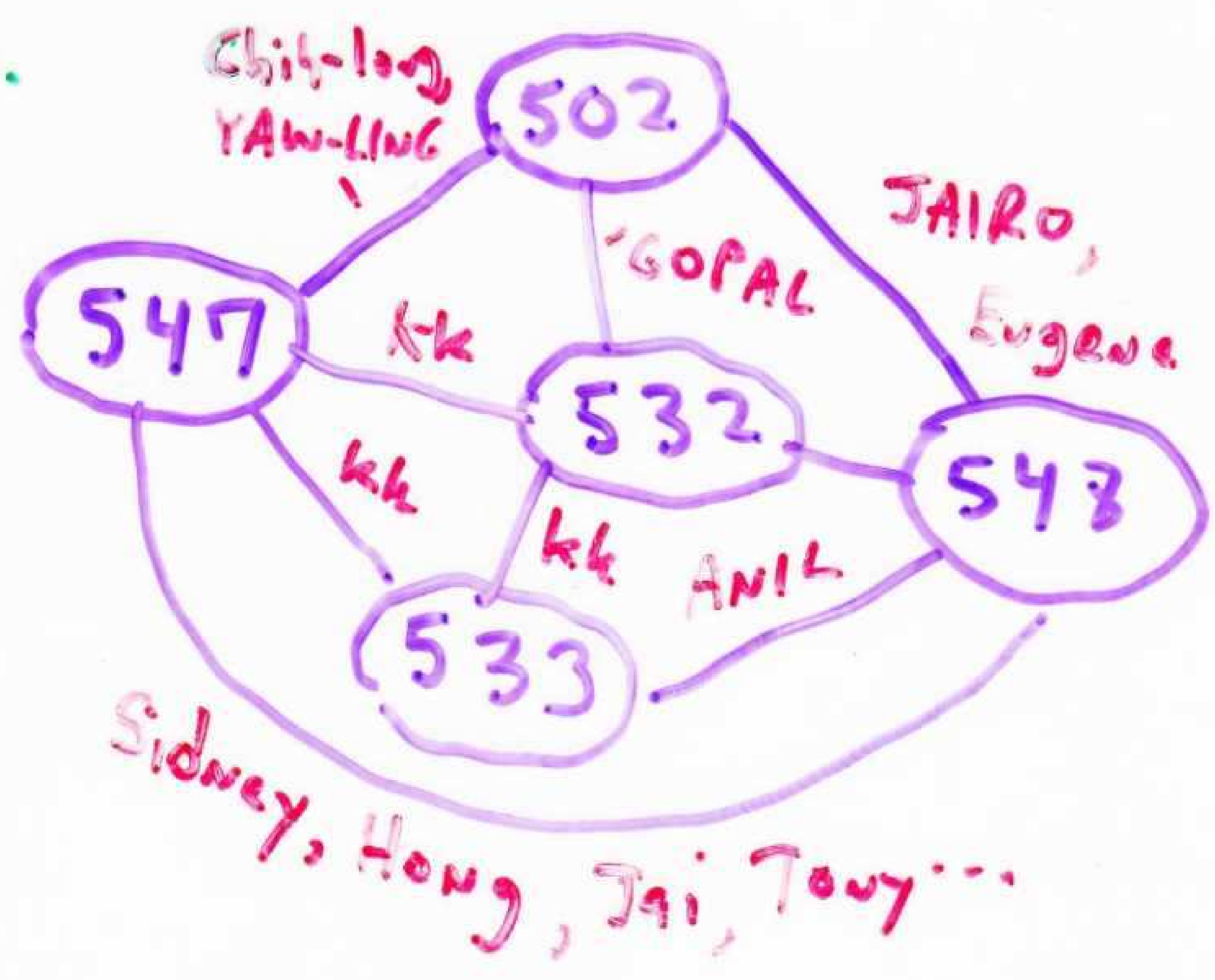
Coloring Graphs

Suppose a school wanted to minimize the number of final exam periods each semester. In an effort to be fair to the students, they want to schedule things so no student has two exams at the same time.



What we want to do is assign a period or color to each vertex such that no edge connects vertices of the same color.

Clearly five colors suffice, since there are only five vertices, but we can do better:

502, 533
547,
548
532

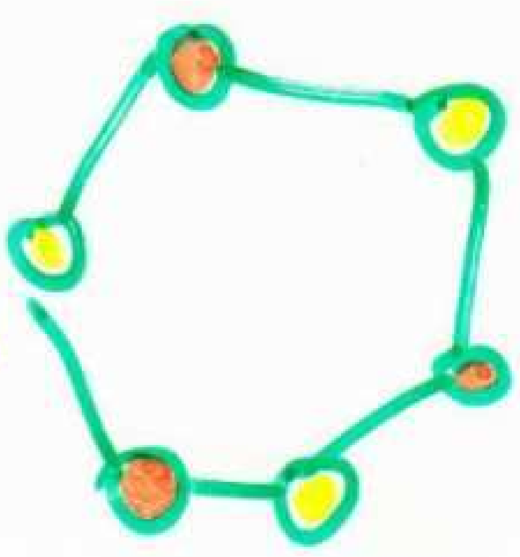
Many scheduling problems reduce to graph coloring

