A New Approach to Auctions and Resilient Mechanism Design

Jing Chen and Silvio Micali
CSAIL, MIT
{jingchen, silvio}@csail.mit.edu

September 17, 2009

Abstract

We put forward a new approach to mechanism design, and exemplify it via a new mechanism guaranteeing significant revenue in *unrestricted combinatorial auctions*. Our mechanism

- succeeds in a new and *very adversarial* collusion model;
- works in a new, *equilibrium-less*, and very strong solution concept;
- benchmarks its performance against the *knowledge that the players have about each other*;
- is *computationally efficient* and *preserves the players' privacy* to an unusual extent.

1 Introduction

A *game* $\mathcal{G}$ consists of a *context* C and a *mechanism* M: that is, $\mathcal{G} = (C, M)$. The context specifies the players' *types*, all possible *outcomes*, and the players' *utilities* in these outcomes. The mechanism specifies the players' *strategy spaces* and how strategies determine outcomes. Let us explain this for combinatorial auctions.

1.1 Unrestricted Combinatorial Auctions

Recall that a *valuation* for a finite set of goods G is a function mapping each subset of G to a non-negative real, and that a *profile* is a vector indexed by the players.

Combinatorial-Auction Contexts. The context of a combinatorial auction with players $1, \ldots, n$ and *non-transferable* goods $g_1, \ldots, g_m$ is defined as follows.

- The players' types consist of a profile of valuations, $TV = TV_1, \ldots, TV_n$, called the *true valuations*. Each valuation TV_i specifies the value that player i truly attributes to any of the 2^m subsets of the goods for sale, and is such that $TV_i(\emptyset) = 0$.
- An outcome Ω specifies who wins which goods and how much everyone pays. Formally, Ω is a pair, $\Omega = (A, P)$:

  A, the *allocation*, is a partition of $g_1, \ldots, g_m$ into $n+1$ disjoint sets, $A = A_0, \ldots, A_n$. (A_0 represents the set of unallocated goods, and, for $i > 0$, A_i represents the set of goods allocated to player i.)

  P, the *price profile*, is a profile of real numbers. (P_i represents the amount player i pays. If P_i is negative, then $-P_i$ represents the amount i receives.)
- Player i's utility in an outcome Ω, $u_i(\Omega)$, consists of his value for the goods he receives, minus the price he pays. That is: letting $\Omega = (A, P)$, $u_i(\Omega) = TV_i(A_i) - P_i$.

Combinatorial-Auction Mechanisms. In a typical, normal-form mechanism M for a combinatorial auction, a player's strategy σ_i, also called a *bid*, consists of a valuation for the goods. (That is, a player's strategy/type/valuation space coincide.) The mechanism also specifies a possibly probabilistic function from strategy profiles to outcomes, the *outcome function*, denoted too by M for simplicity.

The mechanism is played in 3 steps: first, each player i (based on his true valuation TV_i) privately selects a strategy σ_i; second, all players simultaneously announce their selected strategies; third, the outcome function is evaluated on the announced strategy profile $\sigma = \sigma_1, \ldots, \sigma_n$ to yield the final outcome Ω; that is, $\Omega = M(\sigma)$.

Unrestricted Combinatorial Auctions. Combinatorial auctions are very hard, and many restrictions have been considered in the literature for the players' true valuations (e.g., additivity, sub-modularity, single-mindedness, etc.) and the type of goods (e.g., digital goods, for which multiple copies can be made for free, and then sold to multiple players).

We envisage no such restrictions in our paper, and indeed use the term "unrestricted" to emphasize this fact.

1.2 Traditional Mechanism Design

A mechanism designer wishes to guarantee a given property in a given context. The difficulty is that the designer does not fully know the context at hand. A context includes the players' types, knowledge about which, in the purest form of mechanism design, solely lies with the players themselves. The desired property, however, typically depends on the players' types. (E.g., in an auction of a single good, a traditionally desired property consists of allocating the good to the player who values it the most.) Thus a designer strives to find a mechanism "enticing" rational players, that is players acting so as to maximize their utilities (expected utilities, in case of probabilistic mechanisms), to *de facto* "work for him." That is, he wants to design a mechanism M such that, for all appropriate contexts C, *the desired property is guaranteed to hold in a rational play of* (C, M). Traditionally this means that the property should hold "at equilibrium."

Equilibria. An equilibrium is a strategy profile σ such that every player i is better off sticking to his strategy σ_i if he *believes* that all other players stick to theirs. That is, denoting by $-i$ the set of all players but i, a strategy profile σ is an equilibrium if, for all players i and all alternative strategies σ'_i, $u_i(M(\sigma_i \sqcup \sigma_{-i})) \geq u_i(M(\sigma'_i \sqcup \sigma_{-i}))$.

Thus the meaningfulness of an equilibrium crucially depends on the players' (correct) beliefs. A stronger notion, solely depending on the players' rationality (rather than their beliefs), is that of a *dominant-strategy equilibrium.* This is a strategy profile σ^* such that, for every player i, any alternative strategy σ'_i for i, and any strategy subprofile σ_{-i} for the other players: $u_i(M(\sigma^*_i \sqcup \sigma_{-i})) \geq u_i(M(\sigma'_i \sqcup \sigma_{-i}))$. Thus, in a game with a dominant-strategy equilibrium σ^*, the best option for any player i is to choose strategy σ^*_i, no matter what the other players might do.

Traditional Desired Properties. In a combinatorial auction, two quantities are crucial in an outcome $\Omega = (A, P)$: the *social welfare* of Ω, $\text{SW}(\Omega)$, defined as $\sum_i TV_i(A_i)$, and the *revenue* of Ω, $\text{REV}(\Omega)$, defined as $\sum_i P_i$.

Accordingly, one traditionally tries to design combinatorial-auction mechanisms M such that the social welfare and/or the revenue of $M(\sigma)$ are high either for *some* equilibrium σ, or for *all* equilibria σ. Both social welfare and revenue are "in expectation" if M is probabilistic.

DST Mechanisms and the VCG. A normal-form mechanism is said to be *dominant-strategy truthful* (DST for short) if, for every player i, announcing his own true type is a dominant strategy. In combinatorial auctions, the famous VCG mechanism [18, 4, 9] is DST and maximizes social welfare.

On input a profile of valuations V, the VCG returns (a) the allocation A maximizing, over all possible allocations A', $\sum_i V_i(A'_i)$; and (b) the price profile P, where each P_i equals the maximum, over all possible allocations A', of $\sum_{j \in -i} V_j(A'_j)$ minus $\sum_{j \in -i} V_j(A_j)$.

1.3 The Case for Resilient Mechanism Design

A manuscript of Micali and Valiant still unpublished, and therefore referred to as "MV" herein, highlights two main weaknesses of traditional mechanism design, and advocates overcoming them by designing mechanisms in a specific *resilient* way. Let us explain.

Weaknesses of Traditional Mechanisms By aiming to achieve a desired property $\mathbb{P}$ "at equilibrium," traditional mechanism design suffers from two main problems:

1. *Equilibrium Selection* and
2. *Collusion.*

The problem of equilibrium selection arises with the existence of multiple "reasonable" equilibria. Indeed, even if M guarantees that $\mathbb{P}$ holds at each possible equilibrium of the game (C, M), if some players believe that the equilibrium ultimately played out is σ while others believe that it is τ, then the profile of strategies actually selected will be a mixture of σ and τ rather than an equilibrium, and $\mathbb{P}$ may not hold.

Equilibrium selection is not a problem when $\mathbb{P}$ holds at a dominant-strategy equilibrium σ. In this case, in fact, whether or not other equilibria exist, one can confidently predict σ to be the one actually played out by rational players. But then, collusion continues to be a problem, because it prevents traditional mechanisms from *guaranteeing* their desired property $\mathbb{P}$. Indeed equilibria (even dominant-strategy ones) are very fragile notions: they only imply that no *single* player has any incentive to *individually* deviate from his envisaged strategy, but two or more players may have all the incentive in the world to *jointly deviate* from their equilibrium strategies. If they do so, the property $\mathbb{P}$ that was guaranteed at equilibrium, may not hold at all.

The problem of collusion is not only theoretical, but very practical as well. Collusion in auctions is both *well* documented, and *very* disruptive. In particular, as insightfully shown by Ausubel and Milgrom [1], the VCG mechanism, despite being DST, is totally vulnerable to collusion: even two (sufficiently informed) players may totally destroy its social welfare. To justify some of the choices of our paper, it is important to recall their counter-example in its simplest form:

> Consider a combinatorial auction with two goods, g_1 and g_2, three players, 1, 2, and 3, and the following true valuations: $TV_1(\{g_1\}) = TV_2(\{g_2\}) = 1$, $TV_3(\{g_1, g_2\}) = 100$ and $TV_j(S) = 0$ in all other cases. For this context, the allocation giving both goods to player 3 has a social welfare of 100, and is indeed the "best possible." Because the VCG is DST, *if all players were independent,* then the best strategy for them is to bid their true valuations, so that the VCG allocates both goods to 3 for a price of 2 (the other players pay 0). However, assume that players 1 and 2 collude, and that they know that player 3 values only the two goods together and for at most v (as of today, $v = 10^{20}$ should be pretty safe). Then the colluding players are better off bidding the valuations TV_1' and TV_2', where $TV_i'(\{g_i\}) = v$ and $TV_i'(S) = 0$ in all other cases. Because player 3, being independent, will continue to bid his true valuation TV_3, the VCG, on input TV_1', TV_2' and TV_3, will allocate g_1 to player 1 for a price of 0, and g_2 to player 2 for a price of 0 (the price of player 3 will also be 0). Thus, the total social welfare will be 2, rather than 100 (and the total revenue will be 0). This is so despite the fact that the VCG is a DST mechanism maximizing social welfare.

The Ausubel-Milgrom example highlights that, although the "best form" of equilibrium, a dominant-strategy equilibrium still is an equilibrium, and thus offers no guarantee against collusion. Their same example also highlights that the VCG mechanism has no guarantee about revenue, *even in absence of any collusion.* Indeed if TV_1', TV_2' and TV_3 respectively were the true valuations of the three players, the VCG would return 0 revenue, despite "strong competition for the goods."

MV Resiliency MV proposes two principles to address the above weaknesses:

1′ *Dominant-Strategy Truthfulness* and

2′ *Collusion Neutralization.*

Obviously, a DST mechanism dispels any problem of equilibrium selection. Let us thus focus on explaining the second principle. Collusion neutralization aims at guaranteeing the same performance as when, by magic, all collusive players disappeared, leaving the mechanism to be run with just the *independent* players. Ideally, of course, a designer would prefer to be able to "extract additional performance" from the collusive players, but in light of the Ausubel-Milgrom example (where just two colluders can ruin any performance whatsoever), neutralizing collusive players is an attractive goal. According to MV's second principle, the designer of a mechanism has no responsibility if all players are collusive, but is fully responsible for satisfying the desired property so long as a single independent player exists. We stress that, when neutralizing collusive players, not only the designer does not know which players are collusive, but neither do the independent players. Yet, the designer should guarantee the performance of his mechanism solely by providing a proper incentive structure.

