Riemann Surface

Theories and Practice

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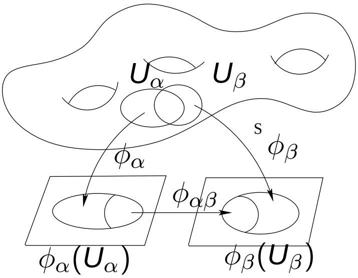
Conformal Structure

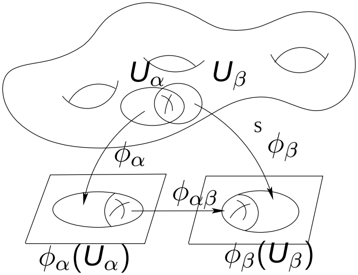
Intuition

- A conformal structure is a structure assigned to a topological manifold, such that angles can be defined.
- It is easy to define angles on the parameter plane.
- But a manifold can't be covered by a single coordinate system, instead it is covered by many local coordinate systems with overlapping.



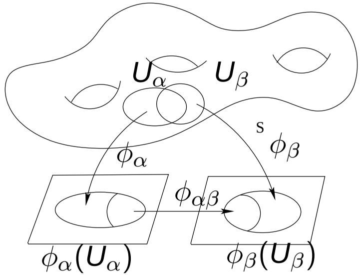
Relativity Idea

- In relativity, the physics law is independent of the local coordinate system of the observer.
- If the transition function from one local coordinates to another is angle preserving, then angle value is independent of the choice of the local chart.
- Therefore, if the manifold is with a special atlas, such that all transition maps are conformal, then angle can be consistently defined on the manifold.



Definition (Conformal Structure)

An atlas is conformal, if all its transition maps are conformal (biholomorphic). A conformal structure is the maximal conformal atlas. A topological surface with an conformal structure is called a Riemann Surface.



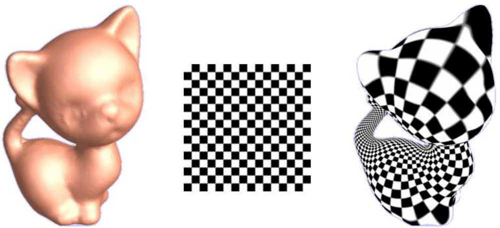
Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u, v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2u}(du^2 + dv^2).$$

The atlas formed by isothermal coordinate systems is an conformal atlas.

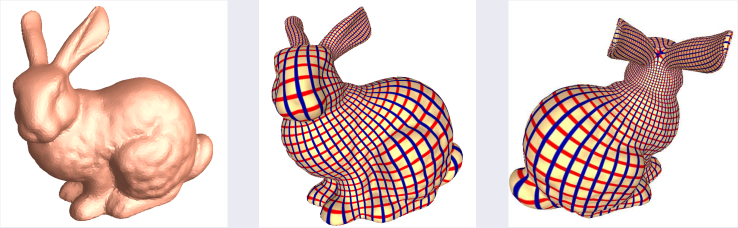


The existence of isothermal coordinates was proven by S.S.Chern.

Theorem (Riemann Surface)

All metric surfaces are Riemann surfaces.

Conformal Structure



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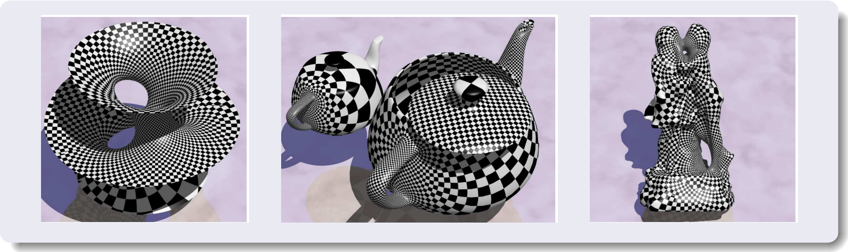
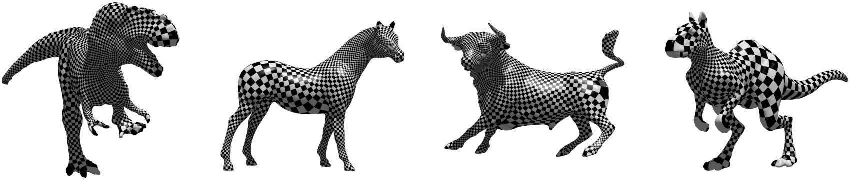




