Surface Topology

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Research

Phenomena → Explanation → Theory → Method → Technology → Product

Purposes

Concepts, theories and algorithms for computing surface topological structure.

Concepts

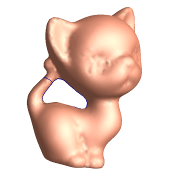
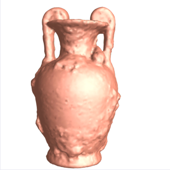
Homology, cohomology, fundamental group, universal covering space, deck transformation group, differential forms, De Rham cohomology.

Algorithms to be covered

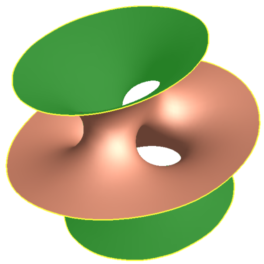
- Computing cut graph and fundamental domain.
- Computing homology group (fundamental group) basis.
- Computing cohmology group basis.
- Computing finite portion of universal covering space.

Topology of surfaces - closed surfaces

A major purpose of geometry is to describe and classify geometric structures.

			
genus 0	genus 1	genus 2	genus 3

Topology of Surface - Surface with Boundaries



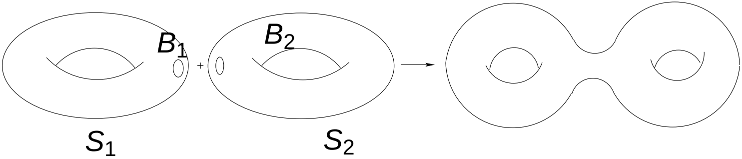
genus 0 with 1 boundary   genus 1 with 3 boundaries

Connected Sum

Definition (Connected Sum)

The connected sum $S_1 \# S_2$ is formed by deleting the interior of disks D_i and attaching the resulting punctured surfaces $S_i - D_i$ to each other by a homeomorphism $h : \partial D_1 \to \partial D_2$, so

$$S_1 \# S_2 = (S_1 - D_1) \cup_h (S_2 - D_2).$$

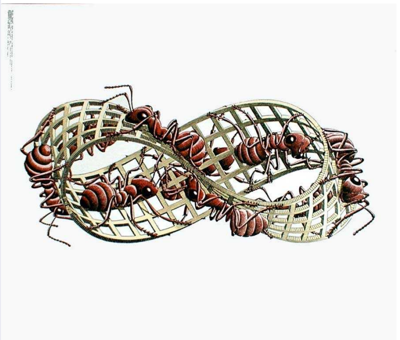


Topology

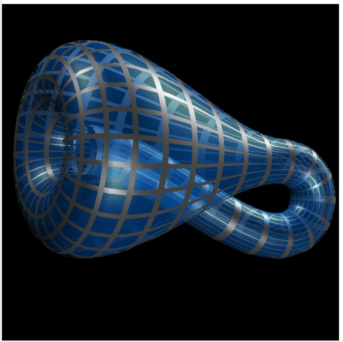
A Genus eight Surface, constructed by connected sum.

Non-Orientable Surfaces

Möbius Band by M.C. Escher.



Klein Bottle



Projective Plane

Definition (Projective Plane)

All straight lines through the origin in $\mathbb{R}^3$ form a two dimensional manifold, which is called the projective plane.

A projective plane can be obtained by identifing two antipodal points on the unit sphere.
A projective plane with a hole is called a crosscap.

Surface Structure Theorem

Theorem (Classification Theorem for Surfaces)

Any closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum of tori, or a sphere with a finite number of disjoint discs removed and with crosscaps glued in their place. The sphere and connected sums of tori are orientable surfaces, whereas surfaces with crosscaps are unorientable.

Any closed surface S is the connected sum

$$S = S_1 \# S_2 \# \cdots S_g,$$

if S is orientable genus g, then S_i is a torus. If S is non-orientable, genus g, then S_i is a projective plane.

Simplical Complex

Definition (Simplex)

Suppose $k+1$ points in the general positions in $\mathcal{R}^n$, $v_0, v_1, \cdots, v_k$, the *standard simplex* $[v_0, v_1, \cdots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \cdots, v_k] = \{x \in \mathcal{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\},$$

we call $v_0, v_1, \cdots, v_k$ as the *vertices* of the simplex σ.

Suppose $\tau \subset \sigma$ is also a simplex, then we say τ is a *facet* of σ.