Although the notion of MV resiliency applies to all kinds of desiderata and contexts, we shall focus on generating revenue in unrestricted combinatorial auctions.

Ultimately, in combinatorial auctions, the performance of any mechanism can be measured via a *benchmark*, a function $\mathbb{B}$ mapping any valuation profile V to a real number. To capture collusion neutralization, $\mathbb{B}$ should not be applied to the entire valuation profile V, but just to the valuation subprofile of the independent players, denoted by V_I.

Accordingly we say that a mechanism M achieves MV-resiliently a fraction c of a revenue benchmark $\mathbb{B}$ if

1″ M is DST and

2″ For any true-valuation profile TV, any set of independent players I and any bid subprofile V_{-I}, the revenue M generates for the profile of bids $TV_I \sqcup V_{-I}$ is at least $c \cdot \mathbb{B}(TV_I)$.

Of course such resiliency is interesting only if $\mathbb{B}$ and c are both reasonable; and of course, "the higher the fraction c, the better the mechanism."

Revenue Limitations of MV Resiliency Although achievable, MV's notion of resiliency has severe limitations vis à vis revenue. Denote by $\mathbb{MSW}_{-\star}$ the benchmark consisting of the maximum social welfare after removing the "star" player (that is the one valuing some subset of the goods more than anyone values any subset)[1]. Then, disregarding small constants, MV proves the following results about guaranteeing revenue in unrestricted combinatorial auctions:

- There exists a probabilistic mechanism that MV-resiliently achieves a logarithmic (in the minimum of n and m) fraction of $\mathbb{MSW}_{-\star}$; and
- No mechanism MV-resiliently achieves more than that.

Since resiliency looks attractive, but logarithmic fractions do not, we shall build resilient mechanisms of a new type.

Let us emphasize that MV's revenue upperbound is very meaningful, because it holds for *all possible* probabilistic DST mechanisms, not just for those of a special form (such as mixtures of deterministic mechanisms). Accordingly, there are no more general forms of probabilistic DST mechanisms to explore: we should instead explore new approaches to resilient mechanism design.

2 Summary of Our Contributions

DST mechanisms may very well be the best way to avoid equilibrium-selection problems. But if we also want to avoid MV's revenue upperbound we must choose at least one of the following two alternatives:

A1. Assuming that more knowledge is available (e.g., that the seller has some Bayesian information about the players' true valuations) or

A2. Adopting a solution concept weaker than dominant-strategy equilibrium.

Actually, in this paper we take *both alternatives*, but *without* violating the purest form of mechanism design (i.e., "all knowledge resides with the players") and *without* introducing any equilibrium-selection problem. In other words, we want to satisfy the spirit of resilient mechanism design, but free it from the specific technical interpretation given by MV.

In this paper, focussing on revenue, we retain MV's notion of collusion neutralization, but otherwise provide new conceptual frameworks and mechanisms.

A New and General Collusive Model We envisage a very adversarial collusion model for the players. (Recall that in an auction the seller is not a player, and thus we do not envisage player-seller collusion.) In particular, we allow for any number of collusive players, partitioned into any numbers of collusive sets. We do not restrict the cardinality of collusive sets, nor the way in which the members of a collusive set coordinate their actions. If they so want, the members of a collusive set may enter binding agreements on how to act.

We insist, however, that all players be *rational,* else prediction of players' behavior and mechanism design itself would be severely impaired. An independent player, that is a player i not belonging to any collusive set, is *individually rational* and acts so as to maximize his traditional utility function u_i from outcomes to real numbers. A collusive set C is *collectively rational,* that is its members coordinate their actions so as to maximize their own *collective utility function* u_C, mapping any outcome to a real number.

To maximize meaningfulness, we want the relationship between u_C and the individual utility functions of C's members to be as general as possible, provided that we do not "transform collusive players into irrational ones." (Indeed, what is the difference between a set C of crazy players and a set C of players rationally maximizing a crazy u_C?[2]) Accordingly, we demand that u_C be *minimally monotone.* Let us explain. Consider two outcomes that are absolutely identical, as far as C's members are concerned, except for member i who receives no goods and pays P'_i (may be negative) in the first outcome, but receives a subset of goods A_i and pays P_i in the second. Then, minimal monotonicity requires that C prefers the first outcome if $-P'_i > TV_i(A_i) - P_i$, and the second if $-P'_i < TV_i(A_i) - P_i$.[3]

Minimal monotonicity is of course a restriction on u_C, but: (a) it is the only restriction to our otherwise general collusion model; and (b) it is a very reasonable restriction.[4]

[1] For any valuation (sub)profile V, letting the "star" player, $\star$, be defined as $\star = \text{argmax}_i \max_{S \subseteq G} V_i(S)$, then $\mathbb{MSW}_{-\star}(V) = \max_A \sum_{i \in -\star} V_i(A_i)$.

[2] Indeed, irrational players may be modeled as taking arbitrary actions, and for any actions taken by C's members, one might find an *ad hoc* collective utility function u_C so as to rationalize their actions as maximizing that u_C.

[3] For example, a minimally monotone u_C may consist of the sum of the individual utilities of C's members. As for a more eccentric example, u_C may be the sum of: the individual utility of C's first member, half of the individual utility of C's second member, a third of the individual utility of C's third member, and so on.

[4] In a sense, since each of them receives exactly the same goods for exactly the same price, the other members of C —if consulted when choosing u_C— should have no reason to object against i's receiving goods that he values more than his price relative to the case when he receives nothing. Indeed, they may even demand (additional) side-payments from a happier i!

A New Knowledge-Based Benchmark Traditional mechanism design works by leveraging the knowledge that each player has about himself, but a different and potentially enormous source of knowledge exists: *the knowledge that the players have about each other.* Focussing on auctions, each player i not only has *internal knowledge*, that is knowledge of his own true valuation TV_i, but also some *external knowledge*, that is some information about TV_{-i}, the other players' true valuations. (This is without any loss of generality, since the external knowledge of i may be "empty.") We prove that even this external knowledge alone is successfully exploitable, even when the designer has no information whatsoever about the players.[5] Indeed:

A main feature of our approach is to benchmark a mechanism's performance against the players' external knowledge.

The external knowledge relevant to our mechanism is "how well each independent player i could sell the goods" if he were the seller, via take-it-or-leave-it offers to the other players. (That is, i's relevant external knowledge is the maximum revenue that i knows he can guarantee via an outcome in which only the players in $-i$ receive goods, everyone only pays if he receives some goods, and no one pays more than his true value for the received goods.) Accordingly, initially assuming that the seller is *totally ignorant,*

Our revenue benchmark is the revenue "known" to the *best informed* independent player.

Players (e.g., in spectrum auctions) may have quite accurate information about their competitors' valuations, yet this source of knowledge was under-utilized in auction design. Aiming at collusion neutralization, we focus on the knowledge of independent players, and we guarantee nothing when they have no "external knowledge." (But, in separate papers, we obtain meaningful results in this setting too.)

A New Solution Concept Although not DST, the mechanism put forward in this paper is immune to any equilibrium-selection problem. The reason is very simple: it relies on an *equilibrium-less* solution concept. At a very high level, it guarantees its goals as long as each player selects a strategy surviving iterated elimination of "essentially strictly dominated strategies."

This elimination and our solution concept are somewhat complex because our mechanism is of extensive-form, and is run in a collusive setting where the players may have little or no information about who colludes with whom, or the collective utility functions of the collusive sets (if any).

After this iterative elimination process is completed, each player is left with a plurality of surviving strategies, and ultimately he chooses one of them to play. Accordingly, it is quite possible that the strategy profile actually played is not an equilibrium at all, yet the mechanism's goals are achieved just the same: any profile of "not-dumb" strategies will do. Therefore, what we call *implementation in surviving strategies* solely relies on common knowledge of *rationality*, and not on the players' beliefs. In such an implementation it is hard to predict precisely which profile of strategies will be ultimately played. But while "strategy predictability" has always been the cornerstone of traditional mechanism design, it has always been a *mean to an end*, not *the end itself.* Ultimately,

in an auction we do not care about predicting strategies,
but care a lot about predicting revenue (or social welfare).

A New Mechanism We exhibit a new mechanism $\mathcal{M}$ for unrestricted combinatorial auctions. Whenever the players select strategies surviving iterated elimination of dominated strategies, $\mathcal{M}$ guarantees, in our general collusion model and without any information about the players, revenue equal to a half of the revenue known to the best informed independent player (minus an arbitrarily small ϵ). Importantly, $\mathcal{M}$ is computationally efficient, makes the use of approximation algorithms compatible with players' incentives, and preserves the players' privacy to an unusual extent.

3 Prior Work

Protecting auctions against collusion has so far envisaged either restricted types of collusion or restricted types of auctions. Notably, *group strategy-proof* mechanisms [13, 17, 6] are robust only against collusive players unable to make side-payments to each other. On the other hand, *c-truthful* mechanisms [7] withstand a general behavior of collusive players, but are restricted to offer any subset of goods S to any player i for a fixed price $p_{S,i}$, and thus cannot produce high revenue unless sufficient information about the players' true valuations is known in advance to the designer. Alternative forms of such mechanisms are free to choose more general outcomes (so as to approximate maximum revenue), but satisfy a weaker notion of collusion resilience and apply to restricted auctions: namely, single

[5] Of course, auction mechanisms are easier to come by if the designer has suitable information about the players' true valuations. But acquiring this information may be too hard. In particular, for auctions of a single good, Cremer and McLean [5] have fully captured the information structure needed to generate the maximum possible revenue, but concluded that acquiring this information would be too difficult for their result to be of practical use.

good in unlimited supply. Resilient mechanisms also exist for a variety of other restricted auctions, in particular those of [14, 2, 3, 10, 15, 8]. By contrast, we do not impose any such restrictions.

Finally, all the above mechanisms are based on dominant-strategy equilibria. A solution concept closer to ours is the classical one of *implementation in undominated strategies*. In essence, in our language, a mechanism M achieves a property $\mathbb{P}$ in undominated strategies if $\mathbb{P}$ holds for any outcome obtained by running M on a profile of undominated strategies. (See Jackson [12] for a formal version.) This notion, however, was never exemplified in any setting of incomplete information, let alone in auctions. Babaioff, Lavi and Pavlov [2] both proposed a *feasible* variant of this notion (in essence, each player can compute his undominated strategies efficiently) and provided the first (and efficient too) mechanism satisfying it for a restricted type of combinatorial auctions. Namely, their mechanisms applies to auctions in which each player i has only two possible values for any subset of the goods: either 0 or a fixed value v_i. In sum, the solution concept of [2] requires less rationality than ours, but their mechanism does not address collusion at all, and does not apply to unrestricted combinatorial auctions.

