Computing Surface Hyperbolic Structure and Real Projective Structures

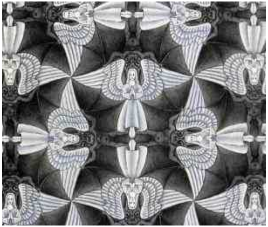
Miao Jin[1] Feng Luo[2] Xianfeng (David) Gu[1]

[1]Department of Computer Science
Stony Brook University

[2]Department of Mathematics
Rutgers University

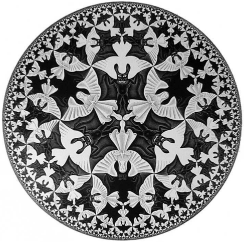
ACM Solid and Physical Modeling Symposium, 2006

M.C.Esher's art works: Angels and Devils



Regular divisin
of the plane

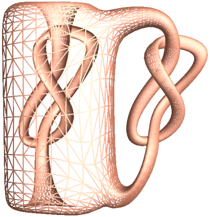
Sphere with Angels
and Devils

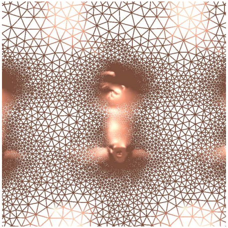
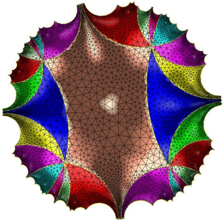


Circle limit IV
Heaven and Hell

Geometries defined on surfaces





Motivation

Main Goals

- Define different geometries on surfaces.
- Systematically generalize planar algorithms to surfaces.

Example: Generalize planar spline schemes to surfaces.

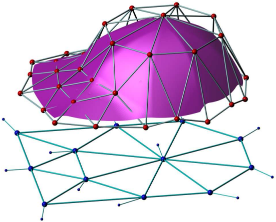
Motivation

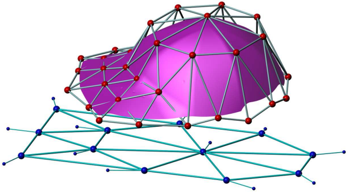
Main Goals

- Define different geometries on surfaces.
- Systematically generalize planar algorithms to surfaces.

Example: Generalize planar spline schemes to surfaces.

Planar Splines

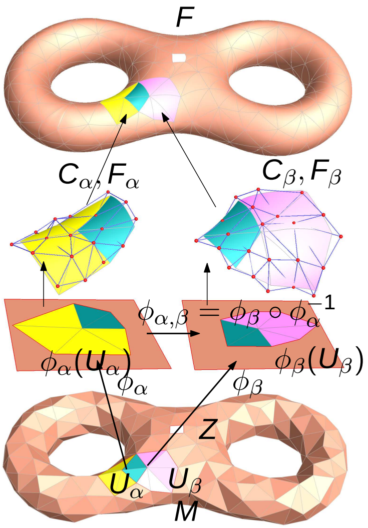




Parametric Affine Invariant

The spline is invariants under the affine transformations of the knots and the parameters.

Manifold SPlines



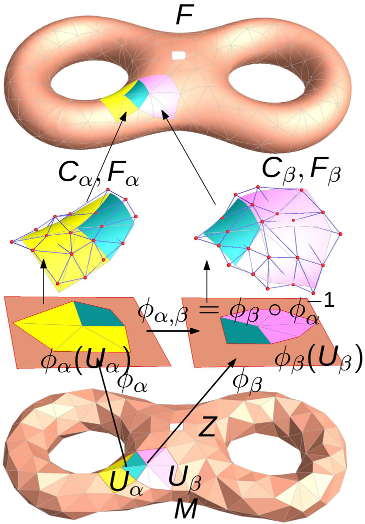
Idea: Geometry Structure

A mesh is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Global Parameterization

Find atlas with special transition functions.

Manifold SPlines



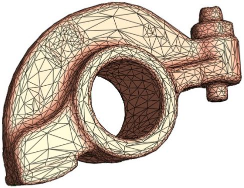
Idea: Geometry Structure

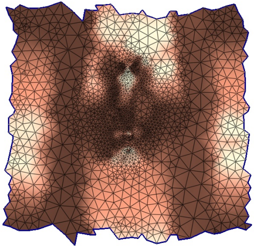
A mesh is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Global Parameterization

Find atlas with special transition functions.

Parameterizations as Finding Metrics

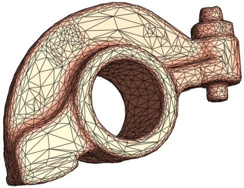


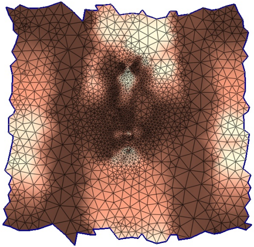


Idea: Flat Metrics

A planar parameterization of a mesh is equivalent to find the edge lengths such that the sum of surrounding angles for each vertex is 2π.

Parameterizations as Finding Metrics

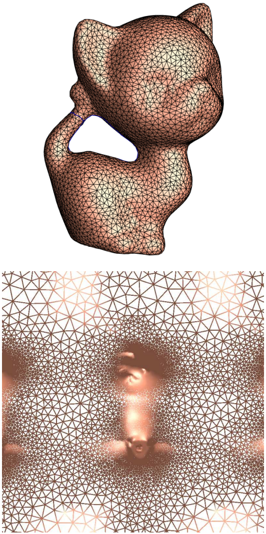




Idea: Flat Metrics

A planar parameterization of a mesh is equivalent to find the edge lengths such that the sum of surrounding angles for each vertex is 2π.

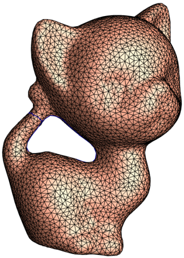
Ricci Flow

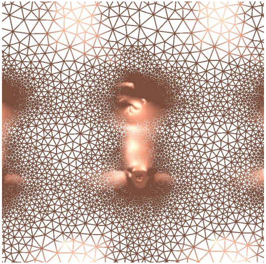


Idea: Ricci Flow

Conformally adjust the edge lengths; the deformation of the edge lengths is driven by the current curvature.

Ricci Flow





Idea: Ricci Flow

Conformally adjust the edge lengths; the deformation of the edge lengths is driven by the current curvature.

Main Ideas

Geometry Structure

A surface is covered by local coordinate charts. Geometric construction is invariant during the transition from one local coordinate to another.

Flat Metrics

Global parameterizations is formulated as finding flat metrics.

Ricci Flow

The conformal deformation of the edge lengths is driven by the curvature.

Erlangen Program - F. Klein 1872

Different geometries study the invariants under different transformation groups.

- Euclidean Geometry : Rigid motion on $\mathbb{R}^2$. Distances between arbitrary two points are the invariants.
- Affine Geometry: Affine transformations. Parallelism and barry centric coordinates are the invariants.
- Real Projective Geometry: Real projective transformations. Collinearity and cross ratios are the invariants.

Delaunay Triangulation: Euclidean geometry.