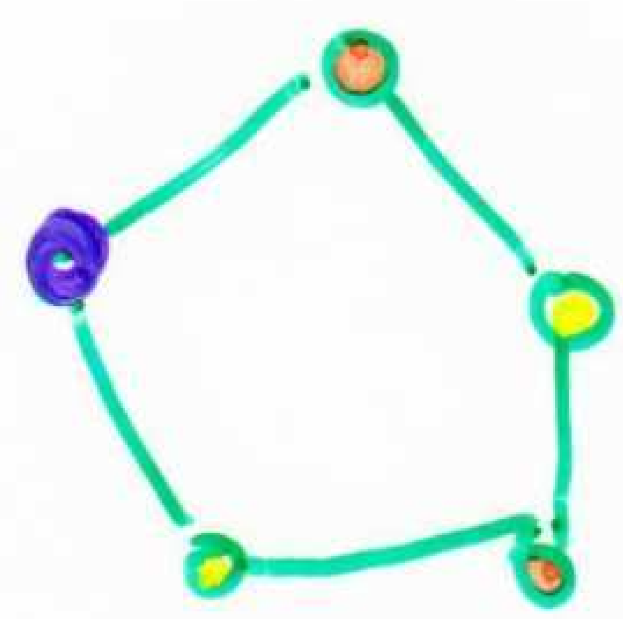
Chromatic Numbers

An assignment of labels to vertices such that no edge links two identically labeled vertices is called a coloring. The minimum number of colors with which it is possible to color a graph is its chromatic number.

The chromatic number of a cycle is 2 or 3, Depending on if it is even or odd.

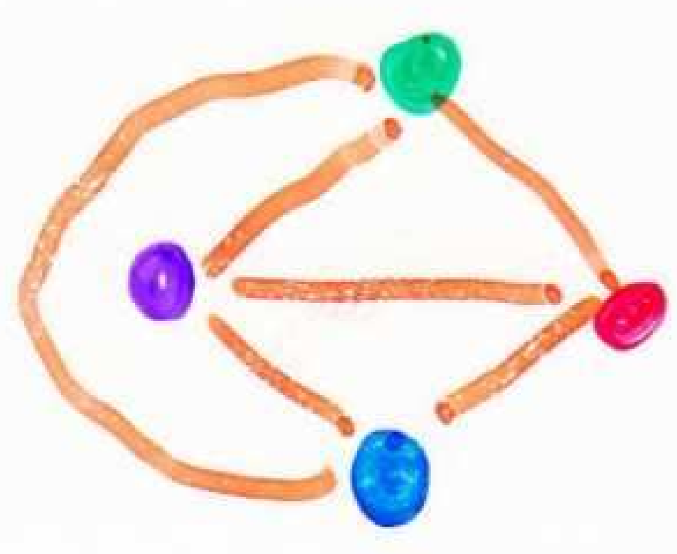
$\chi(G)=2$





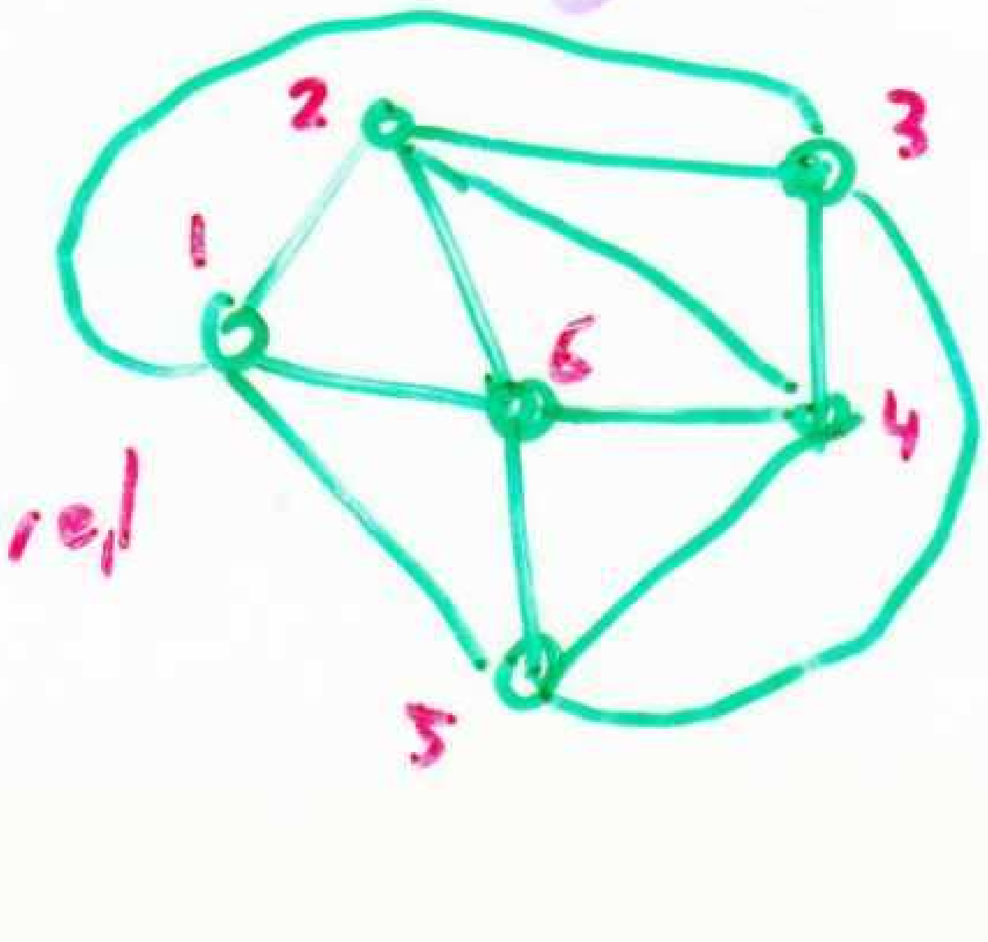
$\chi(G)=3$

The chromatic number of K_N is N:



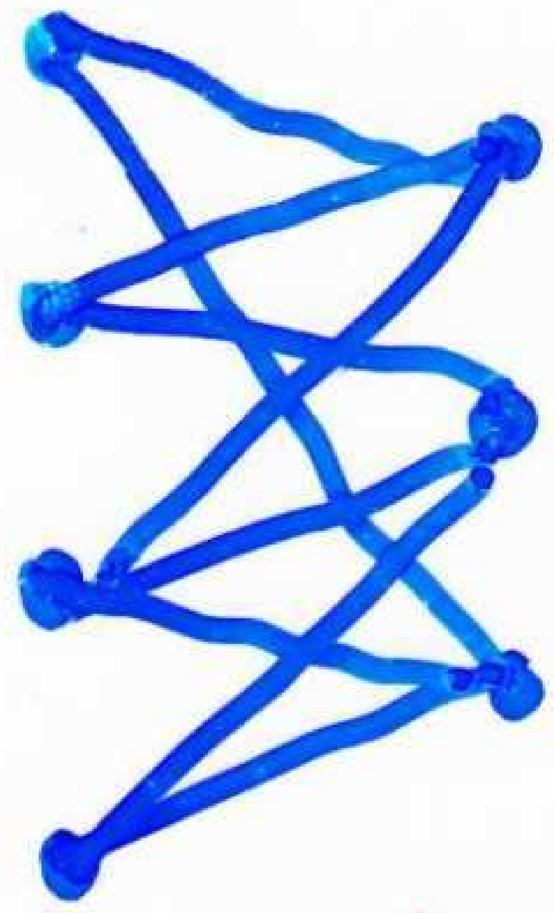
The only graphs of chromatic number 1 have no edges.

It is **not** easy to find the chromatic number of a graph. This can be 3-colored



GBRGBR

A graph G is bipartite if it can be two colored:



Since edges in a bipartite graph alternate stages, every cycle in a bipartite graph must be even.

Further, a graph is bipartite iff it contains only even length cycles.

Finding the chromatic number of a graph is NP-complete, but there are some good ways to get bounds on what it is.

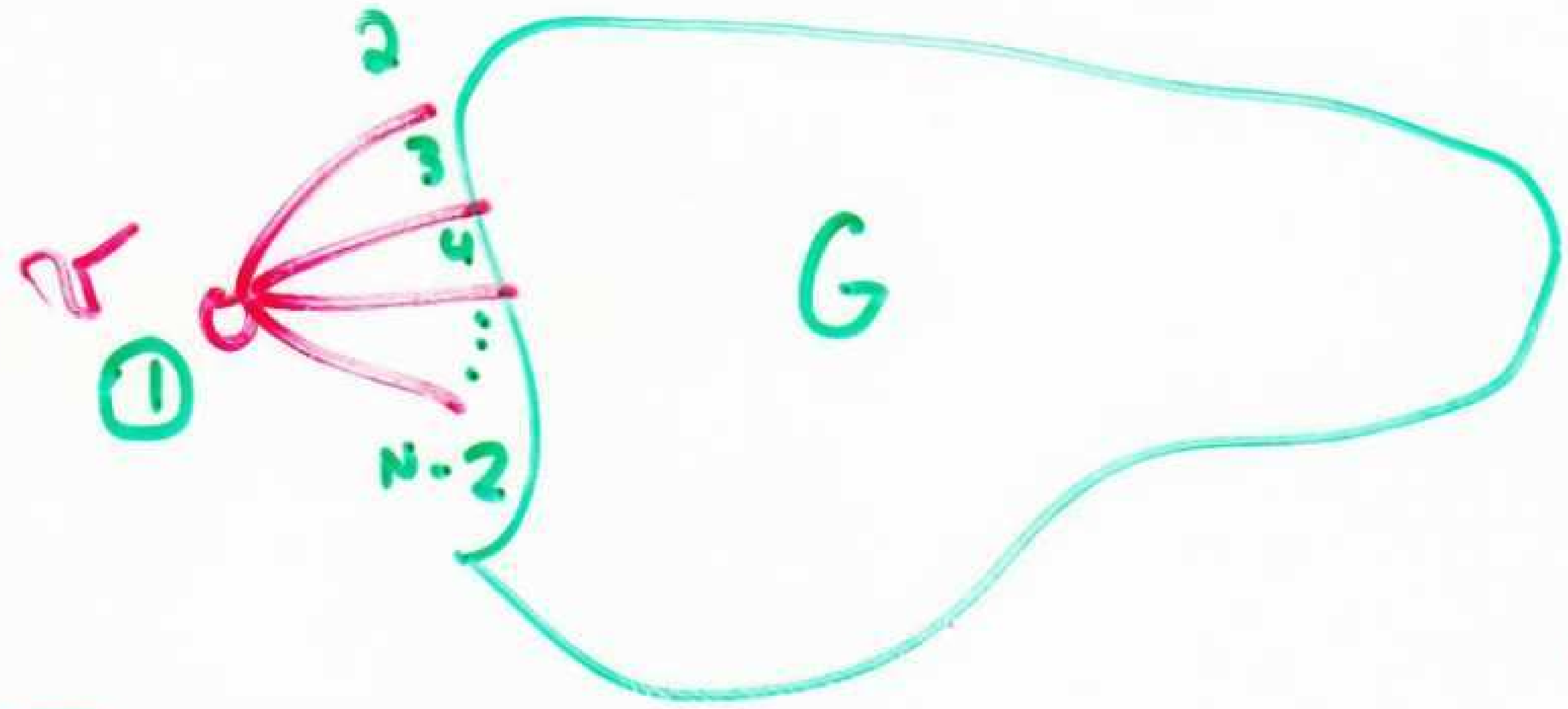
Theorem: For any graph G, the chromatic number $\chi(G) \le 1 +$ Maximum degree of G.

Proof: By induction on chromatic number.

$\chi(G) = 1 \Rightarrow$ maximum degree is 0. ✓

Assume it is true for graphs of chromatic number < N

Now consider a minimal N-chromatic graph, where deleting any vertex leaves an (N-1) chromatic graph.