4 Generalized Contexts and Generalized Auctions

A traditional context for a combinatorial auction can be fully specified by the true-valuation profile TV alone. A *generalized auction context* is instead fully described by the tuple $(TV, (\mathbb{C}, I), u, RK_I, GK_I)$, where: TV continues to be the true-valuation profile, again an original and fundamental object; $(\mathbb{C}, I)$ describes the *collusion structure*, that is the sets of collusive players and the set of independent players; u describes the *generalized utilities*, that is the utility of each *agent*, that is collusive set or independent player, for each possible outcome; RK_I describes the *relevant knowledge*, that is the part of the independent players' knowledge exploited by the mechanism; and GK_I describes the *general knowledge*, that is all the information available to and exploitable by the independent players. Let us now explain.

4.1 Collusion Structure

In our auctions we envision arbitrarily many collusive sets. Our only restriction is their disjointness. Else, saying that a collusive set acts rationally becomes more problematic.

Definition 1. (Collusion Structure.) *A* collusion structure *consists of a pair $(\mathbb{C}, I)$, where $\mathbb{C}$ is a partition of the players, and I is the set of all players i such that $\{i\} \in \mathbb{C}$.*

We refer to a player in I as *independent*, to a player not in I as *collusive*, to any $C \in \mathbb{C}$ of cardinality > 1 as a *collusive set*. We use the term *agent* to denote either an independent player or a collusive set. Since each player i, collusive or not, belongs to a single set in $\mathbb{C}$, for uniformity of presentation we may denote by C_i the set to which i belongs. (A collusion structure specifies separately the set I for convenience and clarity only.)

4.2 Generalized Utilities

Definition 2. (Generalized Utility Function.) *We say that u is a* generalized utility function*, for a set of players with true-valuation profile TV and collusion structure $(\mathbb{C}, I)$, if u is a vector, indexed by the subsets in $\mathbb{C}$, of functions from outcomes to real numbers satisfying the following two properties*

1. *For all $C \in \mathbb{C}$ and all outcomes (A, P) and (A', P') such that $(A_C, P_C) = (A'_C, P'_C)$: $u_C(A, P) = u_C(A', P')$.*
2. *For all $i \in I$: $u_{\{i\}}(A, P) = TV_i(A_i) - P_i$.*

We refer to u_C as C's *collective utility function.* If $i \in I$, we more simply write u_i rather than $u_{\{i\}}$.

Definition 3. (Minimally Monotone Utilities.) *Let u be a generalized utility function for a collusion structure $(\mathbb{C}, I)$. We say that u is* minimally monotone *if, $\forall C \in \mathbb{C}$, $\forall j \in C$, and $\forall$ outcomes (A, P) and (A', P') such that*

$(A_{C\setminus\{i\}}, P_{C\setminus\{i\}}) = (A'_{C\setminus\{j\}}, P'_{C\setminus\{j\}})$ *and* $A'_i = \emptyset$,

we have:

$$u_C(A, P) \geq u_C(A', P') \text{ if and only if } TV_i(A_i) - P_i \geq -P'_i.$$

4.3 Our Relevant Knowledge and Benchmark

Definition 4. (External and Relevant Knowledge.) *Let i be an independent player in a combinatorial auction setting with true-valuation profile TV and collusion structure $(\mathbb{C}, I)$. Then, i's* external knowledge*, denoted by EK_i, is the set of all outcomes (A, P) satisfying three properties:*

1. External Sale Only. $A_i = \emptyset$ *and* $P_i = 0$.

2. No Donation. P_j *is a positive integer whenever* $A_j \neq \emptyset$.

3. Known Feasibility. $P_j = 0$ *if* $A_j = \emptyset$*; else, it is known to i that* $P_j < TV_j(A_j)$.

The relevant knowledge *of i, denoted by RK_i, is defined to be the outcome in EK_i having maximum revenue; that is,*

$$RK_i = \operatorname*{argmax}_{\Omega \in EK_i} \text{REV}(\Omega).$$

Notice that any outcome Ω in EK_i is a way for i to "offer the goods to the other players." Indeed, property 1 guarantees that Ω offers non-empty subsets of the goods only to players other than i. Moreover, property 3 guarantees to i that, *so long as the generalized utility function is minimally monotone*, each offer in Ω will be (rationally) accepted. Thus, Ω essentially is a guaranteed way for i to generate revenue $\text{REV}(\Omega)$ by selling the goods to the other players, no matter what the minimally monotone generalized utility function u may be. Property 2 is a technical requirement with two purposes. The fact that "P_j is positive" is used in our mechanism. (Our mechanism incentivizes players to get some goods for free by providing their relevant knowledge to the seller, and property 2 prevents them from getting goods for free by relying on other players.) The fact that "P_j is an integer" ensures that RK_i is well defined. (If EK_i had infinite cardinality, the outcome with the maximum revenue may not exist.) Of course, if breaking ties proves necessary, RK_i is chosen to be the lexicographically first outcome with maximum revenue.

Definition 5. (The $\mathbb{MEW}$ Benchmark.) *In a generalized context with independent player set I, the* best informed player *in I, $bip(I)$, is so defined:*

$$bip(I) = \operatorname*{argmax}_{i \in I} \text{REV}(RK_i).$$

And the maximum external welfare *benchmark is so defined:*

$$\mathbb{MEW}(RK_I) = \text{REV}(RK_{bip(I)}).$$

Notice that the known-feasibility property of RK_i guarantees that our benchmark consists of "true knowledge." (In future papers we shall also consider benchmarks based on players' *beliefs* rather than true knowledge.)

4.4 General Knowledge

Definition 6. *In a combinatorial auction setting with true-valuation profile TV, collusion structure $(\mathbb{C}, I)$, and generalized, minimally monotone utility function u, the* general knowledge *of an independent player i, denoted by GK_i, consists of (TV_i, the fact that $i \in I$, and) all the information about TV_{-i}, $(\mathbb{C}, I)$, and u_{-i}, known to i.*

Of course, GK_i and RK_i must be *compatible* with each other. In a sense, RK_i should be properly deduced from GK_i. Let us consider two examples.

1. GK_i consists of a subset of $\mathbb{V}_{-i}$ (the set of all possible valuation subprofiles for the players in $-i$) such that $TV_{-i} \in GK_i$. That is, GK_i is the set of all possible candidates, in i's opinion, for the other players' true valuations. Such GK_i is genuine in the sense that one of its candidates is the "right one."[6] In this example, RK_i is deduced from GK_i in two conceptual steps. First, for each valuation subprofile $V \in GK_i$, one computes EK_i^V, the set of all outcomes satisfying properties 1, 2, and 3 of Definition 1 assuming that $TV_{-i} = V$. Then, one computes EK_i, the intersection of all EK_i^V's. Third, one computes RK_i from EK_i as usual. (Thus, if $GK_i = \mathbb{V}_{-i}$, then RK_i is the null outcome.)
2. GK_i consists of a "partial" probability distribution over $\mathbb{V}_{-i}$.[7] For instance, starting with a distribution D assigning positive probability to the actual subprofile TV_{-i}, GK_i is derived from D as follows: if the probability p_V of a subprofile $V \in \mathbb{V}_{-i}$ is positive, then p_V is replaced with a subinterval I_V of $[0, 1]$ that includes p_V. ($I_V = [0, 1]$ is interpreted as i knowing "nothing" about profile V.) In this case, one first computes EK_i^V, as in example 1, for each valuation subprofile V whose subinterval does not coincide with $[0, 0]$, and then computes EK_i and RK_i accordingly.

Why also considering GK_i if our mechanism uses RK_i?

[6]Notice that $GK_i = \mathbb{V}_{-i}$ expresses the fact that i knows "nothing" about TV_{-i}. Also notice that a proper choice of GK_i can precisely express pieces of i's external knowledge such as "player h's valuation for subset S is larger than player j's valuation for subset T."

[7]Of course, GK_i could more simply specify the true probabilistic distribution from which TV_{-i} is drawn, but considering it as a partial probability distribution is more general.

The reason is: *meaningfulness.* Indeed, after a mechanism is chosen, the players will rationally act relying on *all the knowledge available to them*, not just the one that the mechanism relies upon. And by relying on their general knowledge, they may act quite differently than the "mechanism wants." Accordingly, to enhance the meaningfulness of our results, we do not restrict the players' general knowledge at all. That is,

> *Our mechanism achieves our relevant-knowledge benchmark for all possible (but compatible!) general knowledge of the independent players.*

We are fully aware, of course, that better performance could be guaranteed by assuming some suitable restriction for the players' general knowledge. And some of these restrictions may be very realistic and profitable in some future contexts.

4.5 In Sum

Definition 7. *We say that a generalized auction context* $\mathscr{C} = (TV, (\mathbb{C}, I), u, RK_I, GK_I)$ *is* minimally monotone *if* u *is minimally monotone and this fact is common knowledge.*

We say that $(\mathscr{C}, M)$ *is a (minimally monotone) generalized auction if* M *is a mechanism, and* $\mathscr{C}$ *a (minimally monotone) generalized context.*

If $\mathscr{C}$ is a minimally monotone generalized context whose components have not been explicitly specified, then by default we assume that $\mathscr{C} = (TV^{\mathscr{C}}, (\mathbb{C}^{\mathscr{C}}, I^{\mathscr{C}}), u^{\mathscr{C}}, GK_I^{\mathscr{C}}, RK_I^{\mathscr{C}})$.

Remarks Although deducible from the subprofile GK_I, RK_I is explicitly part of the generalized context for clarity.

In this paper, an independent player i's relevant knowledge is non-Bayesian, even when his general knowledge is Bayesian: that is, RK_i always is a way known i to sell the goods to the other players that succeeds with probability 1, whenever the other players are rational. When player i's relevant knowledge is a probability distribution over $\mathbb{V}_{-i}$ more care is needed (the subject of a separate paper).

A generalized context does not define the knowledge of collusive players for two reasons. First, even if each player had his own general knowledge, the "general knowledge of a collusive set C may be hard to predict." (In principle there is no way to guarantee that C's members truthfully reveal their general knowledge to each other. For instance, if C arose from an initial negotiation, then a member i of C might have had incentives to lie about his knowledge in order to enter C and/or influence in his favor the choice of u_C.[8]) Second, we do not need any assumption on the collusive players' general knowledge to achieve our results. The minimal monotonicity of the generalized utility function is all we require.

(Although each GK_i has been defined to consist of information about TV_{-i}, $(\mathbb{C}, I)$ and u, it might also include information about the other players' knowledge.)

5 Our Solution Concept

Implementation in surviving strategies may already be useful in much simpler settings —e.g., settings with complete information and without collusion— and might perhaps benefit from a more gradual presentation. To avoid excessive overlapping, however, we choose to present it directly for the setting at hand: combinatorial auctions with secret valuations and collusive players. We actually focus on a *refinement* of our solution concept maximizing our mechanism's meaningfulness.