Simplicial Complex

Definition (Simplicial complex)

A *simplicial complex* Σ is a union of simplices, such that

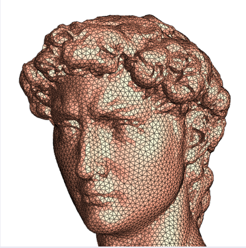
1. If a simplex σ belongs to K, then all its facets also belongs to Σ.
2. If $\sigma_1, \sigma_2 \subset K$, $\sigma_1 \cap \sigma_2 \neq$, then the intersection of σ_1 and σ_2 is also a common facet.

Triangular Mesh

Definition (Triangular Mesh)

A triangular mesh is a surface Σ with a triangulation T,

1. Each face is counter clock wisely oriented with respect to the normal of the surface.
2. Each edge has two opposite half edges.

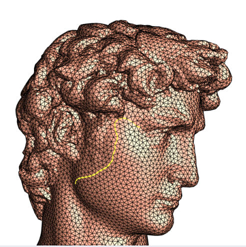


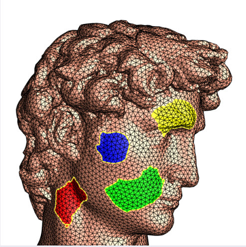
Chain Space

Definition (Chain Space)

A k chain is a linear combination of all k simplicies in Σ, $\sigma = \sum_i \lambda_i \sigma_i, \lambda_i \in \mathcal{Z}$. The n dimensional *chain space* is a linear space formed by all the n chains, we denote n dimensional chain space as $C_n(\Sigma)$

A curve on the mesh is a 1-chain;
A surface patch on Σ is a 2-chain.





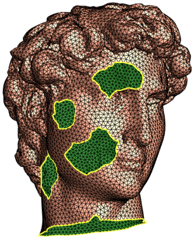
Boundary Operator

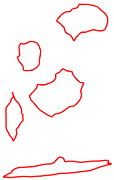
Definition (Boundary Operator)

The n-th dimensional boundary operator $\partial_n : C_n \to C_{n-1}$, is a linear operator, such that

$$\partial_n[v_0, v_1, v_2, \cdots v_n] = \sum_i (-1)^i [v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n].$$

Boundary operator extracts the boundary of the chain.

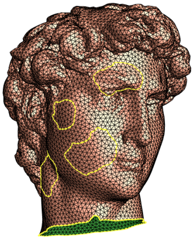




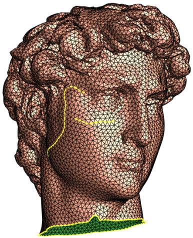
Definition (Closed chain)

A k-chain $\gamma \in C_k(\Sigma)$, if $\partial_k \gamma = 0$, then σ is closed.

A closed 1-chain is a loop. A non-closed 1-chain is with boundary vertices.



closed 1-chain

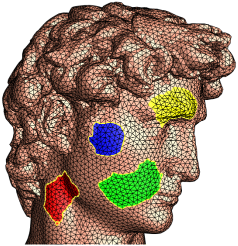


open 1-chain

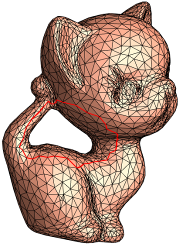
Exact chains

Definition (Exact k-chain)

A k-chain $\gamma \in C_k(\Sigma)$ is exact, if there exists a (k+1)-chain σ, such that $\gamma = \partial_{k+1}\sigma$.



exact 1-chain



closed 1-chain

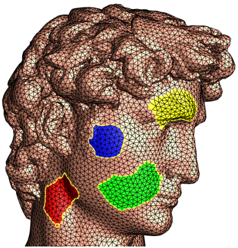
Boundary of Boundary

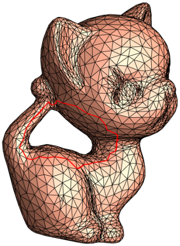
Theorem (Boundary of Boundary)

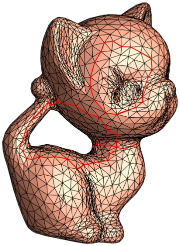
The boundary of a boundary is empty.

$$\partial_{k-1} \circ \partial_k \equiv \emptyset$$

Namely, exact chains are closed. But the reverse is not true.





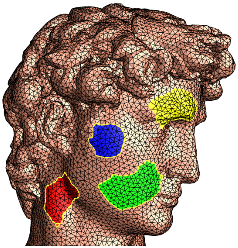


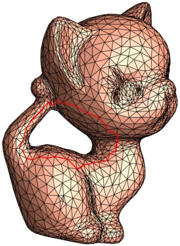
Homology

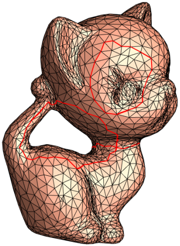
The difference between the closed chains and the exact chains indicates the topology of the surface.