By the assumption G is $(N-1)$-chromatic with maximum degree of at least $(N-2)$

If v is incident on $< N-1$ vertices, it cannot increase $\chi(G)$, since there is a left over color. Thus degree v must be $\geq N-1$ for $G \cup \{v\}$ to be N-chromatic.

To complete the proof, observe that adding more vertices and edges without increasing the chromatic number can only increase the maximum degree. ∎

This bound can be tightened even further:

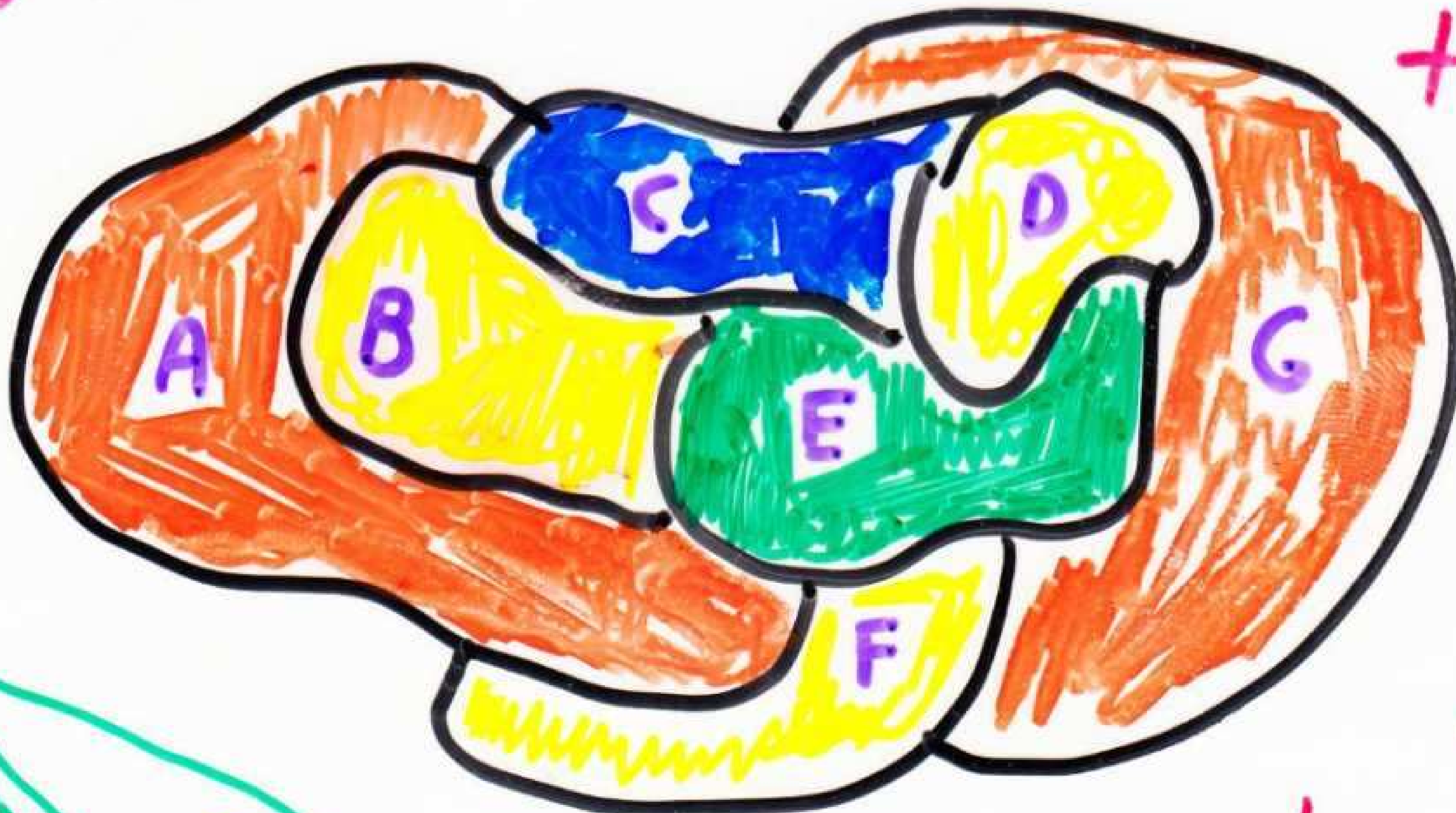
Brook's Theorem: Any graph is N-colorable if its maximum degree is N unless