Implementation in Σ^1/Σ^2_I Strategies In practice, there seem to be different levels of rationality. That is, many players are capable of completing the first few iterations of elimination of "inferior" strategies, but fail to go "all the way." Accordingly, one should prefer mechanisms that guarantee the desired properties for any vector of strategies surviving just the first few iterations. Our mechanism achieves our benchmark for any vector of strategies surviving the following two-round elimination process. First, each agent removes all its *distinguishably dominated strategies* (DD strategies for short), as defined below. Then, each *independent player* further removes all strategies which now become distinguishably dominated. Since in the case of other types of iterative eliminations (e.g., of strictly dominated strategies) it is common to refer to the set of strategy profiles surviving the first iteration as Σ^1, and to that surviving the first two iterations as Σ^2, we call this refinement of our solution concept *implementation in* Σ^1/Σ^2_I *Strategies.*

Let us now provide a bit more motivation for our elimination process. Once a mechanism M is specified, a player i needs to choose a strategy for himself. If i has a single strictly dominant strategy, this choice is easy. Most of the

[8]Perhaps better results may be obtained by restricting the collective knowledge (or the process of coalition formation) but these possibilities are not investigated in this paper.

time, however, such a strategy of i does not exist. When this is the case, i should refine his strategy set by eliminating at least some of his strategies, that is, he should identify some strategies he should never play. To better refine his strategies, i may also need to refine his opponents' strategy sets, relying on his own general knowledge and the common knowledge of rationality. Of course, many such strategy-refining processes $\mathscr{P}$ exist. But we only want those satisfying the following informal properties:

1. *Mechanism Usefulness.* After each player ends his own execution of $\mathscr{P}$, M must achieve its desired property for any profile of strategies σ (not necessarily an equilibrium!) such that each σ_i belongs to i's refined strategy set.
2. *Player Safety.* If i eliminates (in his mind) a strategy σ_j of an opponent j, then a rational j will never play σ_j. If i eliminates a strategy σ_i from his own set, then he will never regret this elimination.

(Note that such an elimination procedure $\mathscr{P}$ needs not be "maximal:" for what we care, it can "stop as soon as Property 1 holds." For instance, if the iterated elimination of all strictly dominated strategies whose binary representation starts with 0 satisfied mechanism usefulness, we would be satisfied. Note too that finding such a $\mathscr{P}$ might be easier if the players were perfectly informed about each other. But in our setting they have *imperfect knowledge* about each other.)

The strategy-refining procedure formalized below satisfies Properties 1 and 2 even in the presence of collusion and with imperfect player knowledge. Indeed, whether a strategy is distinguishably dominated depends on additional factors (such as the collusive sets actually present and their collective utility functions) about which an agent may not have any information beyond what it knows about itself. In particular, some independent players may not know that there are collusive players, while other players (independent or not) may have different knowledge about the actual collusive sets and their collective utility functions. We thus insist that each eliminated strategy for an agent $\mathcal{A}$ must never be played by a rational $\mathcal{A}$, no matter what knowledge it may have (i.e., in any generalized context compatible with $\mathcal{A}$'s knowledge). This of course cannot but increase the number of surviving strategy vectors. Yet, it will not be a problem as long as we can guarantee our benchmark for all such vectors.

Before formalizing our notions, let us quickly recall the type of mechanisms we use, and some needed notation.

Extensive-Form Public-Action Auction Mechanisms. We focus solely on auction mechanisms of extensive form. Thus our mechanisms must specify the *decision nodes* (of a game tree), the player(s) *acting* at each node, the set of *actions* available to each acting player at each node, and the auction outcome (i.e., the allocation A and the price profile P) associated to each *outcome node* (i.e., each terminal node —leaf of the game tree). Our mechanisms may actually specify multiple players to act simultaneously at some decision nodes. Our mechanisms also are of *public action*: that is, each action becomes common knowledge as soon as it is played.[9]

A player i's *strategy* specifies i's action at each decision nodes in which i acts. A *play* of a mechanism M consists of a profile of strategies. If σ is such a play, then

- $H(\sigma)$ denotes the *history* of the play, that is the outcome node finally reached by executing M when each player chooses his actions according to σ_i. (Equivalently, $H(\sigma)$ is the sequences of decision nodes reached during the execution.)
- $M(\sigma)$ denotes the auction outcome (A, P) associated to $H(\sigma)$.

If M is probabilistic, then both $H(\sigma)$ and $M(\sigma)$ are distributions, respectively over node outcomes and auction outcomes.

Formalization. Denoting the set of all deterministic strategies of a player i by Σ_i^0, the set of all deterministic strategy profiles by Σ^0, the set of all deterministic collective strategies of a collusive set C by Σ_C^0, the set of all deterministic strategy vectors of a generalized context $\mathscr{C}$ by $\Sigma_{\mathscr{C}}^0$, and the Cartesian product by $\prod$, we have

$$\Sigma^0 = \prod_i \Sigma_i^0, \quad \Sigma_C^0 = \prod_{i \in C} \Sigma_i^0, \quad \text{and } \Sigma_{\mathscr{C}}^0 = \Sigma^0.$$ [10]

Definition 8. (Σ^1 Strategies.) *Let $(\mathscr{C}, M)$ be a generalized auction, Σ' a set of deterministic-strategy vectors for $(\mathscr{C}, M)$, $\mathcal{A}$ an agent in $(\mathscr{C}, M)$, and $\sigma_{\mathcal{A}}$ and $\sigma'_{\mathcal{A}}$ two strategies of $\mathcal{A}$ in $\Sigma'_{\mathcal{A}}$. We say that $\sigma_{\mathcal{A}}$ is* distinguishably dominated *(by $\sigma'_{\mathcal{A}}$) over Σ' if*

1. *$\exists \tau_{-\mathcal{A}}$* distinguishing *$\sigma_{\mathcal{A}}$ and $\sigma'_{\mathcal{A}}$ over Σ'; that is, $\tau_{-\mathcal{A}} \in \Sigma'_{-\mathcal{A}}$ and $H(\sigma_{\mathcal{A}} \sqcup \tau_{-\mathcal{A}}) \neq H(\sigma'_{\mathcal{A}} \sqcup \tau_{-\mathcal{A}})$;*[11] *and*
2. *$\mathbb{E}[u_{\mathcal{A}}(M(\sigma_{\mathcal{A}} \sqcup \tau_{-\mathcal{A}}))] < \mathbb{E}[u_{\mathcal{A}}(M(\sigma'_{\mathcal{A}} \sqcup \tau_{-\mathcal{A}}))]$ for all subvectors $\tau_{-\mathcal{A}}$ distinguishing $\sigma_{\mathcal{A}}$ and $\sigma'_{\mathcal{A}}$ over Σ'.*

[9] We refrain from using the more standard term "perfect-information" to avoid confusion. Our setting is in fact of "incomplete information." That is, a player's true valuation is not exactly known to his opponents. And mechanisms of "perfect information and incomplete information" would be too much...

[10] Indeed, the latter equality follows because $\Sigma_{\mathscr{C}}^0 = \prod_{C \in \mathbb{C}^{\mathscr{C}}} \Sigma_C^0 = \prod_{C \in \mathbb{C}^{\mathscr{C}}} \prod_{i \in C} \Sigma_i^0 = \Sigma^0$.

[11] If M is probabilistic, then $H(\sigma_{\mathcal{A}} \sqcup \tau_{-\mathcal{A}})$ and $H(\sigma'_{\mathcal{A}} \sqcup \tau_{-\mathcal{A}})$ are distributions over node outcomes, and the inequality means that the two distributions are different.

If Property 1 does not hold, $\sigma_{\mathcal{A}}$ *and* $\sigma'_{\mathcal{A}}$ *are called* equivalent *over* Σ'*. A strategy not distinguishably dominated over* Σ' *is called* undominated *over* Σ'*. We define the sets* $\Sigma^1_{\mathcal{A},\mathscr{C}}$ *and* $\Sigma^1_{\mathscr{C}}$ *as follows:*

- $\Sigma^1_{\mathcal{A},\mathscr{C}}$ *is the set of strategies in* $\Sigma^0_{\mathcal{A},\mathscr{C}}$ *that are undominated over* $\Sigma^0_{\mathscr{C}}$*.*
- $\Sigma^1_{\mathscr{C}} = \prod_{C\in\mathbb{C}^{\mathscr{C}}} \Sigma^1_{C,\mathscr{C}}$.

Note that each $\Sigma^1_{\mathcal{A},\mathscr{C}}$ is uniquely defined up to "renaming/removing" equivalent strategies.

Definition 9. (Compatibility.) *A context* $\mathscr{C}$

- is compatible with an independent player i *if* $TV_i = TV^{\mathscr{C}}_i$*,* $i \in I^{\mathscr{C}}$*,* $RK_i = RK^{\mathscr{C}}_i$*, and* $GK_i = GK^{\mathscr{C}}_i$
- is compatible with a collusive set C *if* $C \in \mathbb{C}^{\mathscr{C}}$ *and* $u^{\mathscr{C}}_C$ *is* C*'s collective utility function.*

Note that, fixing the mechanism M, we have

(1) $\Sigma^1_{\mathcal{A},\mathscr{C}}$ *is the same for any* $\mathscr{C}$ *compatible with* $\mathcal{A}$*.* In fact, the set $\Sigma^0_{\mathscr{C}}$ $(= \Sigma^0)$ is fully determined from M alone, and which strategies of $\mathcal{A}$ are undominated over Σ^0 solely depends on $\mathcal{A}$'s utility function (rather than, say, on the partition of the other players into collusive sets, and their utility functions). Accordingly, we shall more simply write $\Sigma^1_{\mathcal{A}}$ instead of $\Sigma^1_{\mathcal{A},\mathscr{C}}$, and thus $\Sigma^1_{\mathscr{C}} = \prod_{C\in\mathbb{C}^{\mathscr{C}}} \Sigma^1_C$.

(2) $\Sigma^1_{\mathscr{C}}$ *is crucially dependent on* $\mathscr{C}$*.* In fact, although the set of undominated strategies of each agent is "independent of the overall collusive context", a set of players C may be an agent in a collusive context $\mathscr{C}$, but not in another context $\mathscr{C}'$. (For instance, C may consist only of independent players in $\mathscr{C}'$. In this case, let 10 be the number of players in C. Then, in $\mathscr{C}$, Σ^1_C may consist of a single collective strategy. While in $\mathscr{C}'$, each independent player may have 2 uneliminated strategies, so that Σ^1_C consists of 1024 strategy subprofiles.)

Definition 10. (Σ^2_I Strategies and Σ^1/Σ^2_I Plays) *Let* i *be an independent player in a generalized auction. Then,*

- *We say that a strategy* $\sigma_i \in \Sigma^1_i$ *is* globally dominated *if there exists a strategy* $\sigma'_i \in \Sigma^1_i$ *such that, for all generalized contexts* $\mathscr{C}$ *compatible with* i*,* σ'_i *distinguishably dominates* σ_i *over* $\Sigma^1_{\mathscr{C}}$*.*
- *We denote by* Σ^2_i *the set of strategies in* Σ^1_i *which are not globally dominated.*
- *We say that a strategy vector* σ *is a* Σ^1/Σ^2_I play *of a generalized auction* $(\mathscr{C}, M)$ *if*

$$\sigma \in \prod_{i\in I^{\mathscr{C}}} \Sigma^2_i \times \prod_{C\in\mathbb{C}^{\mathscr{C}},|C|>1} \Sigma^1_C.$$

Definition 11. (Implementation in Σ^1/Σ^2_I Strategies.) *Let* $\mathbb{P}$ *be a property of auction outcomes, and* M *an auction mechanism. We say that* M implements $\mathbb{P}$ in Σ^1/Σ^2_I strategies *if, for all generalized contexts* $\mathscr{C}$*, and all* Σ^1/Σ^2_I *plays* σ *of the auction* $(\mathscr{C}, M)$*,* $\mathbb{P}$ *holds for* $M(\sigma)$*.*

Remarks Note that Σ^2_i has been obtained by eliminating Σ^1_i strategies that are distinguishably dominated in a very strong sense. A player having more information about the real context might be able to eliminate more strategies, and thus further refine Σ^2_i.