Any closed 1-chain on genus zero surface is exact.

On tori, some closed 1-chains are not exact.







Closed k-chains form the kernal space of the boundary operator ∂_k. Exact k-chains form the image space of ∂_{k+1}.

Definition (Homology Group)

The k dimensional homology group $H_k(\Sigma, \mathbb{Z})$ is the quotient space of $ker\partial_k$ and the image space of $img\partial_{k+1}$.

$$H_k(\Sigma, \mathbb{Z}) = \frac{ker\partial_k}{img\partial_{k+1}}.$$

Two k-chains γ_1, γ_2 are homologous, if they bound a (k+1)-chain σ,

$$\gamma_1 - \gamma_2 = \partial_{k+1}\sigma$$

Homology Group

Computation

1. The chain space C_1 is a linear space, the oriented edges are the basis. The chain space C_2 is also a linear space, the oriented face are the basis.
2. The boundary operators are linear operators, they can be represented as matrices.

$$\partial_2 = ([f_i, e_j]),$$

$[f_i, e_j]$ is zero if e_j is not on the boundary of f_i; $+1$ if e_j is on the boundary of f_i with consistent orientation; -1 if e_j is on the boundary of f_i with opposite orientation.

3. The basis of $H_1(\Sigma, \mathbb{Z})$ is formed by the eigenvectors of zero eigen values of the matrix

$$\Delta = \partial_2 \circ \partial_2^T + \partial_1^T \circ \partial_1.$$

Homology Group

Algebraic Method

The eigen vectors of Δ can be computed using the Smith norm of integer matrices. It is general for all dimensional complexes, but impractical.

Combinatorial Method

Combinatorial method is efficient and simple. The key is to find a cut graph.

Canonical Homology Basis

Definition (Canonical Homology Basis)

A homology basis $\{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$ is canonical, if

1. a_i and b_i intersect at the same point p.
2. a_i and a_j, b_i and b_j only touch at p.

The surface can be sliced along a set of canonical basis and form a simply connected patch, the fundamental domain. The fundamental domain is with the boundary

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Canonical Homology Basis

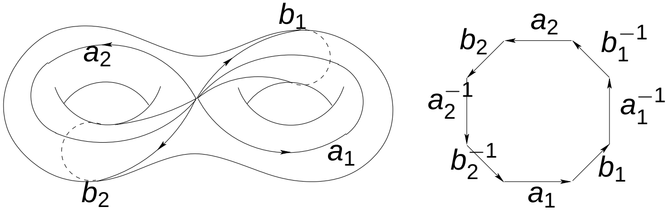
Definition (Canonical Homology Basis)

For genus g closed surface, there exist canonical basis for $\pi_1(M, p_0)$, we write the basis as $\{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$, such that

$$a_i \cdot a_j = 0, a_i \cdot b_j = \delta_i^j, b_i \cdot b_j = 0,$$

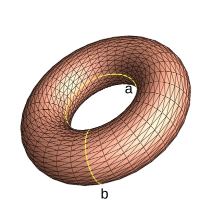
where $\cdot$ represents the algebraic intersection number.

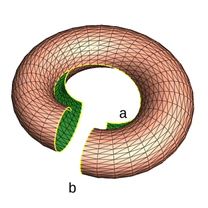
Especially, through any point $p \in M$, we can find a set of canonical basis for $\pi_1(M)$, the surface can be sliced open along them and form a canonical fundamental polygon

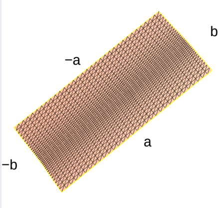


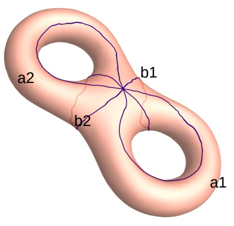
Canonical Homology Basis

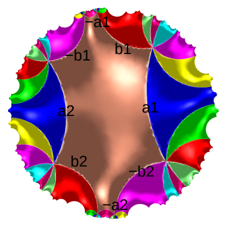
Canonical Homology Basis and Fundamental Domain







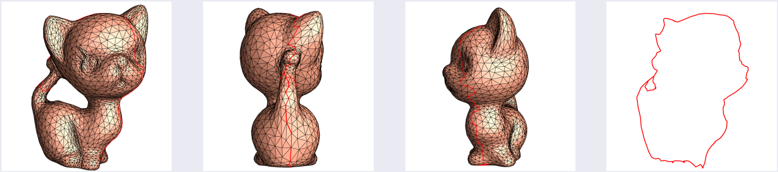




Cut Graph

Definition (cut graph)

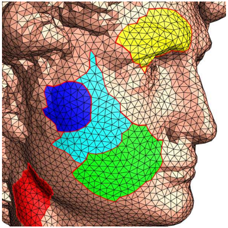
A cut graph G of a mesh Σ is a graph formed by non-oriented edges of Σ, such that Σ/G is a topological disk.