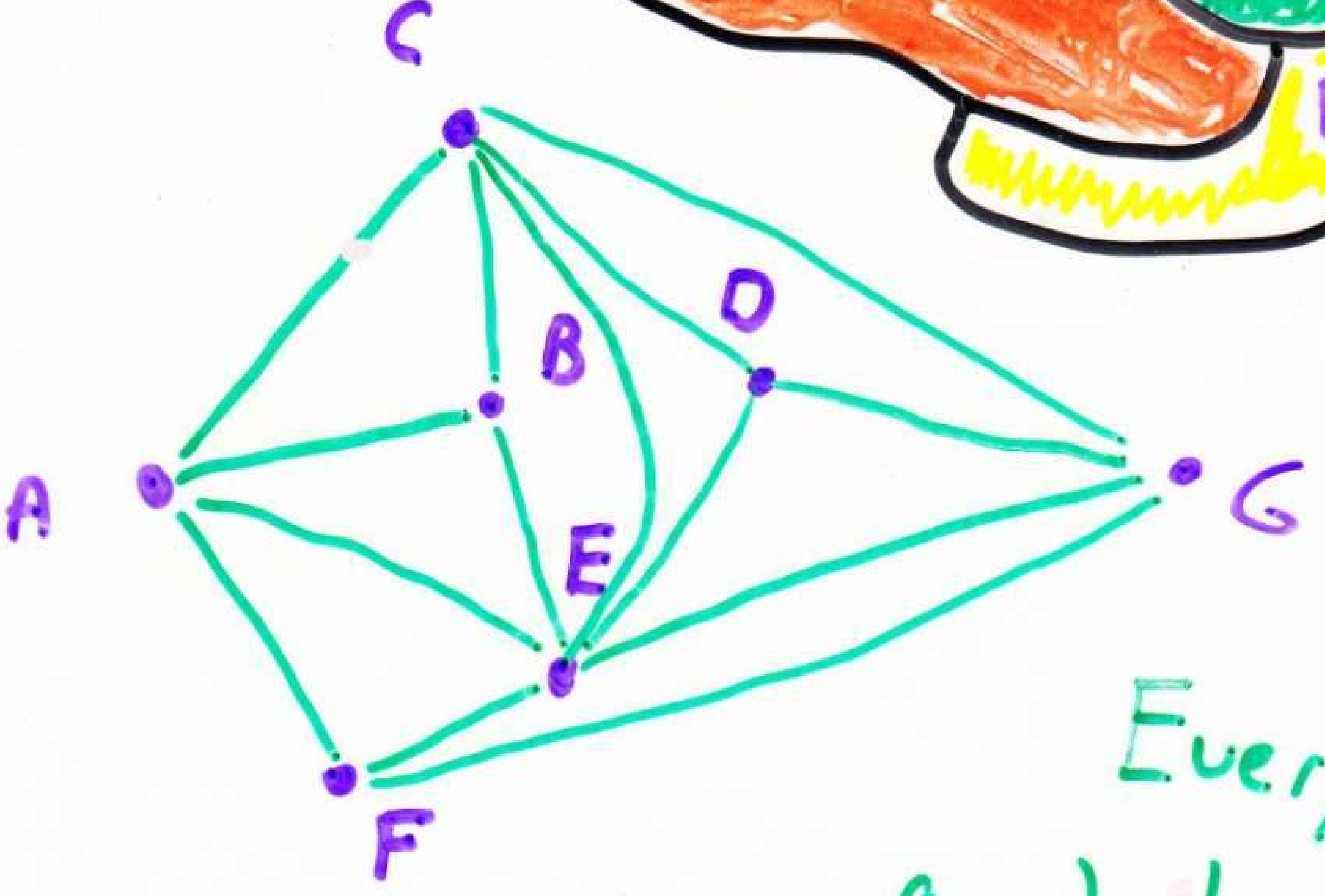
1. it is an odd cycle
2. it contains K_{N+1} as an induced subgraph.

The Four Color Problem

Perhaps the most infamous problem in graph theory is whether the maximum chromatic number of a planar graph is four.

It takes a bit of abstraction to see that map coloring is a graph problem. Give a map, the dual graph has a vertex for each region, with edges connecting adjacent regions

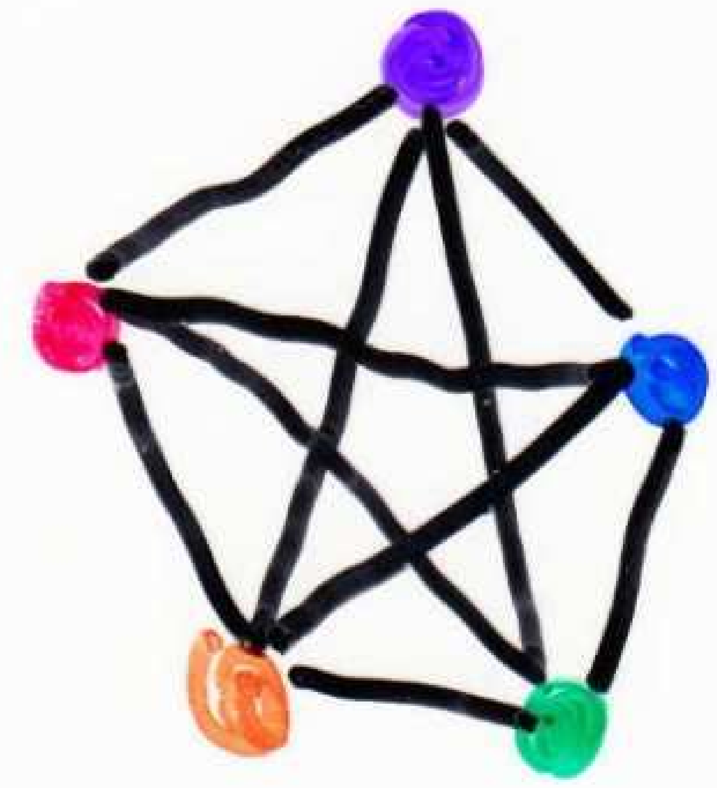




Every planar embedding has a dual graph which is planar.

So the four-color map problem really asks for a vertex coloring of all planar graphs with at most four colors.

By Kuratowski's Theorem, a planar graph cannot have K_5 as an induced subgraph, which would obviously require 5 colors, but that is not enough to prove the theorem:



The history of the problem dates back to Guthrie in 1852.

Kempe proved it in 1879. Heawood broke this proof in 1890, salvaging that 5 colors suffice.

Tait gave a different proof in 1880. Peterson broke this proof in 1891.

Appel & Hakan proved that 4-colors suffice in 1976, by demonstrating that if a counter-example exists it will be one of a finite number of cases, and doing a computer search for 1200 hours on showing none of these cases work. This proof has recieved an unfair amount of scorn in the mathematical community.