If M is probabilistic, then $M(\sigma)$ is a distribution over outcomes, and $\mathbb{P}$ a property of outcome distributions.

6 Our Mechanism

Although requiring some modifications, the basic idea behind our mechanism is very simple: each player i, simultaneously with the others, announces an outcome $\Omega^i = (\alpha^i, \pi^i)$ satisfying properties 1, 2, and 3 of Definition 1, except that we do not require that $\pi^i_j < TV_j(\alpha^i_j)$ whenever $\alpha^i_j \neq \emptyset$. Let $\star$ be the "star player", that is the one who has announced the outcome with the highest revenue. Then, we try to sell the goods according to $\Omega^\star$, so as to generate revenue $R_\star = \text{REV}(\Omega^\star)$. That is, we ask each player $i \in -\star$, receiving some goods in $\alpha^\star$, whether he is willing to buy the subset of goods $\alpha^\star_i$ for price $\pi^\star_i$. If i agrees, the subsale is final. Else, the star player pays a fine equal to $\pi^\star_i$ and the goods in $\alpha^\star_i$ remain unallocated.

Note that the star player may not be the best informed independent player, but our benchmark is achieved *if each independent player i does not "underbid",* that is, if he announces an outcome whose revenue is at least as high as that of RK_i. Thus: can we ensure that an independent player i does not underbid? The problem is that, *depending on the generalized context, underbidding may be the best thing to do.* For instance the context may be such that (1) player i is the best informed player; (2) the second best informed player is j; and yet (3) player j is badly informed about player i: that is, RK_j allocates to i a subset of goods S_i that i highly values for a ridiculously low price. In this case, i would be better off if j announced RK_j and were the star player. In sum, underbidding may be far from being a dominated strategy.

To guarantee that rational independent players do not underbid, we adopt a two-pronged strategy.

First, our mechanism modifies the above basic procedure as follows. Together with RK_i, each player i also announces his (suppositively) favorite subset of the goods, S_i. If i is declared the star player, then a coin toss of the mechanism determines whether the above basic procedure takes place or i receives S_i for free, and all other goods remain unallocated.

Second, the mechanism gives back to the players some amount of revenue: in particular, a *fixed* small amount, $0 < \epsilon < 1$. Each player i actually gets a fraction of ϵ proportional to the revenue of his announced Ω^i. (Formally, this enables us to avoid any ambiguity about relying on "weakly dominated strategies" in our solution concept.)

A precise analysis proves that, for all *independent* players, underbidding does not survive the iterated elimination of distinguishably dominated strategies. The same cannot be said about *collusive* players, but then we do not rely on them for revenue. However, we must ensure that they do not hurt the achievement of our benchmark.

(In $\mathcal{M}$'s description below, real actions occur in "numbered steps" and public updates in "bulleted steps.")

Mechanism $\mathcal{M}$

- Set $A_i = \emptyset$ and $P_i = 0$ for each player i.

1. Each player i simultaneously and publicly announces (1) an outcome $\Omega^i = (\alpha^i, \pi^i)$ such that $\alpha^i_i = \emptyset$, $\pi^i_i = 0$, and π^i_j is 0 whenever $\alpha^i_j = \emptyset$ and a positive integer otherwise; and (2) a subset S_i of the goods.

- Set: $R_i = \text{REV}(\Omega^i)$ for each player i and $\star = \text{argmax}_i R_i$. (We shall refer to player $\star$ as the "star player.")

2. Publicly flip a fair coin.

- (If Heads:) reset $A_\star := S_\star$ and HALT.

3. (If Tails:) Each player i such that $\alpha^\star_i \neq \emptyset$ simultaneously and publicly announces YES or NO.

- Reset:

  (1) for each player i announcing NO, $P_\star := P_\star + \pi^\star_i$;

  (2) for each player i announcing YES, $A_i := \alpha^\star_i$
  and $P_i := \pi^\star_i$; and

  (3) for each player i, $P_i := P_i - \frac{\epsilon R_i}{1+\sum_j R_j}$.

7 Analysis of Our Mechanism $\mathcal{M}$

Our mechanism $\mathcal{M}$ and its analysis assume that a player's true valuation maps subsets of the goods to non-negative numbers (but we could handle negative valuations as well).[12] In the analysis below, all individual strategies, collective strategies, and vectors of strategies are relative to $\mathcal{M}$.

Three Lemmas. Our result follows from the three intuitive but technical lemmas below, proved in our appendix.

Lemma 1. *For all independent players i and all $\sigma_i \in \Sigma^1_i$: if $i \neq \star$ and $\alpha^\star_i \neq \emptyset$ after Stage 1, then in Stage 3 (that is, when $\mathcal{M}$'s coin toss comes up Tails)*

1. i answers YES whenever $TV_i(\alpha^\star_i) > \pi^\star_i$, and

2. i answers NO whenever $TV_i(\alpha^\star_i) < \pi^\star_i$.

Lemma 2. *For all minimally monotone collusive sets C and all $\sigma_C \in \Sigma^1_C$: if $\star \notin C$ after Stage 1, then in Stage 3, for all players i in C,*

1. i answers YES whenever $\alpha^\star_i \neq \emptyset$ and $TV_i(\alpha^\star_i) > \pi^\star_i$, and

2. i answers NO whenever $\alpha^\star_i \neq \emptyset$ and $TV_i(\alpha^\star_i) < \pi^\star_i$.

[12] In traditional auctions, valuations *are* bids, and the seller would immediately dismiss bids associating a subset S of the goods to a negative number (since he has no intention to assign S to a player and also pay him to accept S). The "bidding process" of our mechanism however asks each player i to announce in Step 1 a subset S_i of the goods without mentioning any value for S_i. In principle, therefore, i may have a negative valuation for S_i. And leaving things as they stand, i may have (subtle) reasons to announce such an S_i.

Lemma 3. *$\forall$ independent player i and $\forall\ \sigma_i \in \Sigma_i^2$,*
$\text{REV}(\Omega^i) \geq \text{REV}(RK_i)$ (that is, i does not "underbid").

Notice that, while ruling out underbidding (relative to RK_i) for independent players, our lemmas say nothing about the possibility of "over-bidding." In fact, not only over-bidding needs not be distinguishably dominated, but even be the rational thing to do for some independent players —it all depends on their general knowledge.[13]

Fortunately, our main theorem guarantees that we achieve our revenue benchmark whether or not over-bidding is dominated for all independent players.

Main Theorem

Theorem 1. *For all minimally monotone generalized contexts $\mathscr{C}$ and all Σ^1/Σ_I^2 plays σ of $(\mathscr{C}, \mathcal{M})$, we have*

$$\mathbb{E}[\text{REV}(\mathcal{M}(\sigma))] \geq \frac{\mathbb{MEW}(RK_I) - \epsilon}{2}.$$

Proof. By Definition 2, $bip(I)$ is the independent player "realizing" our benchmark: that is,

$$bip(I) = \operatorname*{argmax}_{i \in I} \text{REV}(RK_i).$$

Notice that players $bip(I)$ and $\star$ may not coincide, and that the following two inequalities hold in any Σ^1/Σ_I^2 play of $(\mathscr{C}, \mathcal{M})$:

(a) $\text{REV}(\Omega^{bip(I)}) \geq \text{REV}(RK_{bip(I)}) = \mathbb{MEW}(RK_I)$.

(b) $R_\star \geq \mathbb{MEW}(RK_I)$.

Indeed, Inequality (a) holds because player $bip(I)$ is independent and, by Lemma 3, he does not underbid; and Inequality (b) holds by Inequality (a) and the fact that $R_\star \geq \text{REV}(\Omega^{bip(I)})$ by the definition of $\star$.

Notice that $\mathcal{M}$ generates revenue only when its coin toss comes up Tails, and when this happens, the revenue generated coincides with $R_\star$, because for each player i such that $\pi_i^\star > 0$, the seller receives $\pi_i^\star$, from i if i announces YES, and from $\star$ if i announces NO. Therefore for any $\sigma \in \Sigma^1/\Sigma_I^2$, the following equality holds for $\mathcal{M}$'s expected revenue:

$$\text{(c)} \quad \mathbb{E}[\text{REV}(\mathcal{M}(\sigma))] > \frac{R_\star - \epsilon}{2},$$

because the total "rebate" given to all players in the last step is $< \epsilon$. Inequalities (b) and (c) imply our thesis.
Q.E.D.

8 Additional Properties of $\mathcal{M}$

Privacy and Deniability/"Tax Freeness" Although traditionally irrelevant to mechanism design, the privacy that a mechanism provides to its players is important.

Consider a second-price auction of a single good where the highest true valuation for the good is \$10M, and the second highest is \$1M. Since the second-price mechanism is DST, we expect the players to bid their true valuations, so that the winner bids \$10M and pays \$1M. Assume, however, that the auction takes place in a country with an overreaching and tyrannical tax code, where the Internal Revenue Service is allowed to collect taxes on "auction utilities." (After all, even lottery gains are taxable in many countries!) In such a country, our winner would own taxes over \$9M. Indeed, unless he pleads "temporary insanity", acting rationally in a DST mechanism, he himself has freely admitted to a utility of \$9M. In such a country, therefore, it is not clear whether the players will bid truthfully in the second-price mechanism. Consider now selling the same good, to the same players and in the same country, via an ascending English auction. In such an auction truthful bidding is not endangered by the tax code: the players who "drop out" reveal their true valuations but are not "taxable" because they have no utility, while the winner never reveals his true valuation, so that he could always credibly *deny* to value the good for more than what he pays. Beyond tax advantages, players value their privacy, and the fact that ascending English auctions preserve the privacy of the winner perhaps provides an additional explanation of why they are more used than the second-price auctions.

[13] For example, assume that the relevant knowledge RK_i of an independent player i specifies allocating to a player j a subset of the goods S for a price p_j. However, i' general knowledge GK_i may include some Bayesian information according to which the probability that $TV_j(S) > 100p_j$ is extraordinarily high, although not equal to 1. In such a case, to maximize his chance of becoming the star player, i may be better off risking a possible answer NO from j and announce an outcome Ω^i allocating S to j for a price $100p_j$.