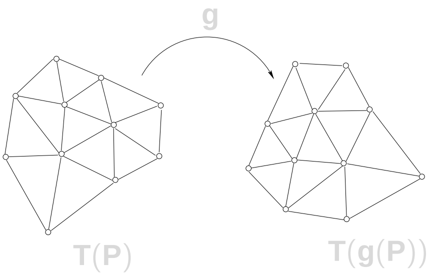
Key concept used in Delaunay triangulation: distance.

Suppose $\mathbf{P}$ is a planar point set, $\mathbf{T}(\mathbf{P})$ is its Delaunay triangulation,

$$\mathbf{P} = \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n\}.$$

Let $\mathbf{g}: \mathbb{R}^2 \to \mathbb{R}^2$ be a planar rigid motion, then,

$$\mathbf{g}(\mathbf{T}(\mathbf{P})) = \mathbf{T}(\mathbf{g}(\mathbf{P})).$$



Delaunay Triangulation: Euclidean geometry.

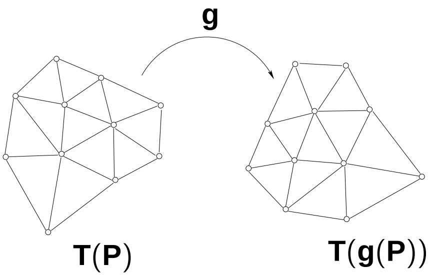
Key concept used in Delaunay triangulation: distance.

Suppose $\mathbf{P}$ is a planar point set, $\mathbf{T}(\mathbf{P})$ is its Delaunay triangulation,

$$\mathbf{P} = \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n\}.$$

Let $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ be a planar rigid motion, then,

$$\mathbf{g}(\mathbf{T}(\mathbf{P})) = \mathbf{T}(\mathbf{g}(\mathbf{P})).$$



Convex Hull: Real projective geometry.

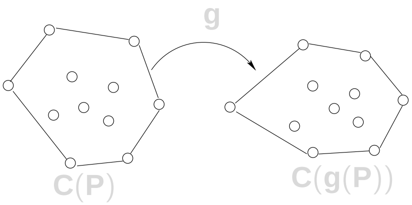
Key concept used in convex hull: collinearity.

Suppose $\mathbf{P}$ is a planar point set, $\mathbf{T}(\mathbf{P})$ is its Delaunay triangulation,

$$\mathbf{P} = \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n\}.$$

Let $\mathbf{g} : \mathbb{RP}^2 \to \mathbb{RP}^2$ be a real projective transformation, then,

$$\mathbf{g}(\mathbf{C}(\mathbf{P})) = \mathbf{C}(\mathbf{g}(\mathbf{P})).$$



Algorithms vs. Geometries

Convex Hull: Real projective geometry.

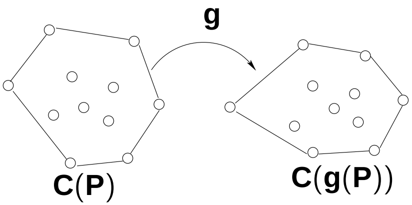
Key concept used in convex hull: collinearity.

Suppose $\mathbf{P}$ is a planar point set, $\mathbf{T}(\mathbf{P})$ is its Delaunay triangulation,

$$\mathbf{P} = \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n\}.$$

Let $\mathbf{g} : \mathbb{RP}^2 \to \mathbb{RP}^2$ be a real projective transformation, then,

$$\mathbf{g}(\mathbf{C}(\mathbf{P})) = \mathbf{C}(\mathbf{g}(\mathbf{P})).$$



(X,G) Invariant Algorithms

Definition ((X,G) invariant Algorithm)

Suppose X is a topological space, G is the transformation group on X. A geometric operator Ω defined on X is (X,G) invariant, if and only if

$$\Omega \circ g = g \circ \Omega, \forall g \in G.$$

Examples:

- Minkowski sum: Translation invariant.
- Voronoi Diagram: Rigid motion invariant.
- Polar form : Affine invariant.

Algorithms vs. Geometries

Central Problem

- Can different geometries be defined on general surfaces?
- Can different planar algorithms be generalized to surface domains directly?

The answers are yes and yes. The major theoretic tool is the *Geometric Structure.*

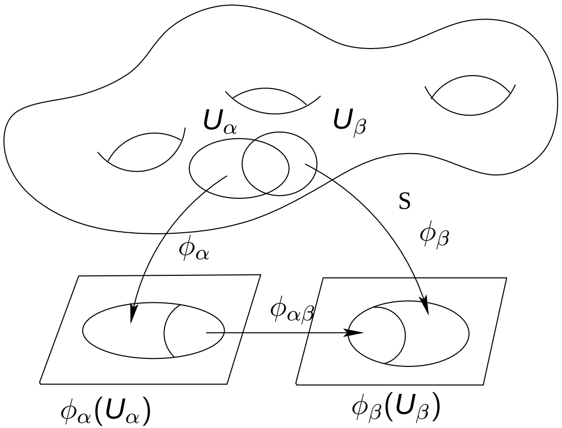
Algorithms vs. Geometries

Central Problem

- Can different geometries be defined on general surfaces?
- Can different planar algorithms be generalized to surface domains directly?

The answers are yes and yes. The major theoretic tool is the *Geometric Structure.*

Manifold



U_α
U_β
S
ϕ_α
ϕ_β
$\phi_{\alpha\beta}$
$\phi_\alpha(U_\alpha)$
$\phi_\beta(U_\beta)$

Definition (Manifold)

A manifold is a topological space Σ covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a chart of Σ, the set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of Σ. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

Transition maps satisfy cocycle condition, suppose $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then

$$\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}.$$

(X, G) structure

Definition ((X,G) Atlas)

Suppose X is a topological space, G is the transformation group of X. A manifold Σ with an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is an *(X, G) atlas* if

1. $\phi_\alpha(U_\alpha) \subset X$, for all charts (U_α, ϕ_α).
2. Transition maps $\phi_{\alpha\beta} \in G$.

Definition (Equivalent (X,G) atlases)

Two (X,G) atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of Σ are *equivalent*, if their union is still an (X,G) atlas of Σ.

Definition ((X,G) structure)

An *(X,G) structure* of a manifold Σ is an equivalent class of its (X,G) atlases.

(X,G) structure

Definition (Equivalent (X, G) atlases)

Two (X, G) atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of Σ are *equivalent*, if their union is still an (X, G) atlas of Σ.

Definition ((X,G) structure)

An *(X, G) structure* of a manifold Σ is an equivalent class of its (X, G) atlases.

Common (X,G) structure

Spherical Structure

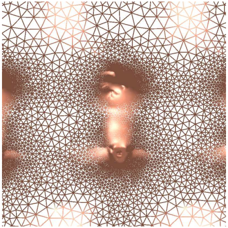
- X: Unit sphere $\mathbb{S}^2$.
- G: Rotation group.
- Surfaces: Genus zero closed surfaces; any open surfaces.

Common (X,G) structure

Spherical Structure

- X: Unit sphere $\mathbb{S}^2$.
- G: Rotation group.
- Surfaces: Genus zero closed surfaces; any open surfaces.