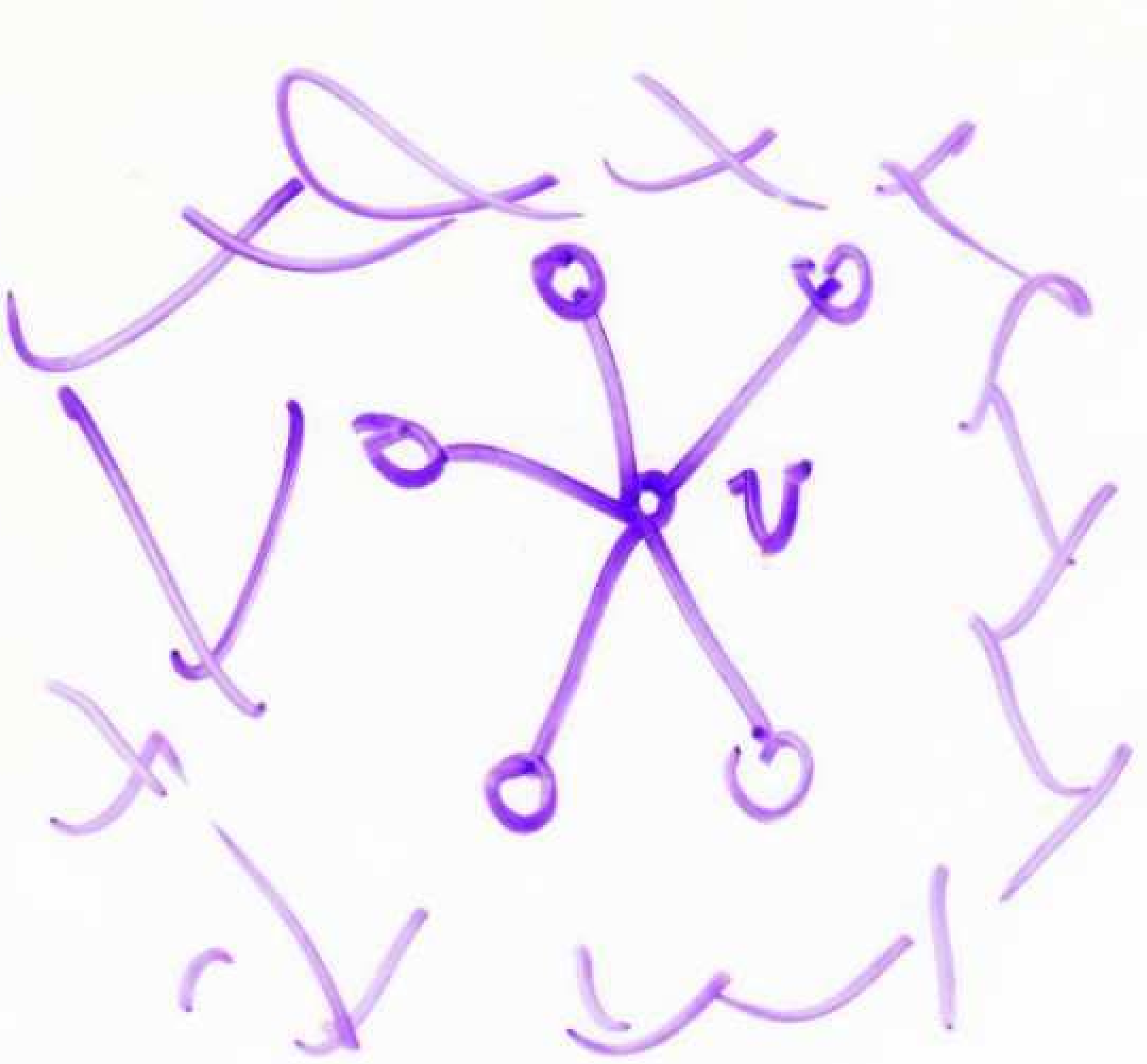
Theorem: 5 Colors Do Suffice to Color any planar graph

Proof: By induction, giving an algorithm to 5-color any planar graph.

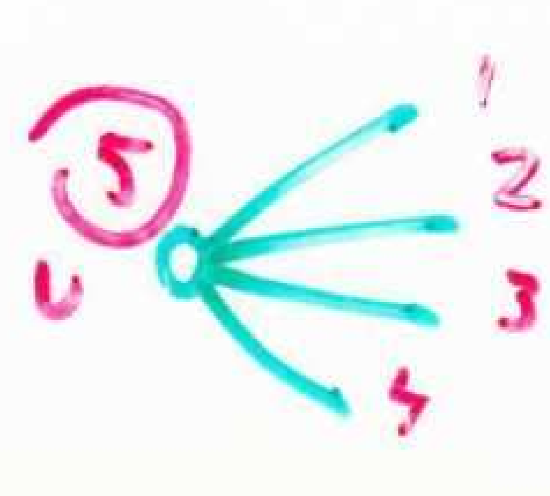
Clearly any planar graph on up to five vertices can be five colored, by assigning each a unique color.

Assume that any planar graph on up to $N-1$ vertices can be colored with five colors.