Our mechanism provides even more deniability/"tax advantages" than ascending English auctions, even when it deals with multiple goods for sale. A player receiving some goods never declares how much he values them. If $\mathcal{M}$'s coin ends up Heads, then the star player receives for free his favorite subset S, but never says anything *himself* about his own valuation for S. (Whatever the other players say about him could be just "hearsay," and over-bidding is not dominated for them.) If $\mathcal{M}$'s coin ends up Tails, then every player who answers YES receives goods that he may always claim to value for exactly what he was offered to pay and indeed paid. In addition our mechanism preserves, to a very large extent, also the privacy of the players who do not receive any goods at all. (And with proper use of envelopes, our mechanism would reveal even less information, without having to trust the auctioneer.)

In sum, loss of privacy may alter the way games are played, and mechanisms should be designed so as to preserve as much privacy as possible. In particular we believe that being "tax-free" is a crucial property of an auction mechanism.

(A general result of Izmalkov, Lepinski, and Micali [11] guarantees that every normal-form mechanism has an extensive-form version that perfectly implements it, in particular without any privacy loss. Note, however, that their construction —although feasible— is practically inconvenient. By contrast, our mechanism $\mathcal{M}$ offers a lot of privacy "for free!"[14])

Computational Efficiency Although traditionally neglected in mechanism design, the computational efficiency of a mechanism is important too.

Unlike the VCG, that requires solving NP-hard problems, our mechanism $\mathcal{M}$ is computationally trivial. Indeed, after each player i reports his Ω^i, $\mathcal{M}$ needs only to sum up the prices in each Ω^i and figure out the outcome with the largest and second-largest revenues.

One might object, however, that $\mathcal{M}$ transfers all hard computations to the players themselves. Not quite so, because:

1. Computing the relevant knowledge *might be easy:* it all depends on the form of general knowledge. (In the extreme, if RK_i coincides with GK_i, there is nothing for player i to compute.)
2. Each player i can always use an efficient algorithm to approximate RK_i *without altering any incentives.*

Let M be a DST auction mechanism whose outcome function f, as for the VCG, is *very hard to compute.* Then, the auctioneer could not use any feasible approximation f' of f, because the so modified mechanism M' may no longer be DST! By contrast, in our $\mathcal{M}$, it is in each independent player i's best interest to report his best possible approximation to RK_i known to him. (Because reporting an outcome known to i to be feasible dominates reporting any outcome with a lower revenue.) Thus, while for the VCG *approximation distorts incentives,* for our $\mathcal{M}$ *approximation is aligned with incentives.* Putting it differently, assume that RK_i is really hard to deduce from GK_i, and that an auction using mechanism $\mathcal{M}$ is announced in two weeks time. Then, after computing day and night for two weeks, each independent player i can only report an approximation to his true RK_i. Yet, $\mathcal{M}$ still achieves our benchmarks, not defined on the *perfect* relevant knowledge, but on the *relevant knowledge actually known to the players at the time of the auction.*

9 Variants and Extensions

Our mechanism relies on rewarding the players by giving them back a fixed and small "rebate" ϵ. Keeping on the spirit of $\mathcal{M}$ there are several alternatives for rewarding the players. In particular, one may consider rewards as percentages of the revenue collected. As for another alternative, one may consider rewarding the star player, when the coin toss ends up Tails, with the difference between $R_\star$ and the "second-highest revenue." (Of course these alternatives reduce our revenue benchmark. However the "sum of social welfare and revenue" will not be affected.)

So far, we have assumed that the seller/designer knows nothing about the players. But it is easy to accommodate a designer having some knowledge about the players. For instance, this knowledge may —in keeping with our overall approach— be modeled as an outcome Ω which he knows to be "realizable." In particular, if $R_\star$ is smaller than $\text{REV}(\Omega)$, then the mechanism tries to sell the goods according to Ω.

Our approach yields significantly better results for simpler auctions, in particular auctions of multiple copies of a single good. Here, in a forthcoming paper, not only we show that it is possible to achieve higher knowledge-benchmarks, but also that it is possible to *aggregate* the knowledge of all players into a greater *total collective knowledge.* In particular, in the case of a single copy of a single good, we can guarantee revenue that is at least equal to that of the second-price mechanism and possible significantly higher if the players are well informed about each other.

In this paper we benchmark against the "guaranteed" knowledge that the players have. In a forthcoming paper with Avinatan Hassidim we shall prove that stronger results are possible in a *more conservative* Bayesian setting, that

[14] Also note that $\mathcal{M}$ could provide further privacy if it first asked each player i to announce just the revenue of Ω^i in Stage 1, and then asked only the star player to reveal both $\Omega^\star$ and $S_\star$. However, this alternative way of proceeding would enable the star player to announce $\Omega^\star$ depending on the revenues announced by the other players, and thus an independent player i may have incentives to underbid.

is, assuming that the players (rather than the designer!) know the distribution from which their true valuations are drawn. This is more in line with the true spirit of mechanism design, where all knowledge lies with the players.

In this paper we show that it is possible to leverage the external knowledge of the independent players. In a most recent result with Paul Valiant, we shall prove that, even in unrestricted combinatorial auctions, it is possible to leverage both the *internal* and external knowledge of *all* players, including the collusive ones. (By a player's internal knowledge we mean his own true valuation.) The result assumes a slightly weaker collusion model and solution concept.

Beyond auctions, our approach also applies to other settings of mechanism design, such as provision of a public good (where all prior mechanisms were highly vulnerable to collusion, and in a way much more overt than in the VCG case). Indeed, we initiate *indirect mechanism design*, a new approach that, with new "strategy spaces," beyond the scope of the revelation principle, bypasses problems of incentive compatibility, computational complexity, and collusion.

10 Acknowledgements

Many thanks to David Easley, Shafi Goldwasser, Avinatan Hassidim, Robert Kleinberg, Eric Maskin, Paul Milgrom, Rafael Pass, Paul Valiant, Avi Wigderson, and an anonymous reviewer for comments and precious encouragement.

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Appendix

A Proofs of Our Three Lemmas

Let us recall and prove our three lemmas.

LEMMA 1. *For all independent players i and all $\sigma_i \in \Sigma_i^1$: if $i \neq \star$ and $\alpha_i^\star \neq \emptyset$ after Stage 1, then in Stage 3 (that is, when $\mathcal{M}$'s coin toss comes up Tails)*

1. *i answers YES whenever $TV_i(\alpha_i^\star) > \pi_i^\star$, and*
2. *i answers NO whenever $TV_i(\alpha_i^\star) < \pi_i^\star$.*

Proof. We restrict ourselves to just prove, by contradiction, the first implication (the proof of the second one is totally symmetric). Assume that there exist an independent player i and a strategy profile σ such that (1) $\sigma_i \in \Sigma_i^1$ and $\sigma_{-i} \in \Sigma_{-i}^0$, and (2) in the execution of σ, i answers NO in Stage 3 and the following property holds:
$\mathcal{P}_i$: $i \neq \star$, $\alpha_i^\star \neq \emptyset$, and $TV_i(\alpha_i^\star) > \pi_i^\star$.
Then, denoting by $\overline{\mathcal{P}_i}$ the negation of $\mathcal{P}_i$, that is,
$\overline{\mathcal{P}_i}$: $i = \star$, or $\alpha_i^\star = \emptyset$, or $TV_i(\alpha_i^\star) \leq \pi_i^\star$,
consider the following alternative strategy for i:

Strategy σ_i'

Stage 1. Run σ_i (with stage input "1" and private inputs TV_i and GK_i) and announce Ω^i and S_i as σ_i does.

Stage 3. If $\overline{\mathcal{P}_i}$, continue running σ_i and answer whatever σ_i does.[15]
If $\mathcal{P}_i$, answer YES.

We derive a contradiction by proving that σ_i is distinguishably dominated by σ_i' over Σ^0, which implies that $\sigma_i \notin \Sigma_i^1$.

Notice that $\mathcal{M}(\sigma_i \sqcup \overline{\tau}_{-i}) = \mathcal{M}(\sigma_i' \sqcup \overline{\tau}_{-i})$ for all subprofiles $\overline{\tau}_{-i} \in \Sigma_{-i}^0$ such that the execution of $\sigma_i \sqcup \overline{\tau}_{-i}$ either satisfies (1) $\overline{\mathcal{P}_i}$, or (2) $\mathcal{P}_i$ and i answers YES in Stage 3. This is so because for such $\overline{\tau}_{-i}$ the executions of $\sigma_i \sqcup \overline{\tau}_{-i}$ and $\sigma_i' \sqcup \overline{\tau}_{-i}$ coincide, and so do their outcomes whenever the coin toss of $\mathcal{M}$ is the same. Therefore such $\overline{\tau}_{-i}$'s do not distinguish σ_i and σ_i' over Σ^0. To prove that σ_i is distinguishably dominated by σ_i' over Σ^0, it suffices to consider the strategy subprofiles $\tau_{-i} \in \Sigma_{-i}^0$ such that the execution of $\sigma_i \sqcup \tau_{-i}$ satisfies $\mathcal{P}_i$ and i announces NO in Stage 3. Notice that, by assumption, $\tau_{-i} = \sigma_{-i}$ is one such subprofile.

For all such τ_{-i}, observe that, since σ_i' coincides with σ_i in Stage 1, the outcome profile Ω is the same in the executions of $\sigma_i \sqcup \tau_{-i}$ and $\sigma_i' \sqcup \tau_{-i}$. Accordingly, R_j is the same in both executions for each player j, and the star player too is the same in both executions. Since (by hypothesis) the execution of $\sigma_i \sqcup \tau_{-i}$ satisfies $\mathcal{P}_i$, so does the execution of $\sigma_i' \sqcup \tau_{-i}$.

We now distinguish two cases, each occurring with probability 1/2.

(1) $\mathcal{M}$'s coin toss comes up *Heads*.
In this case, because only the star player receives goods, we have: $A_i = \emptyset$ and $P_i = 0$ in both executions, and

$$u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i})) = u_i(\mathcal{M}(\sigma_i' \sqcup \tau_{-i})) = 0.$$

[15] The first implication of Lemma 1 specifies that $i \neq \star$ and $TV_i(\alpha_i^\star) > \pi_i^\star$. However, a strategy must be specified in all cases, and thus σ_i' must be specified also when $\overline{\mathcal{P}_i}$.