Computational Method

Heat Flow

Suppose the temperature field on the surface is $T(u,v,t)$, the surface is with a Riemannian metric $\mathbf{g}$, then the temperature will evolve according to the heat flow:

$$\frac{dT(u,v,t)}{dt} = \Delta_{\mathbf{g}} T(u,v,t),$$

at the steady state

$$\Delta_{\mathbf{g}} T(u,v,\infty) \equiv 0,$$

which is called a harmonic function.

Intuition

Harmonic means "the smoothest", a harmonic geometric being can not be further smoothed.

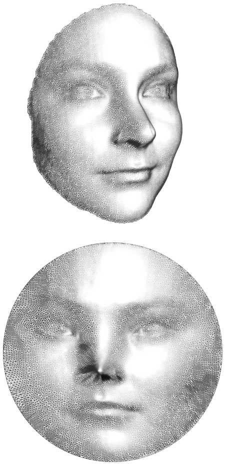
Computational Method

During the heat flow, singularities will be raised. In designing algorithms, special attention needs to be paid for controlling the singularities.

Heat Flow

1. Curvature flow for curves Heat flow acting on the position will deform a simple closed planar curve to a circle and shrink to a point without producing any singularity. The process is linear.
2. Geodesic curvature flow for curves Heat flow acting on the position will deform a closed curve on a surface to a geodesic, which is a local optima. The process is non-linear.
3. Mean curvature flow Heat flow acting on position will deform a surface with appropriate boundary conditions to a minimal surface. The process may produce singularities.

Heat Flow Acting on Linear Maps

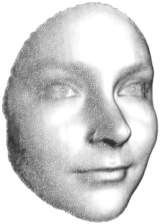


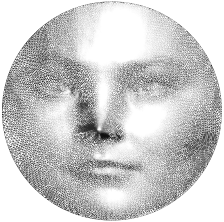
Linear Harmonic Maps

Heat flow acting on the maps

$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t).$$

Heat Flow Acting on Linear Maps





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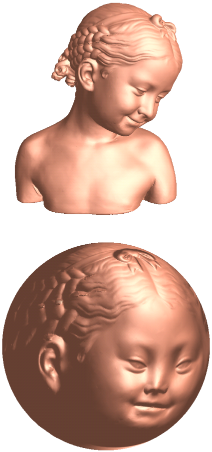
Heat Flow Acting on nonlinear Maps

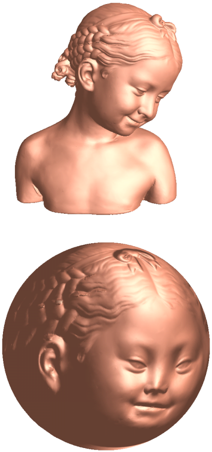


Non-linear Harmonic Maps

Heat flow acting on the maps

$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t) - (\Delta\phi(u,v,t))^{\perp}$$

Heat Flow Acting on nonlinear Maps



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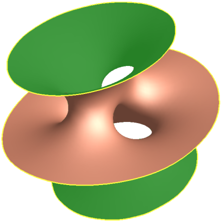
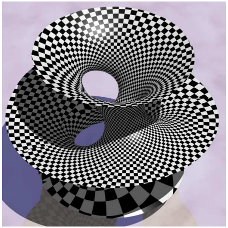