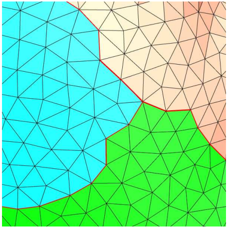
Corner and Wedge

Definition (Wedge)

On a face f, the corner with vertex v is denoted as (f, v). Given a vertex v, the corners are ordered counter-clockwisely. A maximal sequence of adjacent corners without sharp edges form a wedge.



wedges



3 wedges around a vertex

Foundamental Domain

Algorithm for Foundamental Domain

Input : A mesh Σ and a cut graph G.
Output : A fundamental domain $\tilde{\Sigma}$.

1. Label the edges on G as sharp edges.
2. Compute the wedges of Σ formed by the sharp edges.
3. Construct an empty $\bar{\Sigma}$.
4. For each wedge w, insert a vertex v, the vertex position is same as that of the vertex in of the wedge.
5. For each face $f = [v_0, v_1, v_2]$ on Σ, insert a face $\tilde{f} = [w_0, w_1, w_2]$ in $\tilde{\Sigma}$, such that the corner on f at v_i belongs to wedge w_i, $(v_i, f) \in w_i$.

Cut Graph

Algorithm: Cut Graph

Input : A triangular Mesh Σ.
Output: A cut graph G

1. Compute the dual mesh $\bar{\Sigma}$, each edge $e \in \Sigma$ has a unique dual edge $\bar{e} \in \bar{\Sigma}$.
2. Compute a spanning tree $\bar{T}$ of $\bar{\Sigma}$.
3. The cut graph is the union of all edges whose dual are not in $\bar{T}$.

$$G = \{e \in \Sigma | \bar{e} \notin \bar{T}\}.$$

Loop Basis for the Cut Graph

Theorem (Homology Basis)

Suppose Σ is a closed mesh, G is a cut graph of Σ, then the basis of loops of G (assigned with an orientation) is also a homology basis of Σ.

Algorithm: Loop Basis for the Cut Graph

Input : A graph G.
Output: A basis of loops on G.

1. Compute a spanning tree T of G.
2. $G/T = \{e_1, e_2, \cdots, e_n\}$.
3. $e_i \cup T$ has a unique loop, denoted as γ_i.
4. $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ form a basis for all loops of G.

Definition (Homotopy of maps)

Two continuous maps $f_1, f_2 : S \to M$ between manifolds S and M are homotopic, if there exists a continuous map

$$F : S \times [0,1] \to M$$

with

$$\begin{aligned} F|_{S\times 0} &= f_1, \\ F|_{S\times 1} &= f_2. \end{aligned}$$

we write $f_1 \sim f_2$.

Intuition Two closed curves on a surface are homotopic to each other, if they can deform to each other without leaving the surface.

Definition (Homotopy of curves)

Let $\gamma_i : S^1 \to \Sigma, i = 1, 2$ be closed curves on Σ, we say two curves are homotopic if the maps γ_1 and γ_2 are homotopic. we write $\gamma_1 \sim \gamma_2$.

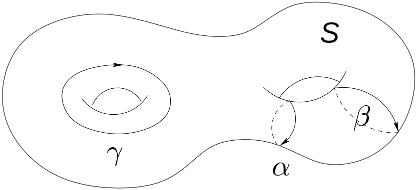


Figure: α is homotopic to β, not homotopic to γ.

Definition (product)

Let $\gamma_1, \gamma_2 : [0,1] \to M$ be curves with
$$\gamma_1(1) = \gamma_2(0),$$
the product of $\gamma_1\gamma_2 := \gamma$ is defined by
$$\gamma(t) := \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

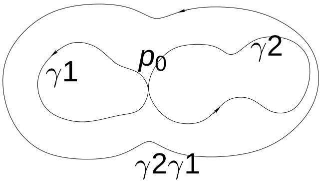


Figure: product of two closed curves.

Fundamental Group

Definition (Fundamental Group)

For any $p_0 \in M$, the fundamental group $\pi_1(M, p_0)$ is the group of homotopy classes of paths $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = p_0$, i.e. closed paths with p_0 as initial and terminal point.