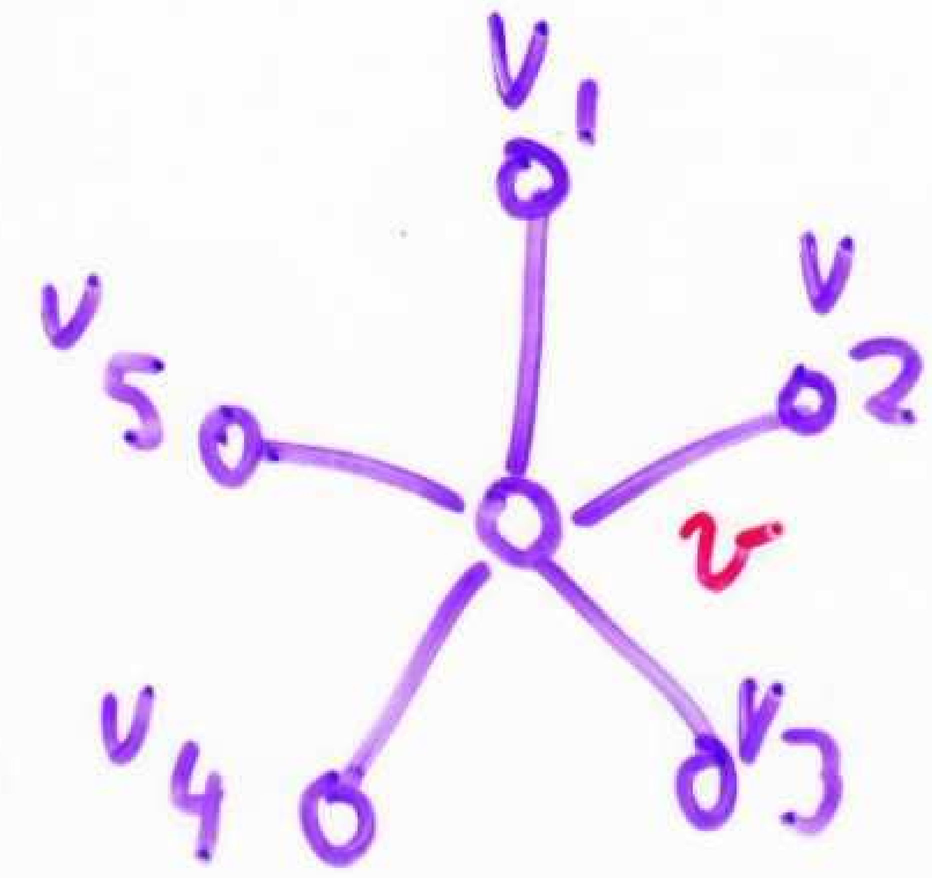
Now, given a planar graph G on N-vertices, there exists at least one vertex v of degree ≤ 5. Deleting it leaves a graph which can be five-colored.



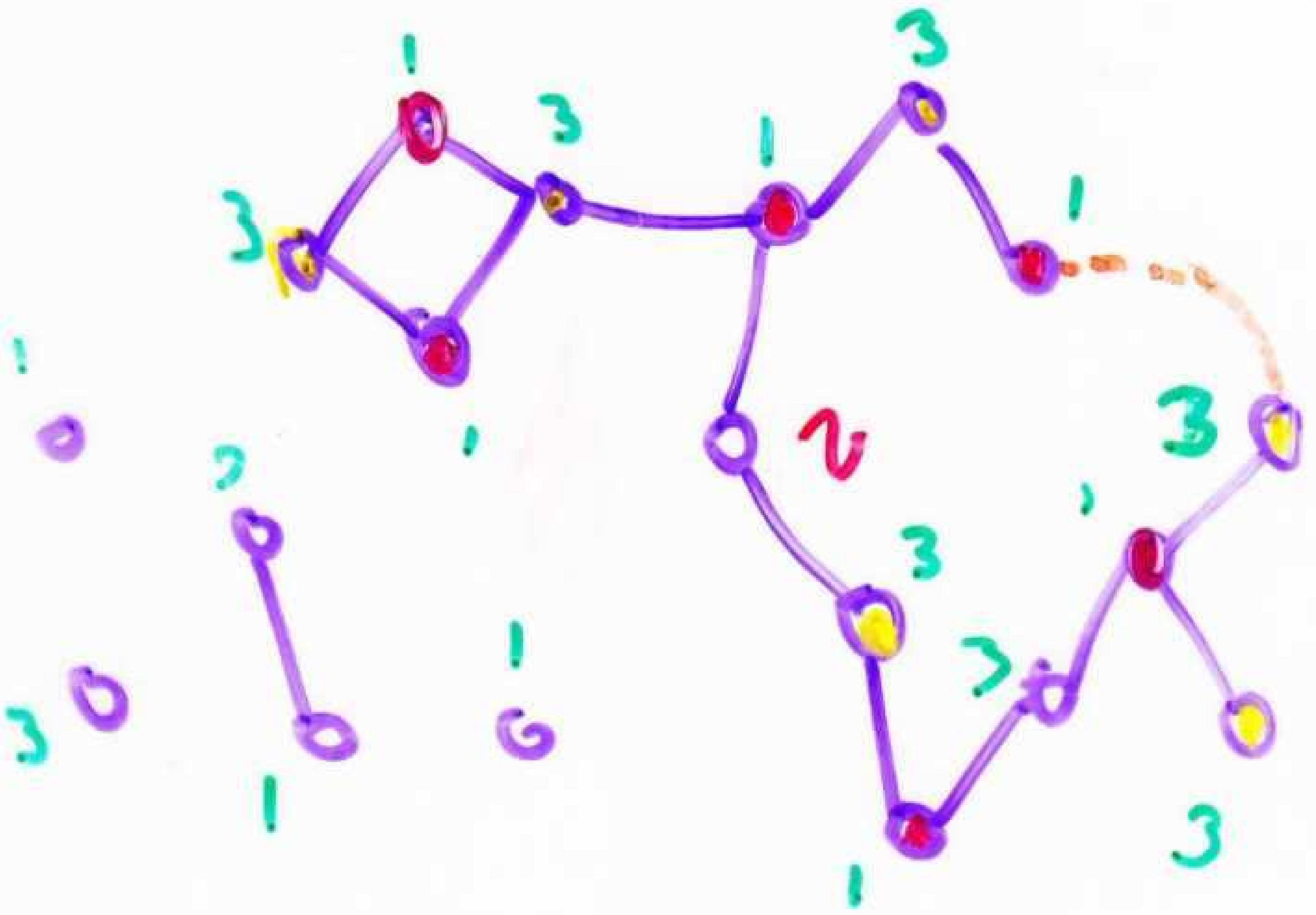
If v is of degree < 5, there are at most 4 distinct colors incident upon it, so the fifth can be used for v.