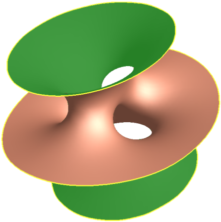


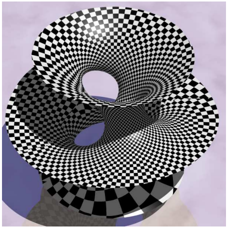
Holomorphic 1-forms

Heat flow acting on 1-forms, the heat flow is

$$\frac{d\omega(u,v,t)}{dt} = \Delta\omega(u,v,t).$$

Heat Flow Acting on Vector Fields (Differential Forms)





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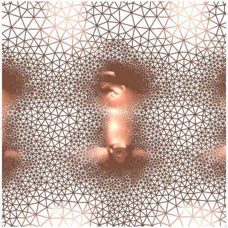
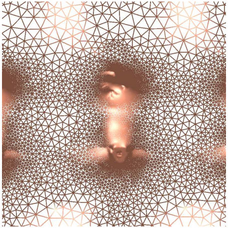
Heat Flow Acting on Metrics



Euclidean Ricci Flow

Heat flow acting on metrics, the curvature satisfies the heat flow

$$\frac{dK(u,v,t)}{dt} = \Delta_{g(t)} K(u,v,t).$$



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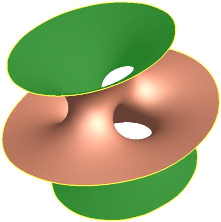

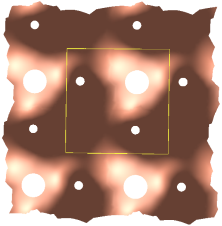

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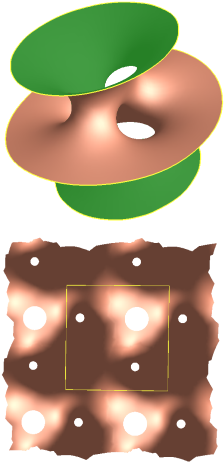

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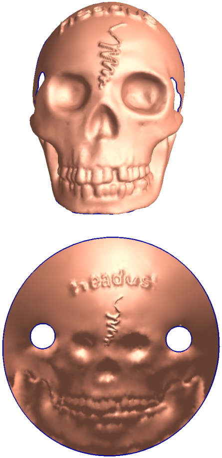

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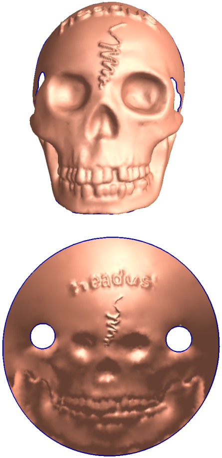

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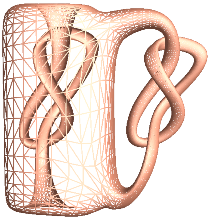

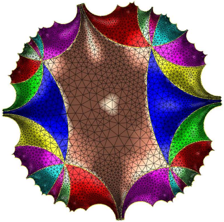

Hyperbolic Ricci Flow

Heat flow acting on metrics, the curvature satisfies the heat flow

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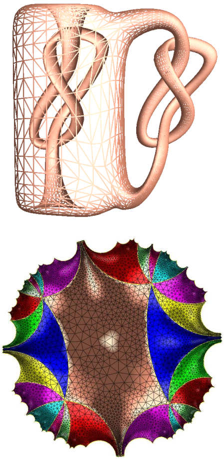
Heat Flow Acting on Metrics



Hyperbolic Ricci Flow

Heat flow acting on metrics, the curvature satisfies the heat flow

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Harmonic Function

Intuition: Harmonic energy measures the *smoothness* of the function.

Definition (Harmonic Energy)

Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a function defined on $\mathbf{R}^2$, the gradient of f is

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

the *harmonic energy* is defined as

$$E(f) := \int_D < \nabla f, \nabla f > dA,$$

where $<,>$ is the inner product defined on the plane, dA is the area element.