(2) $\mathcal{M}$'s coin toss comes up *Tails.*
In this case, because by hypothesis,
(a) $TV_i(\alpha_i^\star) > \pi_i^\star$,
(b) player i answers NO in the execution of $\sigma_i \sqcup \tau_{-i}$ and
(c) i answers YES in the execution of $\sigma'_i \sqcup \tau_{-i}$,
we have: $A_i = \emptyset$ and $P_i = -\frac{\epsilon R_i}{1+\sum_j R_j}$ in the execution of $\sigma_i \sqcup \tau_{-i}$, $A_i = \alpha_i^\star$ $(\neq \emptyset)$ and $P_i = \pi_i^\star - \frac{\epsilon R_i}{1+\sum_j R_j}$ in the execution of $\sigma'_i \sqcup \tau_{-i}$,

$$u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i})) = TV_i(\emptyset) - P_i = \frac{\epsilon R_i}{1+\sum_j R_j}, \text{ and}$$

$$u_i(\mathcal{M}(\sigma'_i \sqcup \tau_{-i})) = TV_i(\alpha_i^\star) - P_i = TV_i(\alpha_i^\star) - \pi_i^\star + \frac{\epsilon R_i}{1+\sum_j R_j} > \frac{\epsilon R_i}{1+\sum_j R_j}.$$

Combining the above two cases yields:
(a) $\mathcal{M}(\sigma_i \sqcup \overline{\tau}_{-i}) \neq \mathcal{M}(\sigma'_i \sqcup \overline{\tau}_{-i})$, since the two outcome distributions differ at A_i.
(b) $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] < \mathbb{E}[u_i(\mathcal{M}(\sigma'_i \sqcup \tau_{-i}))]$.
Therefore σ_i is distinguishably dominated by σ'_i over Σ^0. ■

LEMMA 2. *For all minimally monotone collusive sets C and all $\sigma_C \in \Sigma_C^1$: if $\star \notin C$ after Stage 1, then in Stage 3, for all players i in C,*

1. *i answers YES whenever $\alpha_i^\star \neq \emptyset$ and $TV_i(\alpha_i^\star) > \pi_i^\star$, and*
2. *i answers NO whenever $\alpha_i^\star \neq \emptyset$ and $TV_i(\alpha_i^\star) < \pi_i^\star$.*

Proof. We again restrict ourselves to just prove the first implication, and proceed by contradiction. Assume that there exist an minimally monotone collusive set C, a player $i \in C$, and a strategy vector σ such that $\sigma_C \in \Sigma_C^1$, $\sigma_{-C} \in \Sigma_{-C}^0$, and in σ's execution i answers NO in Stage 3 and the following property holds:
$\mathcal{P}_{i,C} : \star \notin C$, $\alpha_i^\star \neq \emptyset$, and $TV_i(\alpha_i^\star) > \pi_i^\star$.
Then, denoting by $\overline{\mathcal{P}_{i,C}}$ the negation of $\mathcal{P}_{i,C}$, that is,
$\overline{\mathcal{P}_{i,C}} : \star \in C$, or $\alpha_i^\star = \emptyset$, or $TV_i(\alpha_i^\star) \leq \pi_i^\star$,
consider the following alternative collective strategy for C.

Strategy σ'_C

Stage 1. Run σ_C and announce Ω^j and S_j as σ_C does for all $j \in C$.

Stage 3. If $\overline{\mathcal{P}_{i,C}}$, continue running σ_C, and announce whatever σ_C does for all $j \in C$.
If $\mathcal{P}_{i,C}$, continue running σ_C, and announce YES for i and whatever σ_C does for all $j \in C \setminus \{i\}$.

We derive a contradiction by proving that σ_C is distinguishably dominated by σ'_C over Σ^0, which implies $\sigma_C \notin \Sigma_C^1$.

Similar to Lemma 1, to prove that σ_C is distinguishably dominated by σ'_C over Σ^0, it suffices to consider all strategy subvectors $\tau_{-C} \in \Sigma_{-C}^0$ such that the execution of $\sigma_C \sqcup \tau_{-C}$ satisfies $\mathcal{P}_{i,C}$ and i answers NO in Stage 3. Notice that by hypothesis, $\tau_{-C} = \sigma_{-C}$ is one such strategy subvector. For each such τ_{-C}, letting $(A, P) = \mathcal{M}(\sigma_C \sqcup \tau_{-C})$ and $(A', P') = \mathcal{M}(\sigma'_C \sqcup \tau_{-C})$, we have that:

- $A_i = \emptyset$ no matter what the coin toss of $\mathcal{M}$ comes up.

Thus, to show that τ_{-C} distinguishes σ_C and σ'_C over Σ^0, it suffices to show that $A'_i \neq \emptyset$ when the coin toss of $\mathcal{M}$ comes up Tails. Moreover, we have that:

- for all $j \in C \setminus \{i\}$, $(A_j, P_j) = (A'_j, P'_j)$, no matter what the coin toss of $\mathcal{M}$ comes up.

Thus due to C's minimally monotonicity, to show that $\mathbb{E}[u_C(\mathcal{M}(\sigma_C \sqcup \tau_{-C}))] < \mathbb{E}[u_C(\mathcal{M}(\sigma'_C \sqcup \tau_{-C}))]$, it suffices to prove that $u_i(\mathcal{M}(\sigma_C \sqcup \tau_{-C})) = u_i(\mathcal{M}(\sigma'_C \sqcup \tau_{-C})) = 0$ when the coin toss of $\mathcal{M}$ comes up Heads, and that $u_i(\mathcal{M}(\sigma_C \sqcup \tau_{-C})) = \frac{\epsilon R_i}{\sum_j R_j} < u_i(\mathcal{M}(\sigma'_C \sqcup \tau_{-C}))$ when the coin toss of $\mathcal{M}$ comes up Tails. This proof is analogous to the corresponding one of Lemma 1, and is ignored. ■

LEMMA 3. $\forall$ *independent player i and $\forall$ $\sigma_i \in \Sigma_i^2$, $\text{REV}(\Omega^i) \geq \text{REV}(RK_i)$ (that is, i does not "underbid").*
Proof. We proceed by contradiction. Assume that there exists an independent player i and a strategy $\sigma_i \in \Sigma_i^2$ such that $\text{REV}(\Omega^i) < \text{REV}(RK_i)$. Now consider the following alternative strategy for player i.

Strategy $\widehat{\sigma}_i$

Stage 1. Announce the outcome $\widehat{\Omega}^i = (\widehat{\alpha}^i, \widehat{\pi}^i) = RK_i$, and the subset of goods $\widehat{S}_i = \arg\max_{S \subseteq G} TV_i(S)$.

Stage 3. Announce YES, NO, or the empty string as follows:
If $\star = i$ or $\alpha_i^\star = \emptyset$, announce the empty string.
Else, announce YES if $TV_i(\alpha_i^\star) \geq \pi_i^\star$, and announce NO if $TV_i(\alpha_i^\star) < \pi_i^\star$.

We derive a contradiction in two steps, that is by proving two separate claims: namely, (1) $\widehat{\sigma}_i \in \Sigma_i^1$, and (2) σ_i is distinguishably dominated by $\widehat{\sigma}_i$ over $\Sigma_{\mathscr{C}}^1$ for all minimally monotone collusive contexts $\mathscr{C}$ compatible with i. The second fact of course contradicts the assumption that $\sigma_i \in \Sigma_i^2$.

Claim 1: $\widehat{\sigma}_i \in \Sigma_i^1$.
Proof: Proceeding by contradiction, let $\overline{\sigma}_i$ be a strategy such that $\overline{\sigma}_i \neq \widehat{\sigma}_i$ and $\overline{\sigma}_i$ distinguishably dominates $\widehat{\sigma}_i$ over Σ^0. Assume that $\overline{\sigma}_i$ announces $\overline{\Omega}^i \neq \widehat{\Omega}^i$ or $\overline{S}_i \neq \widehat{S}_i$, and let σ_{-i} be the subprofile of strategies in which every player $j \in -i$ announces Ω^j such that $\text{REV}(\Omega^j) = 0$ and $S_j = \emptyset$ in Stage 1, announces YES in Stage 3 if $\Omega^\star = \widehat{\Omega}^i$ and $S^\star = \widehat{S}_i$, and NO otherwise. Notice that σ_{-i} clearly belongs to Σ_{-i}^0. (Indeed Σ^0 consists of what all that the players can do, independent of any rationality consideration.) Notice too however, since $\text{REV}(RK_i) > 0$ by hypothesis, $i = \star$ under the profile $\widehat{\sigma}_i \sqcup \sigma_{-i}$ and $\mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \sigma_{-i}))] = \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon \text{REV}(\widehat{\Omega}^i)}{2(1+\text{REV}(\widehat{\Omega}^i))} > \frac{TV_i(\widehat{S}_i)}{2} = \frac{\max_{S \subseteq G} TV_i(S)}{2}$. While $\mathbb{E}[u_i(\mathcal{M}(\overline{\sigma}_i \sqcup \sigma_{-i}))] \leq \frac{TV_i(\overline{S}_i)}{2} - \frac{\text{REV}(\overline{\Omega}^i)}{2} + \frac{\epsilon \cdot x}{2} \leq \frac{\max_{S \subseteq G} TV_i(S)}{2}$, where $x = 1$ if $\text{REV}(\overline{\Omega}^i) \geq 1$ and $x = 0$ otherwise. Therefore such a $\overline{\sigma}_i$ can not dominate $\widehat{\sigma}_i$ over Σ^0.

Accordingly, if $\overline{\sigma}_i$ dominates $\widehat{\sigma}_i$, it must be that $\overline{\sigma}_i$ announces the same outcome and the same subset of goods as $\widehat{\sigma}_i$ does, and thus coincides with $\widehat{\sigma}_i$ in Stage 1. If the coin toss of $\mathcal{M}$ comes up Heads, then the final outcomes under the profiles $\widehat{\sigma}_i \sqcup \sigma_{-i}$ and $\overline{\sigma}_i \sqcup \sigma_{-i}$ are clearly the same, so are $u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \sigma_{-i}))$ and $u_i(\mathcal{M}(\overline{\sigma}_i \sqcup \sigma_{-i}))$. Let us now consider the case when the coin toss of $\mathcal{M}$ comes up Tails and the two executions run into Stage 3. There, Lemma 1 implies that the only possible difference between $\widehat{\sigma}_i$ and a distinguishably dominating $\overline{\sigma}_i$ consists of what the two strategies announce when $i \neq \star$, $\alpha_i^\star \neq \emptyset$ and $TV_i(\alpha_i^\star) = \pi_i^\star$: namely, $\widehat{\sigma}_i$ answers YES (by definition) and $\overline{\sigma}_i$ answers NO (because it must be different from $\widehat{\sigma}_i$). Therefore $\mathcal{M}(\widehat{\sigma}_i \sqcup \sigma_{-i}) \neq \mathcal{M}(\overline{\sigma}_i \sqcup \sigma_{-i})$, and σ_{-i} distinguishes $\widehat{\sigma}_i$ and $\overline{\sigma}_i$. But the difference between the two outcomes does not translate into any utility difference: indeed, both receiving a utility $\frac{\epsilon R_i}{1+\sum_j R_j}$, accepting a subset of goods and paying what your true valuation for it or receiving no goods at all and paying nothing is equivalent. Therefore no $\overline{\sigma}_i \neq \widehat{\sigma}_i$ can dominate $\widehat{\sigma}_i$ over Σ^0. In sum, $\widehat{\sigma}_i \in \Sigma_i^1$ as we want to show. □

Claim 2: For all minimally monotone collusive contexts $\mathscr{C}$ compatible with i, $\widehat{\sigma}_i$ distinguishably dominates σ_i over $\Sigma_{\mathscr{C}}^1$.
Proof: To prove our claim we consider all strategy subprofiles $\tau_{-i} \in \Sigma_{\mathbb{C}\setminus\{i\}}^1$, where $\mathbb{C}$ denotes the player partition of $\mathscr{C}$, and prove that $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] < \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))]$. (Notice that this actually implies that σ_i is *strictly* dominated by σ_i' over $\Sigma_{\mathscr{C}}^1$.)