$\pi_1(M, p_0)$ is a group with respect to the operation of multiplication of homotopy classes. The identity element is the class of the constant path $\gamma_0 \equiv p_0$.

For any $p_0, p_1 \in M$, the groups $\pi_1(M, p_0)$ and $\pi_1(M, p_1)$ are isomorphic.

If $f : M \to N$ be a continuous map, and $q_0 := f(p_0)$, then f induces a homomorphism $f_* : \pi_1(M, p_0) \to \pi_1(N, q_0)$ of fundamental groups.

Homology vs. Homotopy Groups

Abelianization

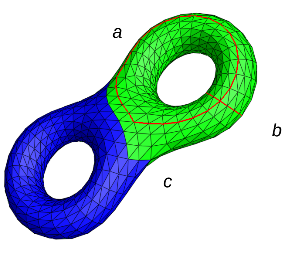
The first fundamental group in general is non-abelian. The first homology group is the abelianization of the fundamental group.

$$H_1(\Sigma) = \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)],$$

where $[\pi_1(\Sigma), \pi_1(\Sigma)]$ is the commutator of π_1, $[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$.

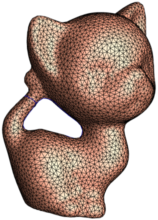
Fundamental group encodes more information than homology group, but more difficult to compute.

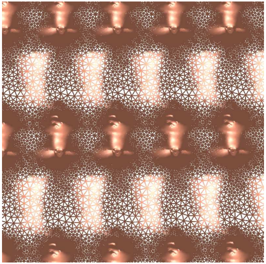
Homotopy Group vs. Homology Group



- c separate the surface to 2 handles.
- c is homotopic to $aba^{-1}b^{-1}$
- c is homologous to zero.

This shows the homotopy group is non-abelian, homotopy group encodes more information than homology group.





Universal Cover

A pair $(\bar{\Sigma}, \pi)$ is a universal cover of a surface Σ, if

- Surface $\bar{\Sigma}$ is simply connected.
- Projection $\pi : \bar{\Sigma} \rightarrow \Sigma$ is a local homeomorphism.

Deck Transformation

A transformation $\phi : \bar{\Sigma} \rightarrow \bar{\Sigma}$ is a deck transformation, if

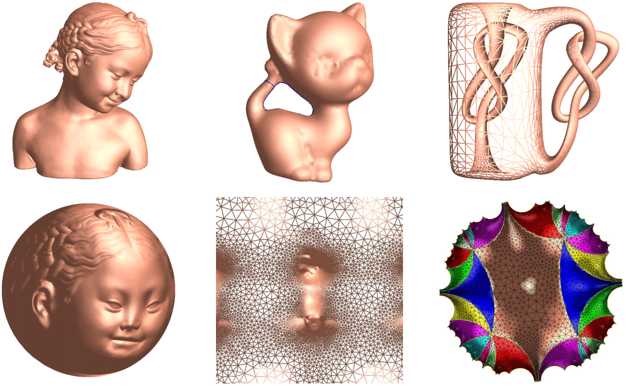
$$\pi = \pi \circ \phi.$$

A deck transformation maps one fundamental domain to

Universal covering space

Theorem (Universal Covering Space)

The universal covering spaces of closed surfaces are sphere (genus zero), plane (genus one) and disk (high genus).



Spherical   Euclidean   Hyperbolic

Intuition A closed curve on the surface is "lifted" to a path in its universal covering space.

Theorem

Suppose $p \in \Sigma$, $(\bar{\Sigma}, \pi)$ *is the universal cover of* Σ,

$$\pi^{-1}(p) = \{\bar{p}_0, \bar{p}_1, \bar{p}_2, \cdots\},$$

a curve $\bar{\gamma}_i$ *connecting* $\bar{p}_0$ *and* $\bar{p}_i$, *a curve* $\bar{\gamma}_j$ *connecting* $\bar{p}_0$ *and* $\bar{p}_j$, $\pi(\bar{\gamma}_i)$ *is homotopic to* $\pi(\bar{\gamma}_j)$ *if and only if* i *equals to* j.

Therefore, there is a one to one map between the fundamental group of Σ and $\pi^{-1}(p)$. A deck transformation maps $\bar{p}_0$ to $\bar{p}_i$. Therefore, the fundamental group is isomorphic to the deck transformation group.

Universal Covering Space

Any topological non-trivial loop on the surface Σ is lifted to a path on its universal cover $\bar{\Sigma}$. The shortest loop is the shortest path.