Gradient on Surfaces

Suppose Σ is a surface with Riemannian metric $\mathbf{g}$, $f : \Sigma \to \mathbb{R}$ is a function defined on Σ. Arbitrarily choose a local coordinate (x_1, x_2), the differential 1-from of f is

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2.$$

Given a tangent vector $\mathbf{v} \in T\Sigma_p$,

$$\mathbf{v} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2},$$

then at point p

$$df(\mathbf{v}) = < \nabla f, \mathbf{v} >_{\mathbf{g}},$$

namely

$$(v_1, v_2) \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = (v_1, v_2) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \nabla f.$$

Gradient on Surfaces

Definition (Gradient)

Given a metric surface Σ with Riemannian metric $\mathbf{g}$, on a local coordinate system (x_1, x_2), a function $f : \Sigma \to \mathbb{R}$ is differentiable. The *gradient* of f is

$$\nabla f = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix}$$

Laplace-Beltrami Operator on Surfaces

Intuition Laplace-Beltrami operator of f equals to divergence of gradient of f. The divergence is

$$div := \frac{1}{\sqrt{G}} \sum_k \frac{\partial}{\partial x_k} \sqrt{G}$$

Definition (Laplace-Beltrami Operator)

Let f be a differentiable function on a Riemannian manifold with metric $\mathbf{g}$, with respect to local coordinates $\{x^i\}$, assume $g_{ij} = < \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >_{\mathbf{g}}, \sum_k g_{ik} g^{kj} = \delta_i^j$, $G = det(g_{ij})$ is the determinant. The Laplace-Beltrami operator is defined as

$$\Delta f = \frac{1}{\sqrt{G}} \sum_k \frac{\partial}{\partial x_k} (\sum_i g^{ik} \sqrt{G} \frac{\partial f}{\partial x^i}).$$

Intuition: Assume a surface M is made of rubber sheet, we map M to N, the harmonic energy measures the *stretching energy* of the mapping.

Definition (Harmonic Energy)

Suppose M and N are metric surfaces embedded in $\mathbb{R}^3$, $\mathbf{f}: M \to N$ is a map, $\mathbf{f} = (f_1, f_2, f_3)$, the harmonic energy of f is defined as

$$E(\mathbf{f}) = \sum_{i=1}^{3} \int_M < \nabla f_i, \nabla f_i > dA_M$$

Harmonic Function

Intuition: A map is harmonic if and only if the Laplacian-Beltrami is zero.

Theorem (Harmonic function)

A function $f : \Sigma \to \mathbb{R}$ is harmonic if and only if

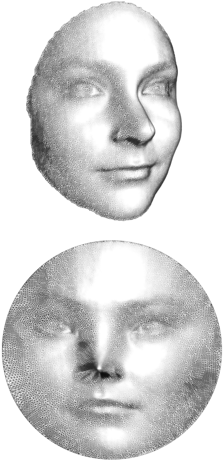
$$\Delta_g f \equiv 0.$$

Definition (Heat Flow)

The heat flow of a function $f : \Sigma \to \mathbb{R}$ is

$$\frac{df}{dt} = -\Delta_g f,$$

Heat Flow Acting on Linear Maps



Linear Harmonic Maps

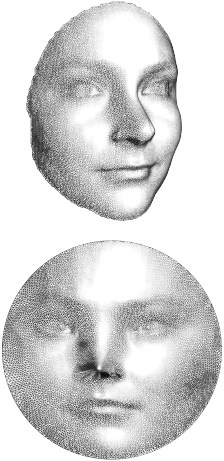
Heat flow acting on the maps

$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t).$$

Theorem (Rado's theorem)

Assume $\Omega \subset \mathbb{R}^2$ is a convex domain with smooth boundary $\partial\Omega$. Given any homeomorphism $\phi : S^1 \to \partial\Omega$, there exists a unique harmonic map $u : D \to \Omega$, such that $u = \phi$ on $\partial D = S^1$ and u is a diffeomorphism.