Arbitrarily fixing such a τ_{-i}, denoting by $\Omega^j = (\alpha^j, \pi^j)$ and $\widehat{\Omega}^j = (\widehat{\alpha}^j, \widehat{\pi}^j)$ the outcomes respectively announced by a player j in the executions of $\sigma_i \sqcup \tau_{-i}$ and $\widehat{\sigma}_i \sqcup \tau_{-i}$, and by R_j and $\widehat{R}_j$ the revenue of Ω^j and $\widehat{\Omega}^j$ respectively, the following four simple observations hold.

O_1: $\forall j \in -i,\ \Omega^j = \widehat{\Omega}^j$ and $R_j = \widehat{R}_j$.

O_2: If $i \neq \star$ in both executions, then the star player is the same in both executions.

O_3: If $i = \star$, then in Stage 3, each player j offered some goods in the outcome announced by player i answers YES if his true valuation for these goods is greater than his price in such outcome, and NO if it is less.

O_4: $\frac{\epsilon R_i}{1+\sum_j R_j} < \frac{\epsilon \widehat{R}_i}{1+\sum_j \widehat{R}_j}$.

Indeed, O_1 holds because outcomes are announced in Stage 1 where all players act simultaneously without receiving any information at all from the mechanism $\mathcal{M}$; O_2 is an immediate implication of O_1; O_3 follows from Lemmas 1 and 2, and the fact that i does not belong to any collusive set; and O_4 follows from O_1, the fact that $R_i < \text{REV}(RK_i)$ (by hypothesis), and the fact that $\widehat{R}_i = \text{REV}(RK_i)$ (by construction).

To establish that $\widehat{\sigma}_i$ distinguishably dominates σ_i over $\Sigma_{\mathscr{C}}^1$, we analyze the following four exhaustive cases, again after arbitrarily fixing $\tau_{-i} \in \Sigma_{\mathbb{C}\setminus\{i\}}^1$.

Case 1: *$i \neq \star$ in the execution of $\sigma_i \sqcup \tau_{-i}$ and $i \neq \star$ in the execution of $\widehat{\sigma}_i \sqcup \tau_{-i}$.*

In this case, by observations O_1 and O_2, $\alpha_i^\star = \widehat{\alpha}_i^\star$ and $\pi_i^\star = \widehat{\pi}_i^\star$. There are four sub-cases.

(a) $\alpha_i^\star = \emptyset$. In this sub-case we have $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] = \frac{\epsilon R_i}{2(1+\sum_j R_j)}$, $\mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))] = \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)}$, and thus

$$\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] < \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))],$$

according to O_4.

(b) $\alpha_i^\star \neq \emptyset$ *and* $TV_i(\alpha_i^\star) = \pi_i^\star$. In this sub-case, no matter whether player i answers YES or NO in Stage 3 of σ_i, we have $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] = \frac{\epsilon R_i}{2(1+\sum_j R_j)} < \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)} = \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))]$.

(c) $\alpha_i^\star \neq \emptyset$ *and* $TV_i(\alpha_i^\star) < \pi_i^\star$. In this sub-case, by Lemma 1, i answers NO in Stage 3 of both executions, and we have $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] = \frac{\epsilon R_i}{2(1+\sum_j R_j)} < \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)} = \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))]$.

(d) $\alpha_i^\star \neq \emptyset$ *and* $TV_i(\alpha_i^\star) > \pi_i^\star$. In this sub-case, by Lemma 1, i answers YES in Stage 3 of both executions, and we have

$$\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] = \frac{TV_i(\alpha_i^\star) - \pi_i^\star + \frac{\epsilon R_i}{1+\sum_j R_j}}{2} < \frac{TV_i(\widehat{\alpha}_i^\star) - \widehat{\pi}_i^\star + \frac{\epsilon \widehat{R}_i}{1+\sum_j \widehat{R}_j}}{2} = \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))].$$

In sum, no matter which sub-case applies, Case 1 implies $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] < \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))]$.

Case 2: *$i \neq \star$ in the execution of $\sigma_i \sqcup \tau_{-i}$* and *$i = \star$ in the execution of $\widehat{\sigma}_i \sqcup \tau_{-i}$.*

In this case, let us first prove that $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] < \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)}$. To this end, we consider the same four sub-cases as above. Namely,

(a) $\alpha_i^\star = \emptyset$. In this sub-case, $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] = \frac{\epsilon R_i}{2(1+\sum_j R_j)}$. Therefore, since $TV_i(\widehat{S}_i) \geq 0$ by definition, we have $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] \leq \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon R_i}{2(1+\sum_j R_j)}$.

(b) $\alpha_i^\star \neq \emptyset$ *and* $TV_i(\alpha_i^\star) = \pi_i^\star$. In this sub-case, no matter whether player i answers YES or NO in Stage 3, we also have $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] = \frac{\epsilon R_i}{2(1+\sum_j R_j)} \leq \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon R_i}{2(1+\sum_j R_j)}$.

(c) $\alpha_i^\star \neq \emptyset$ *and* $TV_i(\alpha_i^\star) < \pi_i^\star$. In this sub-case, player i answers NO in Stage 3, and thus $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] = \frac{\epsilon R_i}{2(1+\sum_j R_j)} \leq \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon R_i}{2(1+\sum_j R_j)}$.

(d) $\alpha_i^\star \neq \emptyset$ *and* $TV_i(\alpha_i^\star) > \pi_i^\star$. In this sub-case, player i answers YES in Stage 3, causing himself to be assigned the subset of goods $\alpha_i^\star$ for price $\pi_i^\star$. Accordingly $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] = \frac{TV_i(\alpha_i^\star) - \pi_i^\star + \frac{\epsilon R_i}{1+\sum_j R_j}}{2} \leq \frac{TV_i(\alpha_i^\star) + \frac{\epsilon R_i}{1+\sum_j R_j}}{2} \leq \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon R_i}{2(1+\sum_j R_j)}$. In fact, $\pi_i^\star$ is always non-negative, and $TV_i(\widehat{S}_i) = \max_{S \subseteq G} TV_i(S)$.

In sum, no matter which sub-case applies, we have $\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] \leq \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon R_i}{2(1+\sum_j R_j)}$. Since $\frac{\epsilon R_i}{2(1+\sum_j R_j)} < \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)}$, we have that

$$\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] < \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)}.$$

Let us now prove that $\frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)} = \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))]$. In this case, i's expected utility in the execution of $\widehat{\sigma}_i \sqcup \tau_{-i}$ is the weighted sum of his utility when $\mathcal{M}$'s coin toss is Heads and his utility when $\mathcal{M}$'s coin toss is Tails.[16] Therefore, denoting by "$\sum_{j:\widehat{NO}}$" the sum taken over every player j who answers NO in Stage 3 of the execution of $\widehat{\sigma}_i \sqcup \tau_{-i}$ (that is, $\mathcal{M}$'s coin toss comes up Tails), we have

$$\begin{aligned} & \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))] \\ = \quad & \frac{TV_i(\widehat{S}_i)}{2} + \frac{-\sum_{j:\widehat{NO}} \widehat{\pi}_j^i + \frac{\epsilon \widehat{R}_i}{1+\sum_j \widehat{R}_j}}{2} \end{aligned}$$

By definition of RK_i and compatibility, $\forall j \in -i$ such that $\widehat{\alpha}_j^i \neq \emptyset$, $\widehat{\pi}_j^i < TV_j(\widehat{\alpha}_j^i)$. Thus by observation O_3 every such player j answers YES in Stage 3: in our notation $\sum_{j:\widehat{NO}} \widehat{\pi}_j^i = 0$. Accordingly, we have

$$\mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))] \quad = \quad \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)}$$

[16] Both individual utilities are expected, if the strategies of the other players are probabilistic.

as desired.

Therefore Case 2 implies

$$\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] < \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))].$$

Case 3: *$i = \star$ in the execution of $\sigma_i \sqcup \tau_{-i}$* and *$i = \star$ in the execution of $\widehat{\sigma}_i \sqcup \tau_{-i}$.*

In this case, similar to Case 2, i's expected utility in the execution of $\sigma_i \sqcup \tau_{-i}$ is the weighted sum of his utility when $\mathcal{M}$'s coin toss is Heads and his utility when $\mathcal{M}$'s coin toss is Tails. Therefore, similar to Case 2, denoting by "$\sum_{j:NO}$" the sum taken over every player j who answers NO in Stage 3 of the execution of $\sigma_i \sqcup \tau_{-i}$, we have

$$\begin{aligned} &\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] \\ = \quad &\frac{TV_i(S_i)}{2} + \frac{-\sum_{j:NO} \pi_j^i + \frac{\epsilon R_i}{1+\sum_j R_j}}{2} \end{aligned}$$

Since $\sum_{j:NO} \pi_j^i \geq 0$, we have that

$$\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] \quad \leq \quad \frac{TV_i(S_i)}{2} + \frac{\epsilon R_i}{2(1+\sum_j R_j)}$$

Let us now analyze i's expected utility in the execution of $\widehat{\sigma}_i \sqcup \tau_{-i}$. Same as in Case 2, and by observation O_3, we have that

$$\mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))] \quad = \quad \frac{TV_i(\widehat{S}_i)}{2} + \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)}$$

According to our construction of $\widehat{\sigma}_i$, we have that $TV_i(\widehat{S}_i) = \max_{S \subseteq G} TV_i(S)$. Therefore by O_4,

$$\mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))] > \frac{TV_i(S_i)}{2} + \frac{\epsilon \widehat{R}_i}{2(1+\sum_j \widehat{R}_j)} > \frac{TV_i(S_i)}{2} + \frac{\epsilon R_i}{2(1+\sum_j R_j)}.$$

In sum, Case 3 implies

$$\mathbb{E}[u_i(\mathcal{M}(\sigma_i \sqcup \tau_{-i}))] < \mathbb{E}[u_i(\mathcal{M}(\widehat{\sigma}_i \sqcup \tau_{-i}))].$$

Case 4: *$i = \star$ in the execution of $\sigma_i \sqcup \tau_{-i}$* and *$i \neq \star$ in the execution of $\widehat{\sigma}_i \sqcup \tau_{-i}$.*

Fortunately, this case can never happen. Since $\textsc{rev}(\widehat{\Omega}^i) > \textsc{rev}(\Omega^i)$ (by construction) and $\forall j \in -i \; \Omega^j = \widehat{\Omega}^j$ (by observation O_1), we have that if $i = \star$ in the execution of $\sigma_i \sqcup \tau_{-i}$, it must be true that $i = \star$ also in the execution of $\widehat{\sigma}_i \sqcup \tau_{-i}$.

Having finished to analyze all possible cases, we conclude that σ_i is distinguishably dominated by $\widehat{\sigma}_i$ over $\Sigma^1_{\mathscr{C}}$. □

Since both Claims 1 and 2 hold, so does Lemma 3. ■