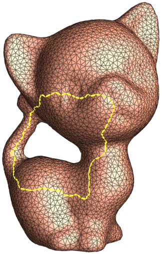
The homotopy group of Σ can be traversed by connecting $\bar{p}_0$ to $\bar{p}_k$'s on $\bar{\Sigma}$ and project to Σ.

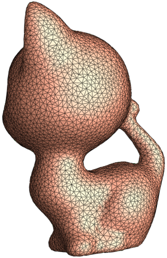
The number of fundamental domains on a universal cover grows exponentially fast for high genus surfaces.

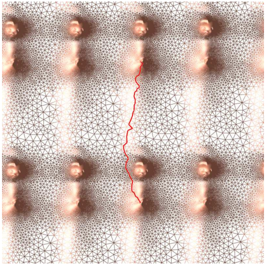
Universal covering space

Loop Lifting

Any nontrival closed loop γ on Σ is lifted to an open curve $\bar{\gamma}$ on $\bar{\Sigma}$. The homotopy class of γ is determined by the starting and ending points of $\bar{\gamma}$.

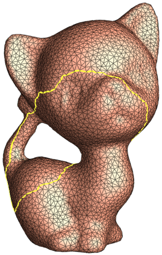


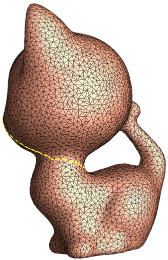


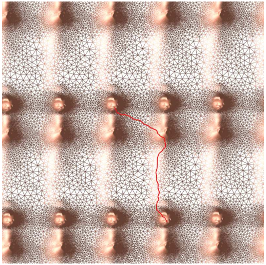


Shortest Loop

Shortest loop on surface is lifted to a shortest path on the universal cover.







Algorithm Universal Cover

Input : A mesh Σ.
Output: A finite portion of the universal cover $\bar{\Sigma}$.

1. Compute a cut graph G of Σ. We call a vertex on G with valence greater than 2 a knot. The knots divide G to segments, assign an orientation to each segment, labeled as $\{s_1, s_2, \cdots, s_n\}$.
2. Slice Σ along G to get a fundamental domain $\tilde{\Sigma}$, the boundary is composed of $\pm s_k$'s.
3. Initialize $\bar{\Sigma} \leftarrow \tilde{\Sigma}$, book keep $\partial\bar{\Sigma}$ using $\pm s_k$'s.
4. Glue a copy of $\tilde{\Sigma}$ to current $\bar{\Sigma}$ along only one segment $s_k \in \partial\bar{\Sigma}$, $-s_k \in \partial\tilde{\Sigma}$, $\bar{\Sigma} \leftarrow \bar{\Sigma} \cup_{s_k} \tilde{\Sigma}$.
5. Update $\partial\bar{\Sigma}$, if $\pm s_i$ are adjacent in $\partial\bar{\Sigma}$, glue the boundary of $\bar{\Sigma}$ along s_i. Repeat this step until no adjacent $\pm s_i$ in the boundary.
6. Repeat gluing the copies of $\tilde{\Sigma}$ until $\bar{\Sigma}$ is large enough.

Simplicial Cohomology

Definition (Cochain Space)

A k cochain is a linear function

$$\omega : C_k \to \mathcal{Z}.$$

The k cochain space $C^k(M, \mathcal{Z})$ is linear space formed by all linear functionals defined on $C_k(M, \mathcal{Z})$. The k-cochain is also called k form.

Definition (Coboundary)

The coboundary operator $\delta_k : C^k(M, \mathcal{Z}) \to C^{k+1}(M, \mathcal{Z})$ is a linear operator, such that

$$\delta_k \omega := \omega \circ \partial_{k+1}, \omega \in C^k(M, \mathcal{Z}).$$

For example, ω is a 1-form, then $\delta_1 \omega$ is a 2-form, such that

$$\begin{aligned} \delta_1 \omega([v_0, v_1, v_2]) &= \omega(\partial_2 [v_0, v_1, v_2]) \\ &= \omega([v_0, v_1]) + \omega([v_1, v_2]) + \omega([v_2, v_0]) \end{aligned}$$

Intuition Coboundary operator is similar to differentiation. δ_0 is gradient operator, δ_1 is curl operator.

Definition (Closed forms)

A n-form ω is closed, if $\delta_n\omega = 0$.

Definition (Exact forms)

A n-form ω is exact, if there exists a $n-1$ form σ, such that $\omega = \delta_{n-1}\sigma$.