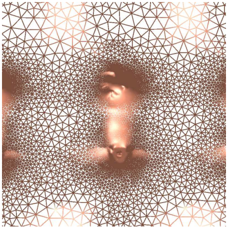
Common (X,G) structure



Affine Structure

- X: Real plane $\mathbb{R}^2$.
- G: Affine transformation group.
- Surfaces: Genus one closed surface and open surfaces.

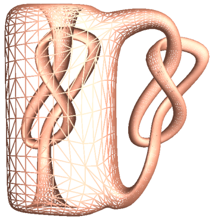
Common (X,G) structure

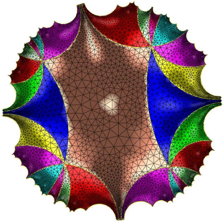


Affine Structure

- X: Real plane $\mathbb{R}^2$.
- G: Affine transformation group.
- Surfaces: Genus one closed surface and open surfaces.

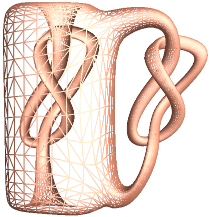
Common (X,G) structure

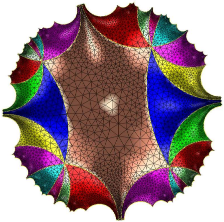




Hyperbolic Structure

- X: Hyperbolic plane $\mathbb{H}^2$.
- G: Möbius transformation group.
- Surfaces: with negative Euler number.

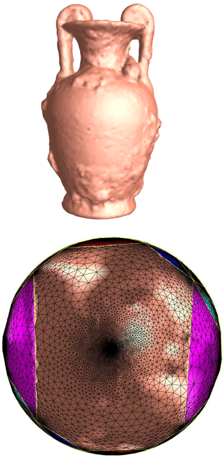




Hyperbolic Structure

- X: Hyperbolic plane $\mathbb{H}^2$.
- G: Möbius transformation group.
- Surfaces: with negative Euler number.

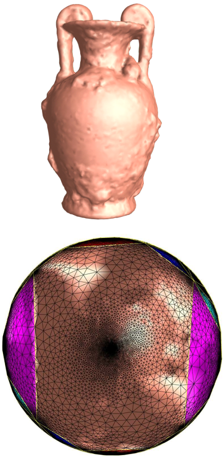
Common (X,G) structure



Real Projective Structure

- X: Real projective plane $\mathbb{RP}^2$.
- G: Real projective transformation group.
- Surfaces: any surface.

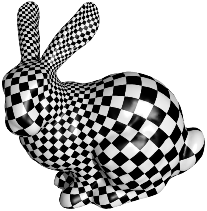
Common (X,G) structure

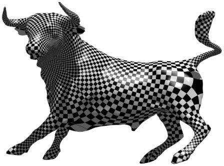


Real Projective Structure

- X: Real projective plane $\mathbb{RP}^2$.
- G: Real projective transformation group.
- Surfaces: any surface.

Pseudo (X,G) structure

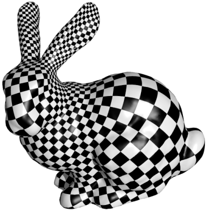


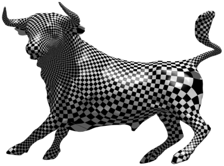


Conformal Structure

- X: Complex plane $\mathbb{C}$.
- G: Biholomorphic maps.
- Surfaces: any surface.

Pseudo (X,G) structure

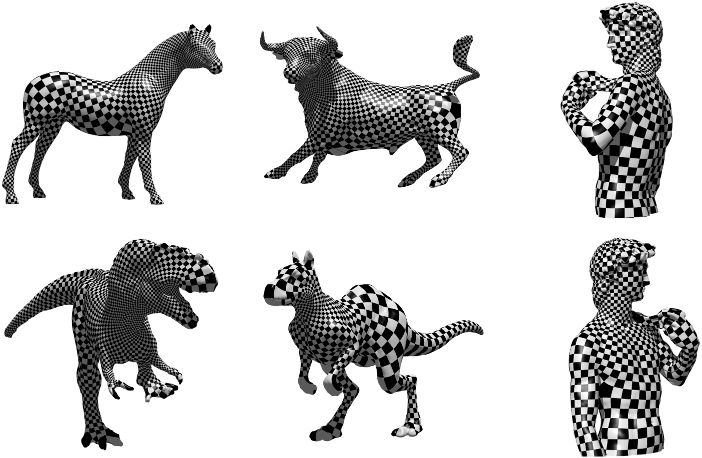




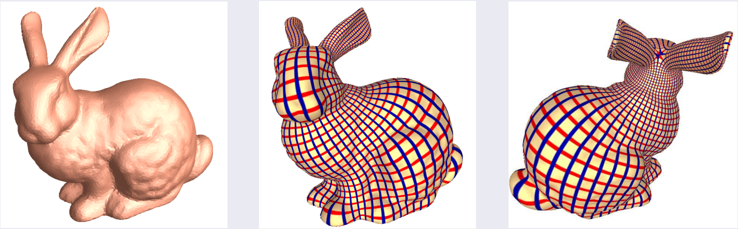
Conformal Structure

- X: Complex plane $\mathbb{C}$.
- G: Biholomorphic maps.
- Surfaces: any surface.

Conformal Structure



Global Tensor Product Structure



Relations Among Geometric Structures

Relations

- Conformal structure (A holomorphic 1-form) induces affine structure.
- Hyperbolic structure induces conformal structure.
- Hyperbolic structure induces real projective structure.

(X,G) structure

Theorem

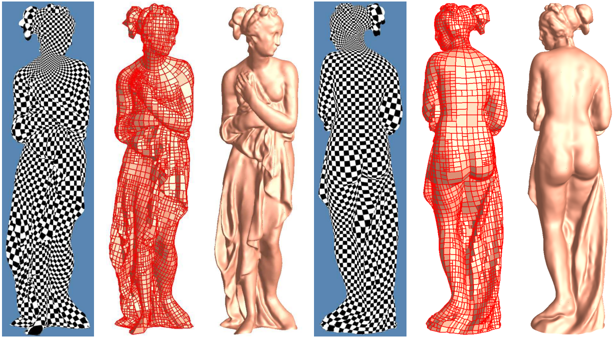
Suppose a manifold with an (X, G) structure, then any (X, G) invariant algorithms can be generalized on the manifold.

Corollary (Manifold Splines - Gu,He,Qin 2005)

Spline schemes based on polar forms can be defined on a manifold, if and only if the manifold has an affine structure.

Manifold Splines SPM2005

Manifold TSplines GMP2006



Topological Obstructions

Theorem (Benzécri 1959)

If a closed surface admits an affine structure, it has zero Euler class.

Real projective structure

- Real projective structure is general, it exists for all surfaces.
- Real projective structure is simple, all transitions are linear rational functions.
- Real projective structure is suitable for designing manifold spline schemes.

Topological Obstructions

Theorem (Benzécri 1959)

If a closed surface admits an affine structure, it has zero Euler class.

Real projective structure

- Real projective structure is general, it exists for all surfaces.
- Real projective structure is simple, all transitions are linear rational functions.
- Real projective structure is suitable for designing manifold spline schemes.

Previous Works

Related Works

1. Geometric Structures
2. Circle packing
3. Ricci flow
4. Conformal surface parameterizations

There are too many related works, we only briefly review the most related ones. Detailed information can be found in the survey papers in each field.