Heat Flow Acting on Linear Maps



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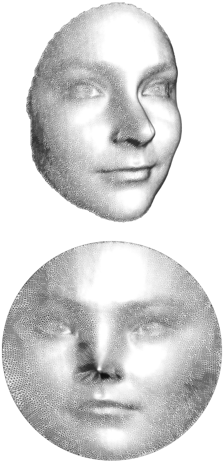


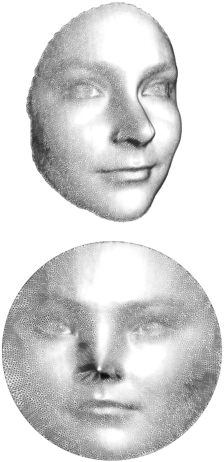
Finite Element Method

Given a mesh Σ, for an edge e_{ij} connecting vertices v_i and v_j, suppose two angles against e are α, β, the define *edge weight* as

$$w_{ij} = \frac{1}{2}(\cot\alpha + \cot\beta)$$

suppose a map $\phi : \Sigma \to \mathbb{R}^2$, then the discrete energy is

$$E(\phi) = \sum_{e_{ij}} w_{ij} |\phi(v_i) - \phi(v_j)|^2.$$



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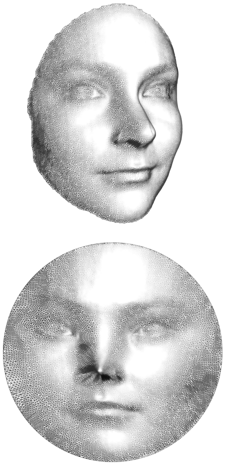


Finite Element Method

Discrete Laplace-Beltrami operator

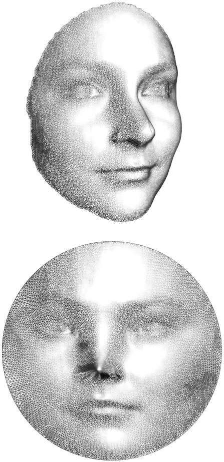
$$\Delta\phi(v_i) = \sum_{e_{ij}} w_{ij}(\phi(v_i) - \phi(v_j)),$$

Heat flow

$$\phi(v_i) - = \Delta\phi(v_i)\epsilon,$$

where ϵ is a small constant.

Heat Flow Acting on Linear Maps



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Intuition A conformal map from complex domain to the complex plane preserving angles.

Definition (Holomorphic function)

A complex function $f : (x, y) \to (u, v)$ is holomorphic, if it satisfies the Riemann-Cauchy equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Conformal maps are stronger than Harmonic maps. Harmonic maps between two homeomorphic surfaces exist, but conformal maps may not exist.

Spherical Conformal Maps

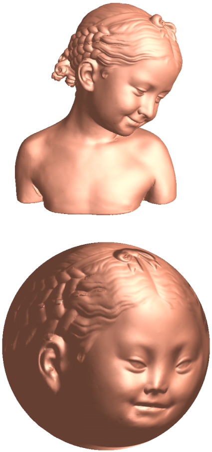
Non-linear Harmonic Maps

Heat flow acting on the maps

$$\frac{d\phi(u,v,t)}{dt} = \Delta\phi(u,v,t) - (\Delta\phi(u,v,t))^{\perp}$$

Theorem (Heat Flow for Topological Sphere)

The heat flow of a map from a closed genus zero surface to the unit sphere converges to a conformal map under normalization constraints. The conformal map is a diffeomorphism.



