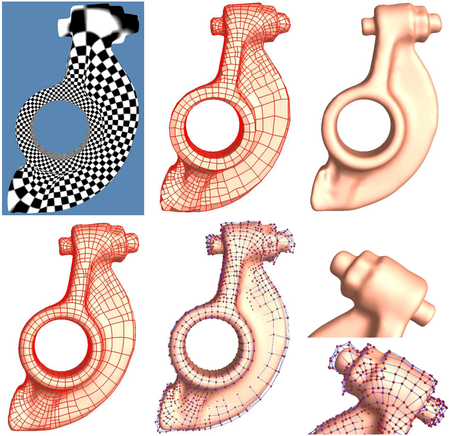


Figure: Close-up view of the reconstructed details

Manifold T-spline Examples



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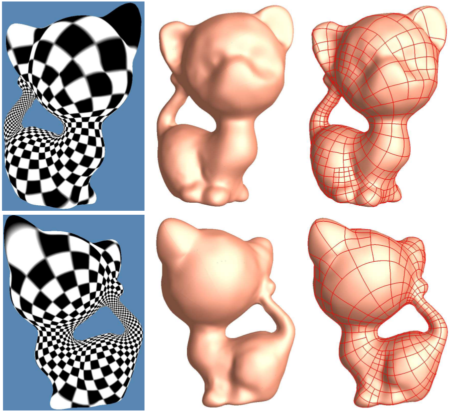

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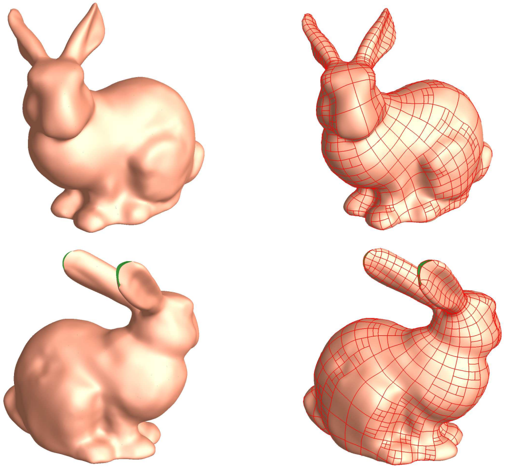

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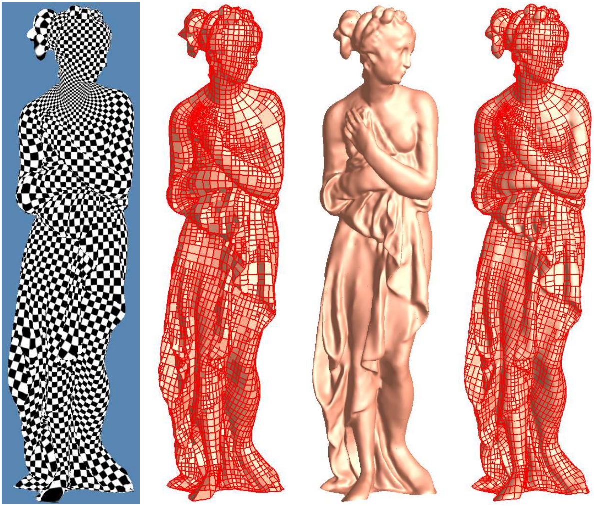


Manifold T-spline Examples

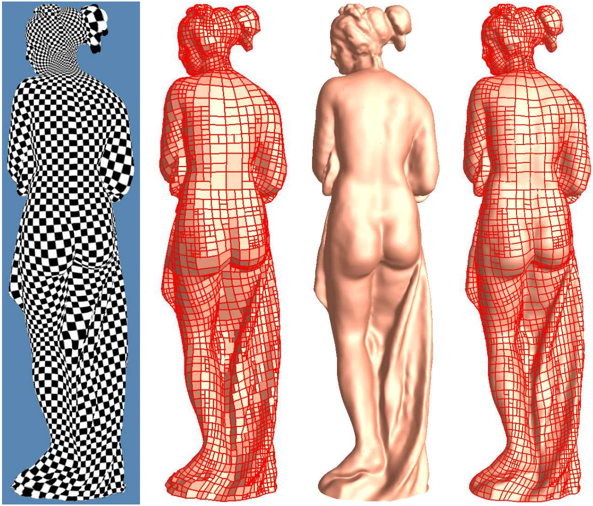


Table: Statistics of test cases. N_p, # of points in the polygonal mesh; N_c, # of control points; *rms*, root-mean-square error; L_∞, maximal error. The execution time measures in minutes.

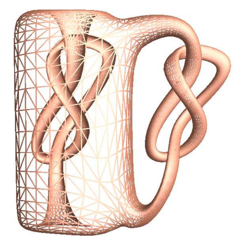
Object	N_p	N_c	*rms*	L_∞	Time
David	200,000	7,706	0.08%	0.74%	39m
Bunny	34,000	1,304	0.09%	0.81%	18m
Iphegenia	150,000	9,907	0.06%	0.46%	53m
Rocker Arm	50,000	2,121	0.04%	0.36%	26m
Kitten	40,000	765	0.05%	0.44%	12m

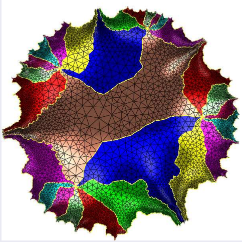
Future Directions

- Manifold Splines with single singularity.
- Manifold Splines which are polynomials everywhere with C^k continuity.
- Planar splines based on projective invariants.

Thanks

For more information, please email to gu@cs.sunysb.edu.





Thank you!