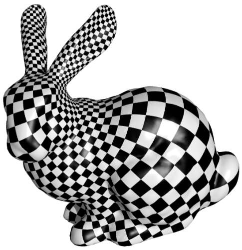
Conformal Metric

Definition

Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. $e^{2\lambda}$ is called the conformal factor.



Angles are invariant measured by conformal metrics.

Conformal Metrics on a Surface

Given a surface Σ with a Riemannian metric $\mathbf{g}$, find a function $\lambda : \Sigma \to \mathbb{R}$, such that $e^{2\lambda}\mathbf{g}$ is one of the followings:

- Uniform flat metric

  $$\bar{K} \equiv 0,$$

  for interior points

  $$\bar{k}_g \equiv const$$

  for boundary points. The constant values are determined by the conformal structure of Σ.
- Uniformization metric

  $$\bar{K} \equiv const,$$

  for interior points

  $$\bar{k}_g \equiv 0.$$

The tool to calculate the above metrics is Ricci flow.

Surface Ricci Flow

Definition (Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = (\bar{K} - K)g_{ij}, \bar{K} = \frac{2\pi\chi(\Sigma)}{S(\Sigma)}$$

$\chi(\Sigma)$ is the Euler number, $S(\Sigma)$ is the total area of Σ.

Theorem (Hamilton 1982)

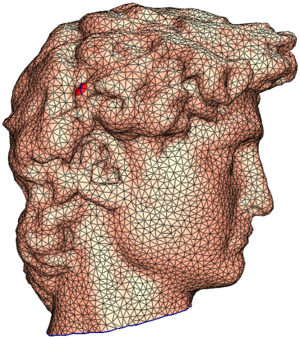
For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

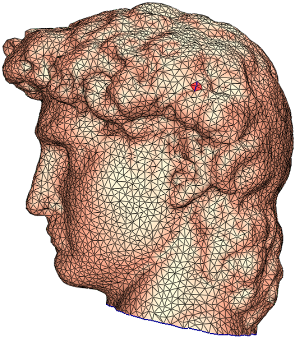
Theorem (Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Generic Surface Model - Triangular Mesh

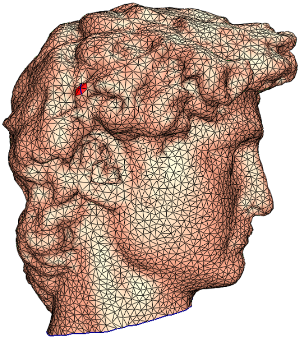
- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.

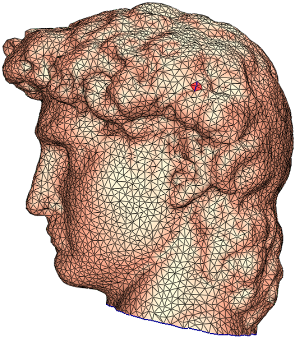




Generic Surface Model - Triangular Mesh

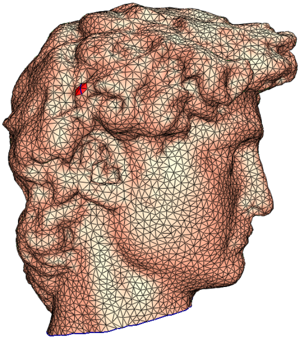
- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.

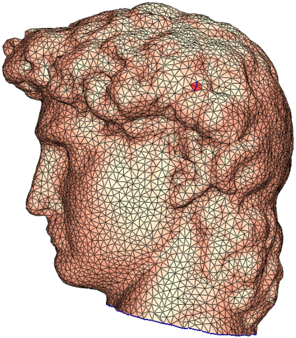




Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.





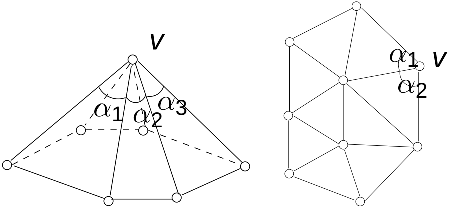
Curvature

- Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

- Discrete Gauss-Bonnet theorem

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$

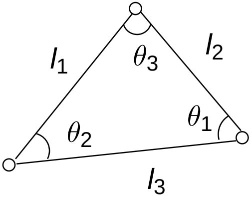


Discrete Metrics

Metric

- Discrete Metric: $l : E = \{all\ edges\} \to \mathbb{R}^1$, satisfies triangular inequality.
- Metrics determine curvatures by cosine law.

$$\cos\theta_i = \frac{l_j^2 + l_k^2 - l_i^2}{2 l_j l_k}, l \neq j \neq k \neq i$$



Metrics vs. Curvatures

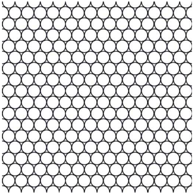
- All metrics for a mesh $\mathbf{L}(\Sigma)$ form a convex polytope.
- All admissible curvature configurations for a mesh $\mathbf{K}(\Sigma)$ also form a convex polytope.
- The mapping from the metrics to the curvatures

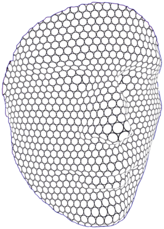
$$\Phi : \mathbf{L}(\Sigma) \to \mathbf{K}(\Sigma),$$

  is not one to one.
- The mapping from a conformal class of metrics to the curvatures is a homeomorphism.

Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.





Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

Circle Packing Metric

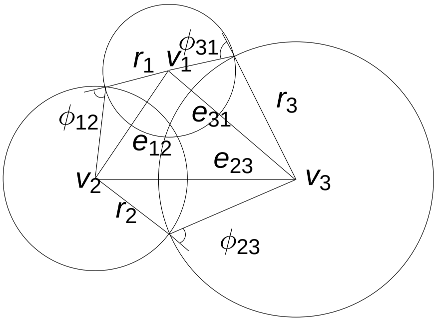
CP Metric

We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$



Definition (Conformal Circle Packing Metrics)

Two circle packing metrics $\{\Sigma, \Phi_1, \Gamma_1\}$ and $\{\Sigma, \Phi_2, \Gamma_2\}$ are conformal equivalent, if

- The radii of circles are different, $\Gamma_1 \neq \Gamma_2$.
- The intersection angles are same, $\Phi_1 \equiv \Phi_2$.

In practice, the circle radii and intersection angles are optimized to approximate the induced Euclidean metric of the mesh as close as possible.

Discrete Euclidean Ricci flow

Definition (Discrete Ricci flow)

A mesh Σ with a circle packing metric $\{\Sigma, \Gamma, \Phi\}$, where $\Gamma = \{\gamma_i, v_i \in V\}$ are the vertex radii, $\Phi = \{\Phi_{ij}, e_{ij} \in E\}$ are the angles associated with each edge, the discrete Ricci flow on Σ is defined as

$$\frac{d\gamma_i}{dt} = (\bar{K}_i - K_i)\gamma_i,$$

where $\bar{K}_i$ are the target curvatures on vertices. If $\bar{K}_i \equiv 0$, the flow with normalized total area leads to a metric with constant Gaussian curvature.