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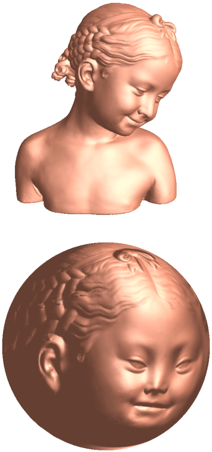
Discrete Approximation

Heat flow acting on the maps

$$\phi(v_i) - = (\Delta\phi(v_i) - \Delta\phi(v_i)^{\perp})\epsilon$$

where $\Delta\phi(v_i))^{\perp}$ is defined as

$$< \Delta\phi(v_i), \phi(v_i) > \phi(v_i).$$



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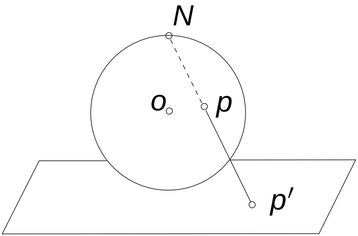
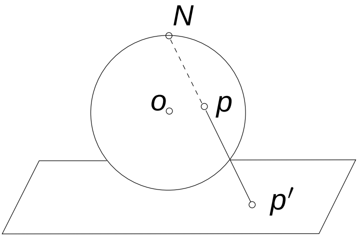
Spherical Conformal Maps



Stereo graphic projection

A conformal map from the unit sphere $p(x, y, z)$ to the complex plane

$$p' = \frac{2}{2-z}p,$$



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Möbius Transform

A Möbius transform on the complex plane $\phi : \mathbb{C} \to \mathbb{C}$ is

$$\phi(z) = \frac{az+b}{cz+d}, ad - bc = 1,$$

where $a, b, c, d \in \mathbb{C}$

Theorem (Conformal Automorphism Group)

The conformal maps from a unit sphere to itself (or the complex plane) differ by a Möbius map.

Normalization

In order to remove the Möbius ambiguity, spherical harmonic map in normalized

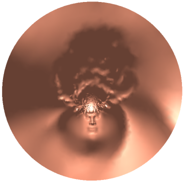
1. Compute the mass center of the image,

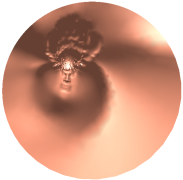
$$\mathbf{c} = \sum_{v_i} \phi(v_i),$$

2. Normalize

$$\phi(v_i) = \frac{\phi(v_i) - \mathbf{c}}{|\phi(v_i) - \mathbf{c}|}$$

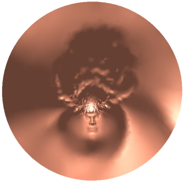
Riemann Mapping Theorem

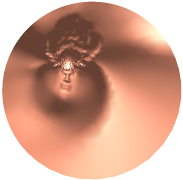




Topological Disk Conformal Mapping

1. Double cover
2. Conformally map the doubled surface to the unit sphere
3. Use the sphere Möbius transformation to make the mapping symmetric.
4. Use stereographic projection to map each hemisphere to the unit disk.





Möbius Transformation

A Möbius transformation from the unit disk to itself is a conformal map

$$\phi(z) = e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 z}.$$

Theorem (Riemann Mapping)

Any metric topological disk can be conformally mapped to the unit disk, the mapping is unique up to a Möbius transformation.

Definition (Holomorphic 1-form)

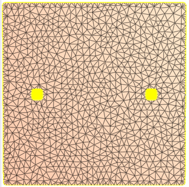
Suppose Σ is a Riemann surface, $\{z_\alpha\}$ is a local complex parameter, a holomorphic 1-form ω has a local representation as

$$\omega = f(z_\alpha)dz_\alpha,$$

where $f(z_\alpha)$ is a holomorphic function.

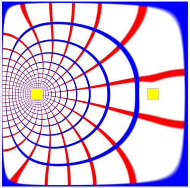
Locally, ω is the derivative of a holomorphic function. Globally, it is not.