Theorem

$$\delta^n \circ \delta^{n-1} \equiv 0.$$

Therefore, all exact forms are closed. The curl of gradient is zero.

Intuition The difference between exact forms and closed forms indicates the topology of the surface.

Definition (Cohomology Group)

The n dimensional cohomology group of Σ is defined as

$$H^n(\Sigma, \mathbb{R}) = \frac{ker\,\delta_n}{img\delta_{n-1}}.$$

Two 1-forms ω_1, ω_2 are cohomologous, if they differ by a gradient of a 0-form f,

$$\omega_1 - \omega_2 = \delta_0 f.$$

Homology vs. Cohomology

Duality

$H_1(\Sigma)$ and $H^1(\Sigma)$ are dual to each other. Suppose ω is a 1-form, σ is a 1-chain, then the pair

$$<\omega, \sigma> := \omega(\sigma),$$

is a bilinear operator.

Definition (Dual Cohomology Basis)

Suppose a homology basis of Σ is $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, the dual cohomology basis is $\{\omega_1, \omega_2, \cdots, \omega_n\}$, if and only if

$$<\omega_i, \gamma_j> = \delta_i^j.$$

Dual Cohomology Basis

Algorithm for Dual Cohomology basis

Input : A homology basis $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$.
Output : A dual cohomology basis $\{\omega_1, \omega_2, \cdots, \omega_n\}$.

1. Compute the spanning tree of the faces T_f.
2. Traverse T_f, push the faces to a stack S_f during the traversing.
3. Compute the cut graph G_c and its spanning tree T_c. Suppose $G_c/T_c = \{e_1, e_2, \cdots, e_n\}$.
4. Assign $\omega_i(e_j) = \delta_i^j$, for any edge e on T_c, $\omega_i(e) = 0$.
5. Popup the first face f from the stack S_f, $\partial f = h_0 + h_1 + h_2$. If all values of ω_i on h_k's have been assigned, then continue; otherwise, arbitrarily assign ω_i values on those h_k's which haven't been assigned, such that $\omega_i(h_0) + \omega_i(h_1) + \omega_i(h_2) = 0$.
6. Repeat poping up the faces from the stack, until the stack is empty.

Double Cover

Intuition By gluing two copies of an open surface along their corresponding boundaries, a symmetric closed surface is obtained.

Algorithm Double Cover

Input : an open mesh Σ
Output : the doubled closed mesh $\bar{\Sigma}$.

1. Make a copy of Σ, reverse the orientation of each face to get Σ'.
2. Suppose $e \in \partial\Sigma$, then $-e$ must be in $\partial\Sigma'$. Glue Σ and Σ' along their corresponding opposite boundary oriented edges.

$$\bar{\Sigma} = (\Sigma \cup \Sigma')/(\partial\Sigma \ni e \sim -e \in \partial\Sigma')$$

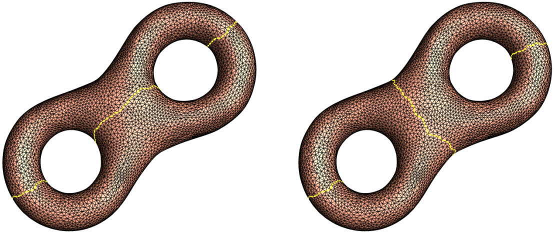


Pants Decomposition

Definition (Pants)

A genus zero surface with three boundaries is called a pair of pants.

Any closed high genus surface can be decomposed as χ number of pants, where χ is the Euler number. There are $3g - 3$ cuts on the surface.



Pants Decomposition

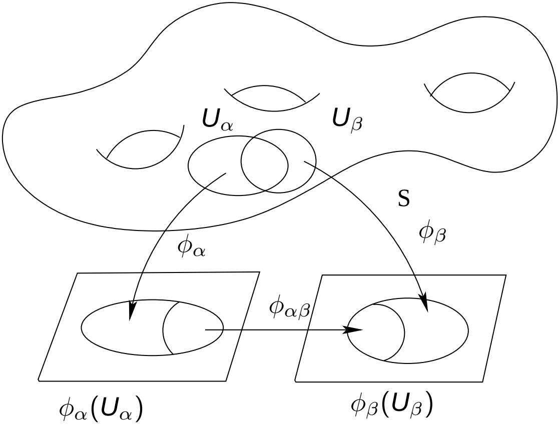
Intuition If there is a loop which is not homotopic to the boundary, slice the surface along it.

Algorithm Pants Decomposition

Input : a mesh Σ
Output : the pants decomposition.