Idea

Metric deformation is driven by curvature.

Discrete Euclidean Ricci flow

Theorem (Chow and Luo 2002)

A discrete Euclidean Ricci flow $\{\Sigma, \Gamma, \Phi\} \to \{M, \bar{\Gamma}, \Phi\}$ *converges.*

$$|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},$$

and

$$|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},$$

where c_1, c_2 *are positive numbers.*

Definition

Let $u_i = ln\gamma_i$, the Ricci energy is defined as

$$f(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \sum_{i=1}^{n} (K_i - \bar{K}_i) du_i,$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, $\mathbf{u}_0 = (0, 0, \cdots, 0)$.

Derivative Euclidean Cosine Law

Theorem (Ricci Energy)

Euclidean Ricci energy is Well defined and convex, namely, there exists a unique global minimum.

Proof.

In an Euclidean triangle, with angles $(\theta_1, \theta_2, \theta_3)$ and radius $(\gamma_1, \gamma_2, \gamma_3)$, let $u_i = ln\gamma_i$, according to Euclidean cosine law,

$$\frac{\partial\theta_i}{\partial u_j} = \frac{\partial\theta_j}{\partial u_i}.$$

Therefore $\omega = \sum \theta_i du_i$ is a closed 1-form. The Euclidean Ricci energy is well defined. Direct computation verifies that Hessian matrix is positive definite. □

Derivative Euclidean Cosine Law

Theorem (Ricci Energy)

Euclidean Ricci energy is Well defined and convex, namely, there exists a unique global minimum.

Proof.

In an Euclidean triangle, with angles $(\theta_1, \theta_2, \theta_3)$ and radius $(\gamma_1, \gamma_2, \gamma_3)$, let $u_i = ln\gamma_i$, according to Euclidean cosine law,

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.$$

Therefore $\omega = \sum \theta_i du_i$ is a closed 1-form. The Euclidean Ricci energy is well defined. Direct computation verifies that Hessian matrix is positive definite. □

Newton's method for Euclidean Ricci energy

Gradient descent Method

Ricci flow is the gradient descent method for minimizing Ricci energy,

$$\nabla f = (K_1 - \bar{K}_1, K_2 - \bar{K}_2, \cdots, K_n - \bar{K}_n).$$

Newton's method

The Hessian matrix of Ricci energy is

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial K_i}{\partial u_j}.$$

Newton's method can be applied directly.

Ricci Flow for Uniform Flat Metric

Suppose Σ is a closed genus one mesh,

1. Compute the circle packing metric (Γ, Φ).
2. Set the target curvature to be zero for each vertex

$$\bar{K}_i \equiv 0, \forall v_i \in V$$

3. Minimize the Euclidean Ricci energy using Newton's method to get the target radii $\bar{\Gamma}$.
4. Compute the target flat metric.

Algorithm : uniform flat metric for open surfaces

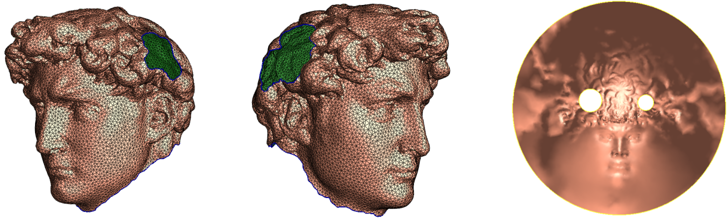
Given a surface Σ with genus g and b boundaries, then it Euler number is

$$\chi(\Sigma) = 2 - 2g - b.$$

Suppose the boundary of Σ is a set of closed curves

$$\partial\Sigma = C_1 \cup C_2 \cup C_3 \cdots C_b.$$

The total curvature for each C_i is denoted as $2m_i\pi, m_i \in \mathbb{Z}$, and $\sum_{i=1}^{b} m_i = \chi(\Sigma)$. The target curvature for interior vertices are zeros.



Algorithm : uniform flat metric for open surfaces

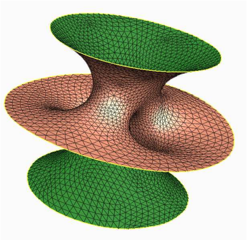
Euclidean Ricci flow for open surfaces

- Use Newton's method to minimize the Ricci energy to update the metric.
- Adjust the boundary vertex curvature to be proportional to the ratio between the current lengths of the adjacent edges and the current total length of the boundary component.
- Repeat until the process converges.

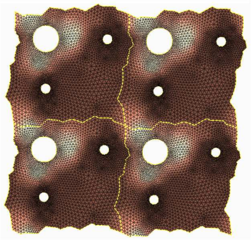
Embedding

1. Determine the planar shape of each triangle using 3 edge lengths.
2. Glue all triangles on the plane along their common edges by rigid motions. Because the metric is flat, the gluing process is coherent and results in a planar embedding.

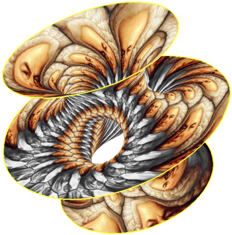
Euclidean Uniform Flat Metric



original surface
genus 1, 3 boundaries

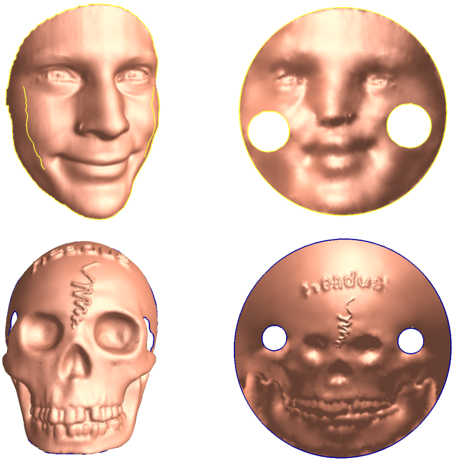


universal cover
embedded in $\mathbb{R}^2$

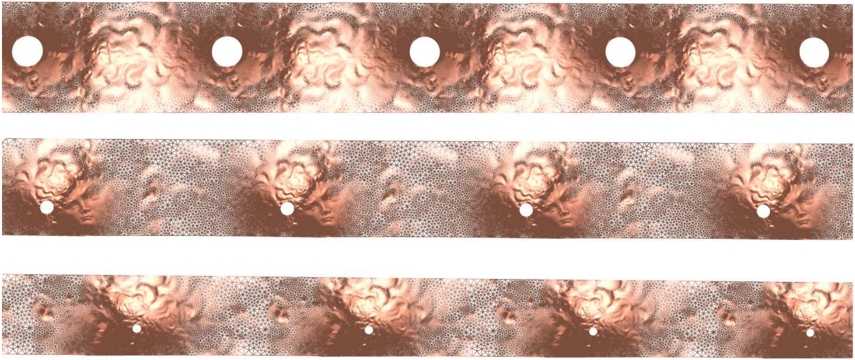


texture mapping

Euclidean Uniform Flat Metric

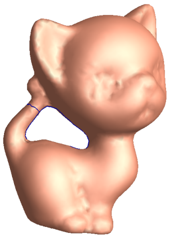


Euclidean Uniform Flat Metric

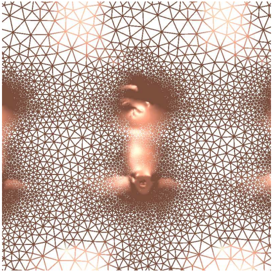


Different boundaries are mapped to straight lines.