Original Surface



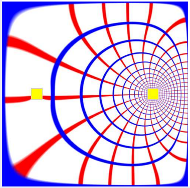
Holomorphic 1-forms

One basis holomorphic 1-form



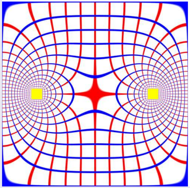
Holomorphic 1-forms

Another one basis holomorphic 1-form

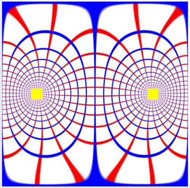


Holomorphic 1-forms

Summation of ω_1 and ω_2

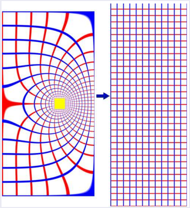


Difference between ω_1 and ω_2



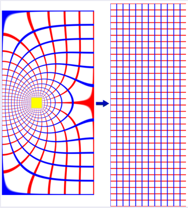
Holomorphic 1-forms

Holomorphic 1-form induces a conformal parameterization.



Holomorphic 1-forms

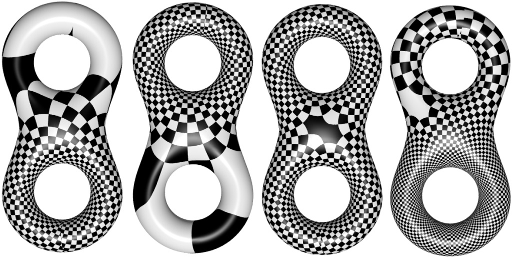
Holomorphic 1-form induces a conformal parameterization.



Holomorphic 1-forms

Theorem (Holomorphic 1-forms)

All holomorphic 1-forms form a linear space $\Omega(\Sigma)$ *which is isomorphic to the first cohomology group* $H^1(\Sigma, \mathbb{R})$.

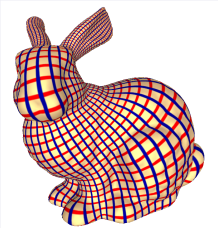


Holomorphic 1-forms

Holomorphic 1-form ω can be treated as two real 1-forms $\omega = (\omega_0, \omega_1)$.

Furthermore, we can treat each 1-form as a vector field, such that

1. $curl\,\omega_0 \equiv 0$
2. $div\,\omega_0 \equiv 0$
3. $\omega_1 = \mathbf{n} \times \omega_0$, where **n** is the normal field.



Holomorphic 1-forms

Intuition Hodge star operator rotates a vector field about the normal a right angle.

Definition (Hodge Star)

Hodge star operator is defined in the following:

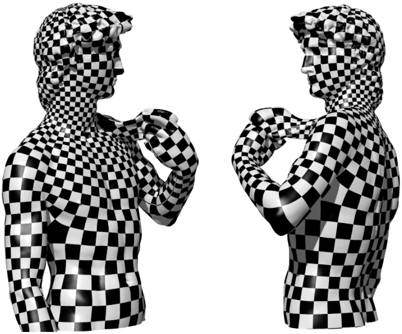
$$*dx = dy, *dy = -dx,$$

Definition (harmonic 1-form)

Suppose Σ is a Riemann surface, ω is differential 1-form, locally ω is the derivative of a harmonic function. Symbolically,

$$d\omega = 0, *d*\omega = 0.$$

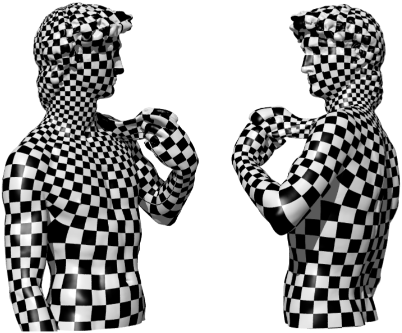
Globally , such harmonic function doesn't exist.



Holomorphic 1-forms

Theorem (Hodge)

Each cohomologous class has a unique harmonic 1-form.