1. Compute a homology basis of Σ.
2. Select on loop γ which is not homotopic to any boundary loop.
3. Slice the surface along γ.
4. Repeat step 2 and 3, until all loops are homotopic to boundary loops.

Manifold



U_α
U_β
S
ϕ_α
ϕ_β
$\phi_{\alpha\beta}$
$\phi_\alpha(U_\alpha)$
$\phi_\beta(U_\beta)$

Definition (Manifold)

A manifold is a topological space Σ covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a chart of Σ, the set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of Σ. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

Transition maps satisfy cocycle condition, suppose $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then

$$\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}.$$

Intuition Change discrete piecewise linear setting to smooth setting, replace discrete forms by differential forms.

Definition

Differential 0-form A function (0-form) $f : \Sigma \to \mathbb{R}$ has local representations on chart (x_α, y_α) as

$$f_\alpha(x_\alpha, y_\alpha).$$

On chart (x_β, y_β)

$$f_\beta(x_\beta, y_\beta).$$

Then on the overlapping regions

$$f_\alpha(x_\alpha(x_\beta, y_\beta), y_\alpha(x_\beta, y_\beta)) \equiv f_\beta(x_\beta, y_\beta).$$

Differential one-forms

Definition (Differential 1-form)

Suppose Σ is a surface with a differential structure $\{U_\alpha, \phi_\alpha\}$ with (u_α, v_α), then a real different one-form ω has the parametric representation on local chart

$$\omega = f_\alpha(u_\alpha, v_\alpha)du_\alpha + g(u_\alpha, v_\alpha)dv_\alpha,$$

where f_α, g_α are functions with C^∞ continuity.
On different chart $\{U_\beta, \phi_\beta\}$,

$$\omega = f_\beta(u_\beta, v_\beta)du_\beta + g(u_\beta, v_\beta)dv_\beta$$

then

$$(f_\alpha, g_\alpha)\begin{pmatrix} \frac{\partial u_\alpha}{\partial u_\beta} & \frac{\partial u_\alpha}{\partial v_\beta} \\ \frac{\partial v_\alpha}{\partial u_\beta} & \frac{\partial v_\alpha}{\partial v_\beta} \end{pmatrix} = (f_\beta, g_\beta)$$

Exterior Differentiation

Definition (Wedge)

A special operator $\wedge$ can be defined on differential forms, such that

$$\begin{aligned} f \wedge \omega &= f\omega \\ \omega \wedge \omega &= 0 \\ \omega_1 \wedge \omega_2 &= -\omega_2 \wedge \omega_1 \end{aligned}$$

Definition (Exterior differentiation)

The so called exterior differentiation operator d can be defined on differential forms, such that

$$\begin{aligned} df(u,v) &= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv \\ d(\omega_1 \wedge \omega_2) &= d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2 \end{aligned}$$

The exterior differential operator d is the generalization of *curl* and *divergence* on vector fields.

De Rham Cohomology

Theorem

$d \circ d \equiv 0.$

Example:

$$
\begin{aligned}
d \circ df &= d\left(\frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv\right) \\
&= \left(\frac{\partial^2 f}{\partial v \partial u} - \frac{\partial^2 f}{\partial u \partial v}\right) dv \wedge du
\end{aligned}
$$

Definition (Closed differential forms)

A n-form ω is closed, if $d_n\omega \equiv 0$.

Definition (Exact differential forms)

A n-form ω is exact, if there exists a (n-1)-form τ, such that $d_{n-1}\tau = \omega$.

Intuition differential 1-form can be treated as vector fields. The difference between curl free vector fields and the gradient fields indicates the topology of the surface.

De Rham Cohomology

Intution The difference between closed forms and exact forms indicates the topology of the surface.

Definition (De Rham Cohomology Group)

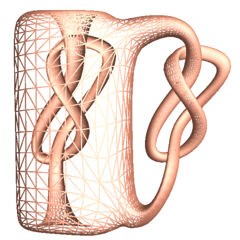
The first De Rham cohomology group $H^n(\Sigma, \mathbb{R})$ is

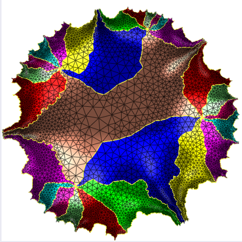
$$H^n(\Sigma, \mathbb{R}) := \frac{ker d_n}{img d_{n-1}}.$$

Simplicial cohomology groups are isomorphic to De Rham cohomology groups.

Simplicial cohomology can be interpreted as the finite element version of De Rham cohomology.

For more information, please email to gu@cs.sunysb.edu.





Thank you!