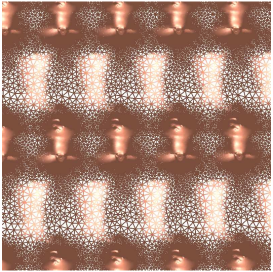
Euclidean Uniform Flat Metric



original surface



fundamental domain



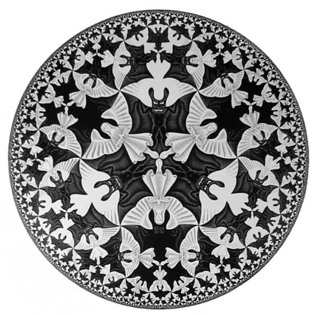
universal cover

Conformal Model : Poincaré Disk

Poincaré disk

A unit disk $|z| < 1$ with the Riemannian metric

$$ds^2 = \frac{4dzd\bar{z}}{(1-\bar{z}z)^2}.$$

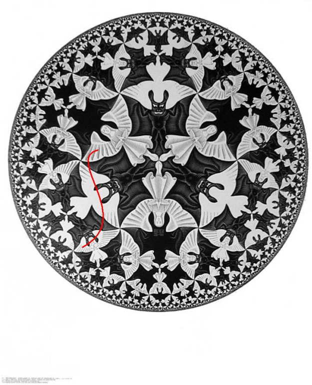


Conformal Model : Poincaré Disk

Poincaré disk

The rigid motion is the Möbius transformation

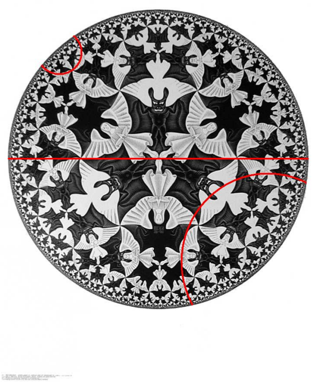
$$e^{i\theta}\frac{z-z_0}{1-\bar{z}_0 z}.$$



Conformal Model : Poincaré Disk

Poincaré disk

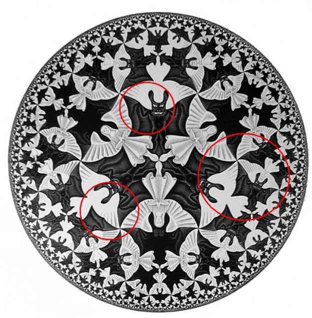
The hyperbolic line through two point z_0, z_1 is the circular arc through z_0, z_1 and perpendicular to the boundary circle $|z| = 1$.



Conformal Model : Poincaré Disk

Poincaré disk

A hyperbolic circle (c,γ) on Poincare disk is also an Euclidean circle (C,R) on the plane, such that $\mathbf{C} = \frac{2-2\mu^2}{1-\mu^2|\mathbf{c}|^2}$, $R^2 = |\mathbf{C}|^2 - \frac{|\mathbf{c}|^2-\mu^2}{1-\mu^2|\mathbf{c}|^2}, \mu = \frac{e^r-1}{e^r+1}$.



Discrete Hyperbolic Ricci Flow

Definition (Discrete Hyperbolic Ricci Flow)

Let

$$u_i = ln \tanh \frac{\gamma_i}{2},$$

Discrete hyperbolic Ricci flow for a mesh Σ is

$$\frac{du_i}{dt} = \bar{K}_i - K_i, \bar{K}_i \equiv 0,$$

the Euler number of Σ is negative, $\chi(\Sigma) < 0$.

Discrete Hyperbolic Ricci flow

Theorem (Discrete Hyperbolic Ricci flow, Chow and Luo 2002)

A hyperbolic discrete Ricci flow $(M, \Gamma, \Phi) \rightarrow (M, \bar{\Gamma}, \Phi)$ *converges,*

$$|K_i(t) - \bar{K}_i| < c_1 e^{-c_2 t},$$

and

$$|\gamma_i(t) - \bar{\gamma}_i| < c_1 e^{-c_2 t},$$

where c_1, c_2 *are positive numbers.*

Definition (Discrete Hyperbolic Ricci Energy)

The discrete Hyperbolic Ricci energy is defined as

$$f(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i.$$

Discrete hyperbolic Ricci flow is the gradient descendent method to minimize the discrete hyperbolic ricci energy.

Derivative hyperbolic Cosine Law

Theorem (Hyperbolic Discrete Ricci Energy)

Discrete hyperbolic Ricci energy is well defined and convex, namely, there exists a unique global minimum.

Proof.

In a hyperbolic triangle, with angles $(\theta_1, \theta_2, \theta_3)$ and radius $(\gamma_1, \gamma_2, \gamma_3)$, $u_i = ln \tanh \frac{\gamma_i}{2}$, according to hyperbolic cosine law,

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.$$

Therefore $\omega = \sum \theta_i du_i$ is a closed 1-form. The hyperbolic Ricci energy is convex. Direct computation verifies the Hessian matrix is positive definite. □

Derivative hyperbolic Cosine Law

Theorem (Hyperbolic Discrete Ricci Energy)

Discrete hyperbolic Ricci energy is well defined and convex, namely, there exists a unique global minimum.

Proof.

In a hyperbolic triangle, with angles $(\theta_1, \theta_2, \theta_3)$ and radius $(\gamma_1, \gamma_2, \gamma_3)$, $u_i = ln \tanh \frac{\gamma_i}{2}$, according to hyperbolic cosine law,

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.$$

Therefore $\omega = \sum \theta_i du_i$ is a closed 1-form. The hyperbolic Ricci energy is convex. Direct computation verifies the Hessian matrix is positive definite. □

Algorithm: Computing Hyperbolic uniformization metric

Hyperbolic Ricci Energy Optimization

1. Set target curvature $K(v_i) \equiv 0$.
2. Optimize the hyperbolic Ricci energy using Newton's method, with the constraint the total area is preserved.

Flattening Mesh in Hyperbolic Space

1. Determine the shape of each triangle.
2. Glue the hyperbolic triangles coherently by Möbius transformation.

Key: all computations use hyperbolic geometry.

Algorithm: Computing Hyperbolic uniformization metric

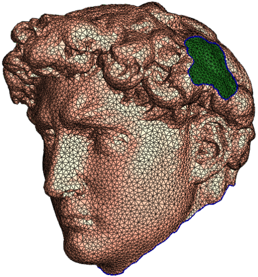
Hyperbolic Ricci Energy Optimization

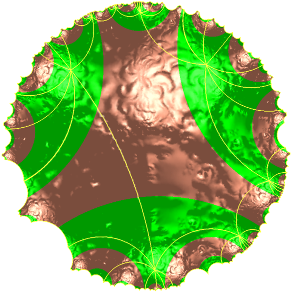
1. Set target curvature $K(v_i) \equiv 0$.
2. Optimize the hyperbolic Ricci energy using Newton's method, with the constraint the total area is preserved.