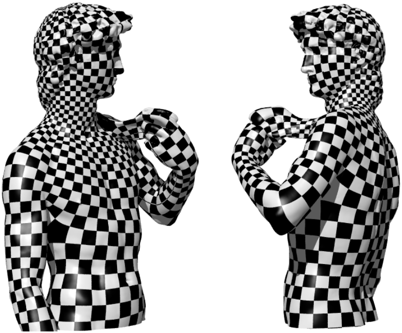
Holomorphic 1-forms

Algorithm for Holomorphic 1-forms

Input : A triangle mesh Σ.
Output : Basis for holomorphic 1-forms

1. Compute cohomology basis $\{\omega_1, \omega_2, \cdots, \omega_n\}$.
2. Heat flow to deform ω_i to harmonic 1-forms.
3. Compute hodge star of ω_i's.
4. return holomorphic 1-form basis

$$\{\omega_1 + \sqrt{-1}*\omega_1, \omega_2 + \sqrt{-1}*\omega_2, \cdots, \omega_n + \sqrt{-1}*\omega_n\}$$



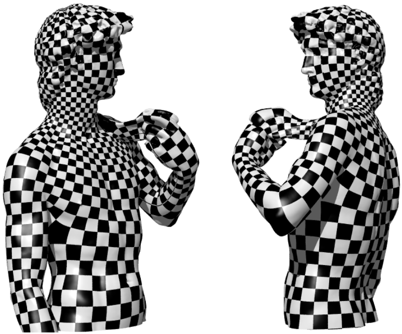
Heat Flow for 1-forms

Suppose $\omega : \{Edges\} \to \mathbb{R}$ is a closed 1-form. Let $f : \{Vertices\} \to \mathbb{R}$ is a function, then

$$f- = \Delta(\omega + df) \times \epsilon,$$

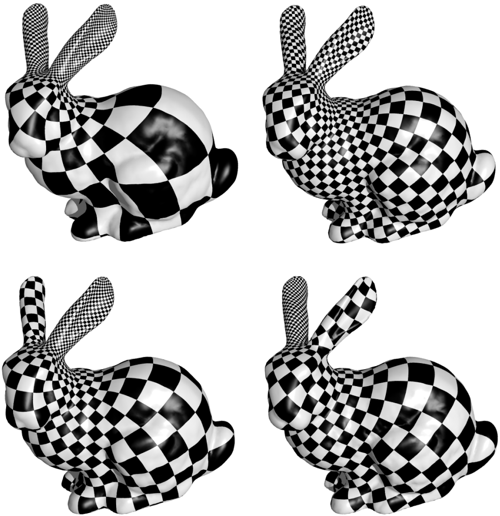
where $\Delta(\omega + df)(v_i)$

$$\sum_{e_{ij}} w_{ij}(\omega(e_{ij}) + f(v_j) - f(v_i)).$$



Holomorphic 1-forms

Choose the best cohomology class to optimize the distortion,

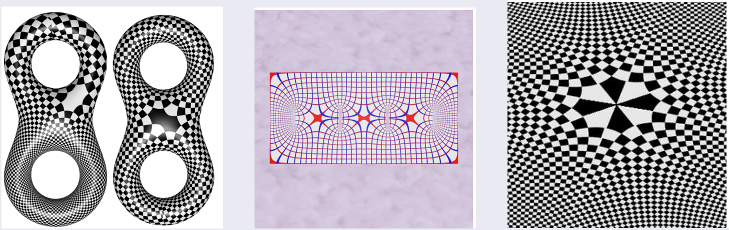


Zero Point

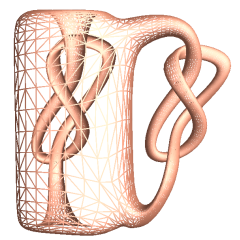
Zero Point

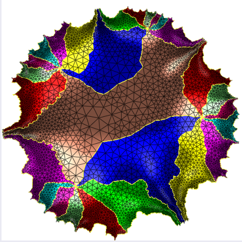
In a neighborhood of a zero point, the holomorphic 1-form has the form zdz. According to Hopf theorem, in general, there are totally $\chi(\Sigma)$ zero points for a holomorphic 1-form.

The position of zero points are intrinsic to the conformal structure, can not be fully prescribed.



For more information, please email to gu@cs.sunysb.edu.





Thank you!