Flattening Mesh in Hyperbolic Space

1. Determine the shape of each triangle.
2. Glue the hyperbolic triangles coherently by Möbius transformation.

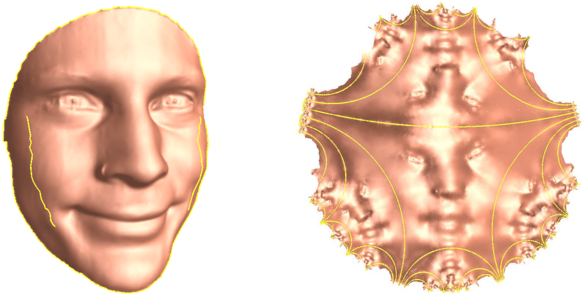
Key: all computations use hyperbolic geometry.





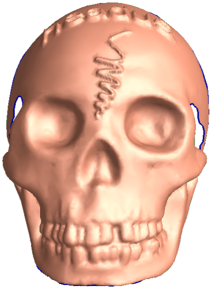
Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.

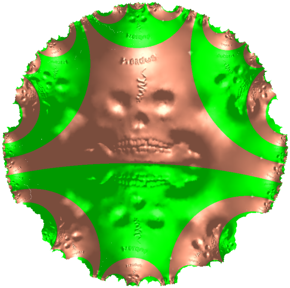
Hyperbolic Uniformization Metric



Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.

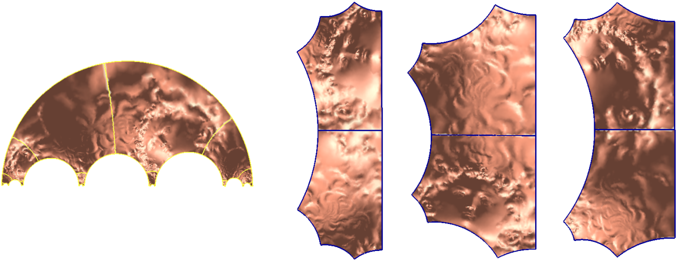
Hyperbolic Uniformization Metric





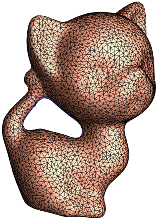
Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.

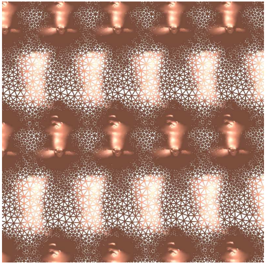
Hyperbolic Uniformization Metric



Embedding in the upper half plane hyperbolic space model. Different period embedded in the hyperbolic space. The boundaries are mapped to hyperbolic lines.

Universal Covering Space and Deck Transformation





Universal Cover

A pair $(\bar{\Sigma}, \pi)$ is a universal cover of a surface Σ, if

- Surface $\bar{\Sigma}$ is simply connected.
- Projection $\pi : \bar{\Sigma} \to \Sigma$ is a local homeomorphism.

Deck Transformation

A transformation $\phi : \bar{\Sigma} \to \bar{\Sigma}$ is a deck transformation, if

$$\pi = \pi \circ \phi.$$

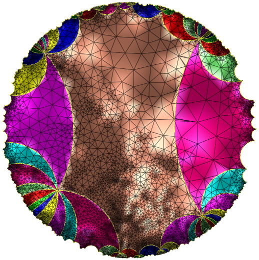
A deck transformation maps one period to another.

Fuchsian Group

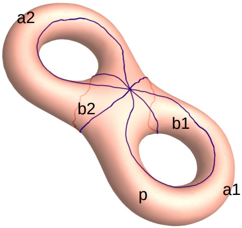
Definition (Funchsian Group)

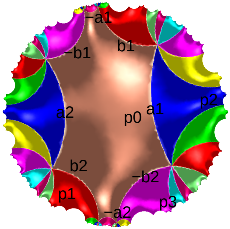
Suppose Σ is a surface, $\mathbf{g}$ is its uniformization metric, $(\bar{\Sigma}, \pi)$ is the universal cover of Σ. $\mathbf{g}$ is also the uniformization metric of $\bar{\Sigma}$. A deck transformation of $(\bar{\Sigma}, \mathbf{g})$ is a Möbius transformation. All deck transformations form the Fuchsian group of Σ.

Fuchsian group indicates the intrinsic symmetry of the surface.



Fuchsian Group





The Fuchsian group is isomorphic to the fundamental group

	$e^{i\theta}$	z_0
a_1	$-0.631374 + i0.775478$	$+0.730593 + i0.574094$
b_1	$+0.035487 - i0.999370$	$+0.185274 - i0.945890$
a_2	$-0.473156 + i0.880978$	$-0.798610 - i0.411091$
b_2	$-0.044416 - i0.999013$	$+0.035502 + i0.964858$

Klein Model

Another Hyperbolic space model is Klein Model, suppose **s**, **t** are two points on the unit disk, the distance is

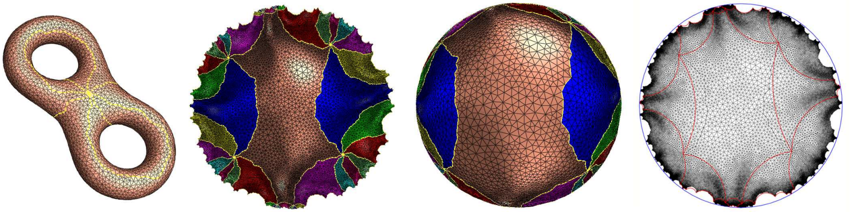
$$d(s,t) = arccosh \frac{1 - \mathbf{s}\cdot\mathbf{t}}{\sqrt{(1-\mathbf{s}\cdot\mathbf{s})(1-\mathbf{t}\cdot\mathbf{t})}}$$

Poincaré vs. Klein Model

From Poincaré model to Klien model is straight froward

$$\beta(z) = \frac{2z}{1+\bar{z}z}, \beta^{-1}(z) = \frac{1-\sqrt{1-\bar{z}z}}{\bar{z}z},$$

Assume ϕ is a Möbius transformation, then transition maps $\beta\circ\phi\circ\beta^{-1}$ are real projective.

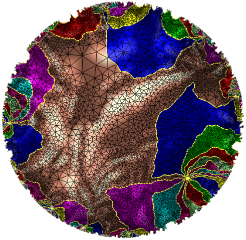


Real projective structure

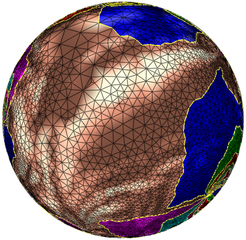
The embedding of the universal cover in the Poincaré disk is converted to the embedding in the Klein model, which induces a real projective atlas of the surface.

Hyperbolic and Real Projective Structure

Surface

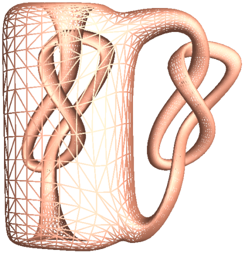


Hyperbolic Structure

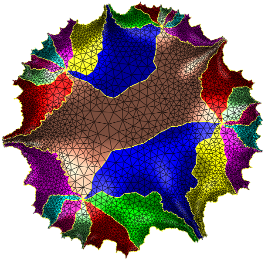


Projective Structure

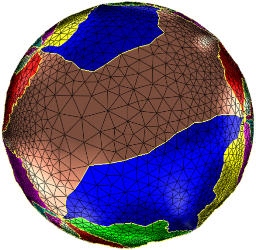
Hyperbolic and Real Projective Structure



Surface, courtesy of Cindy Grimm

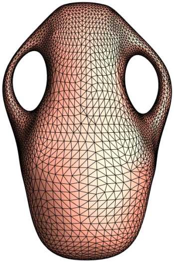


Hyperbolic Structure

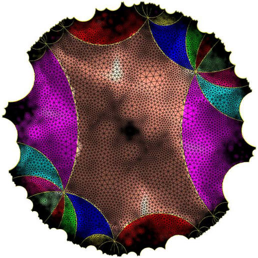


Projective Structure

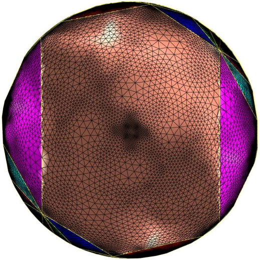
Hyperbolic and Real Projective Structure



Surface



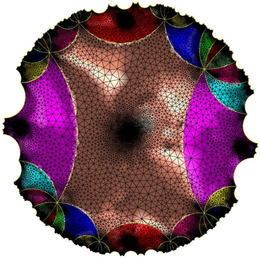
Hyperbolic Structure



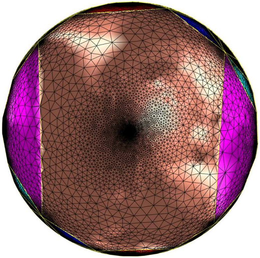
Projective Structure

Hyperbolic and Real Projective Structure

Surface

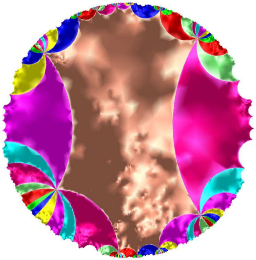


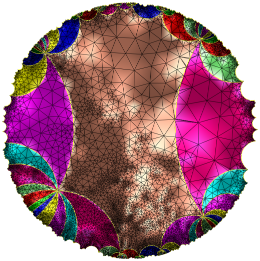
Hyperbolic Structure



Projective Structure

Hyperbolic Uniformization Metric

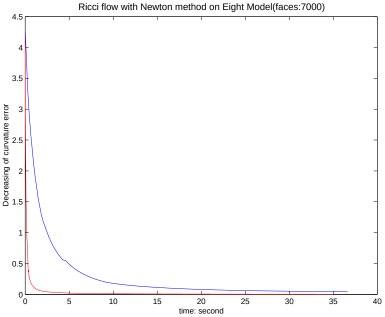




Performance

performance

- Based on OpenMesh library on Windows platform, [Sovakar and Kobbelt 2005].
- Eight model ($7k$ faces),10^2 seconds on $1.7G$ CPU with 1G RAM laptop.

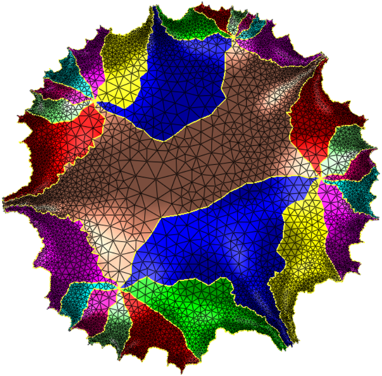


Curvature error vs. running time. Red curve Newton's method; Blue curve : gradient decent method.

Challenges

Challenges

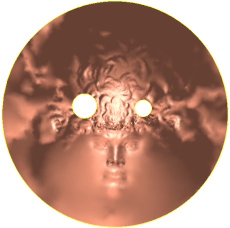
- Intrinsically nonlinear method.
- Intrinsically the conformal factor may be exponential.
- Determine the optimal initial circle packing metric.
- Embed universal cover in the Poincaré disk.

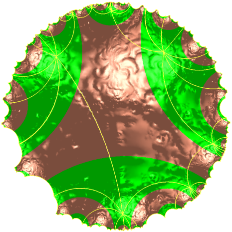


Summary

Contributions

- Introduce general geometric structures
  - Different geometries can be defined on surfaces.
  - Planar algorithms can be systematically generalized to surfaces.
- Ricci flow method to compute special metrics.
  - Uniform flat metric.
  - Uniformization metric.
- Algorithms to compute geometric structures
  - hyperbolic structure
  - real projective structure

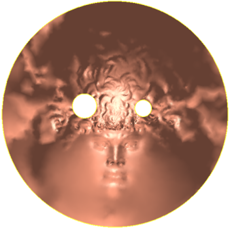


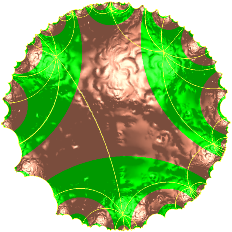


Summary

Contributions

- Introduce general geometric structures
  - Different geometries can be defined on surfaces.
  - Planar algorithms can be systematically generalized to surfaces.
- Ricci flow method to compute special metrics.
  - Uniform flat metric.
  - Uniformization metric.
- Algorithms to compute geometric structures
  - hyperbolic structure
  - real projective structure

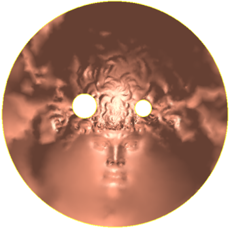


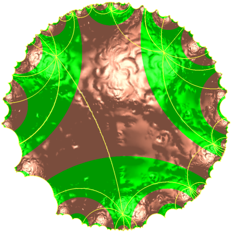


Summary

Contributions

- Introduce general geometric structures
  - Different geometries can be defined on surfaces.
  - Planar algorithms can be systematically generalized to surfaces.
- Ricci flow method to compute special metrics.
  - Uniform flat metric.
  - Uniformization metric.
- Algorithms to compute geometric structures
  - hyperbolic structure
  - real projective structure

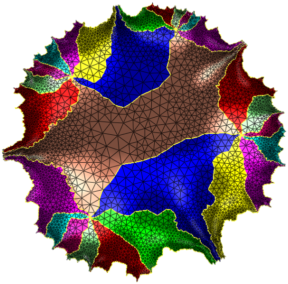




Future Works

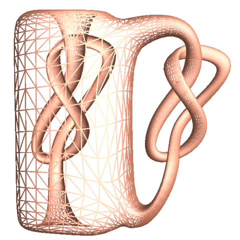
Future Works

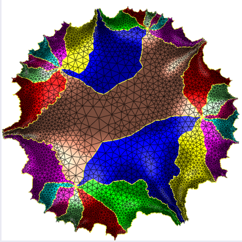
- Design spline schemes based on real projective geometry.
- Hirearchical approach for Ricci energy optimization.
- Surface classification using Fuchsian group.
- Generalize planar geometric algorithms to surface domains using geometric structures.
- Ricci flow on 3-manifolds.



Thanks

For more information, please email to gu@cs.sunysb.edu.





Thank you!