Now, give a planar embedding of G with a five coloring of $G - \{v\}$. The embedding defines an ordering around v:

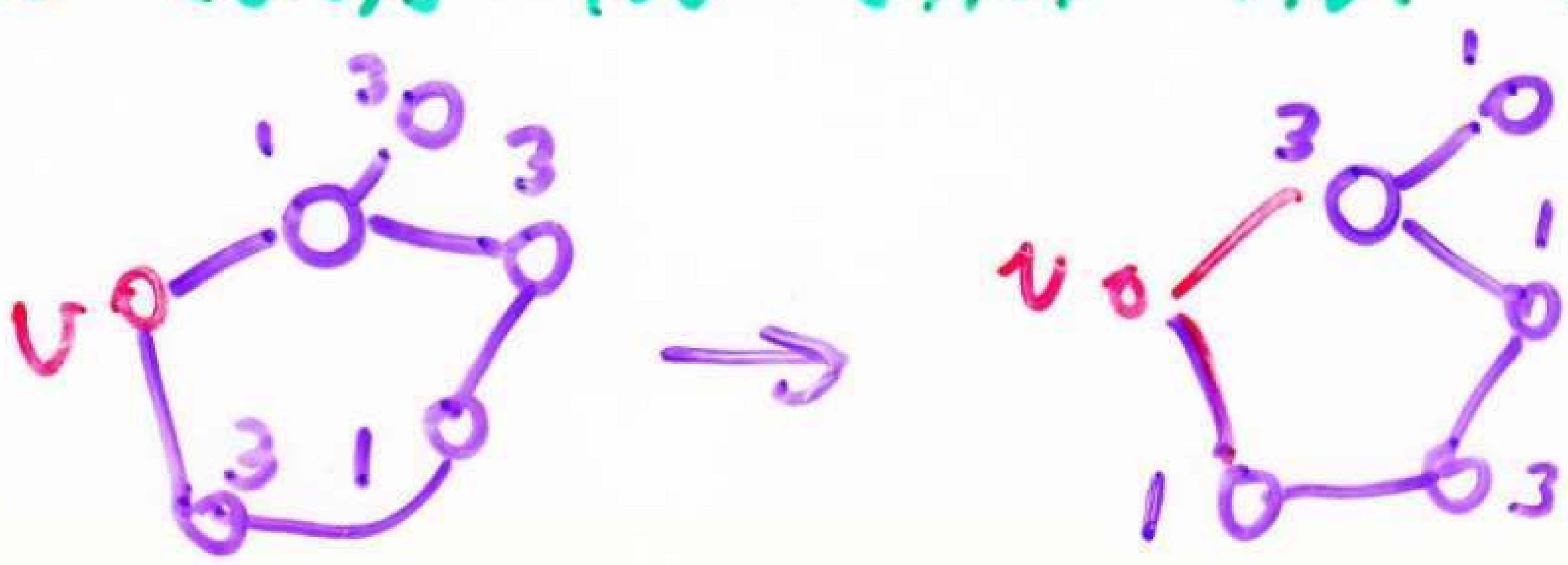


Now induce a subgraph of G with all vertices colored 1 & 3 (or any two colors not neighbors around v.)

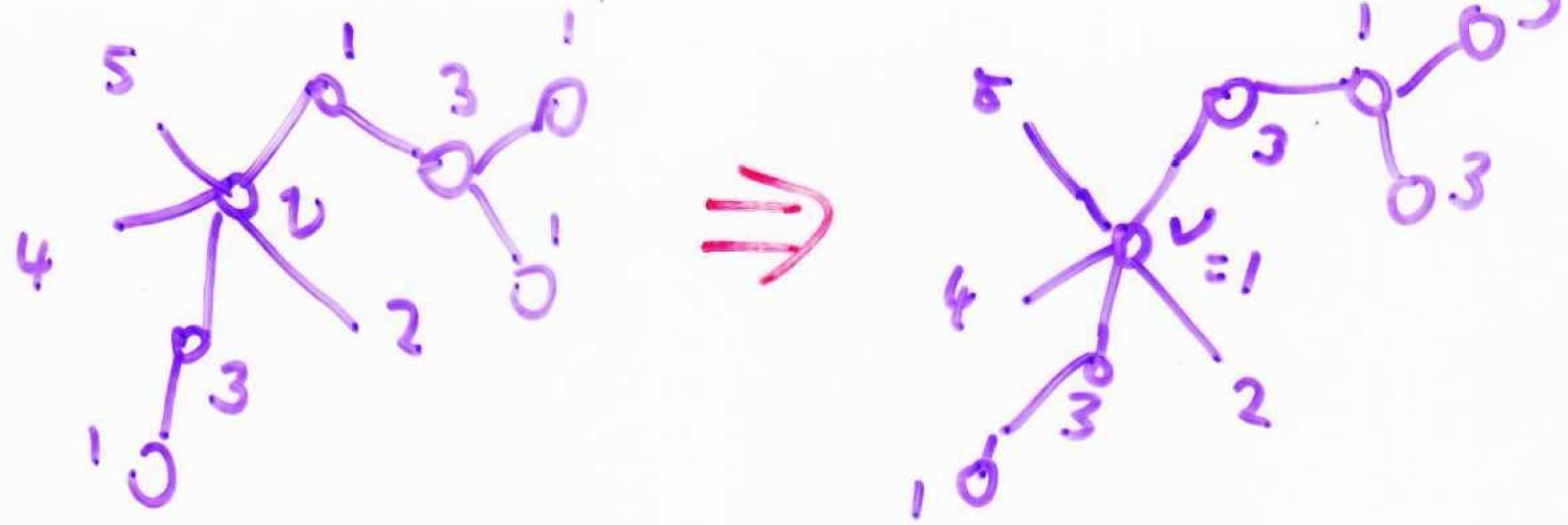


Either v_1 & v_3 belong to the same connected component of this subgraph, or they don't.

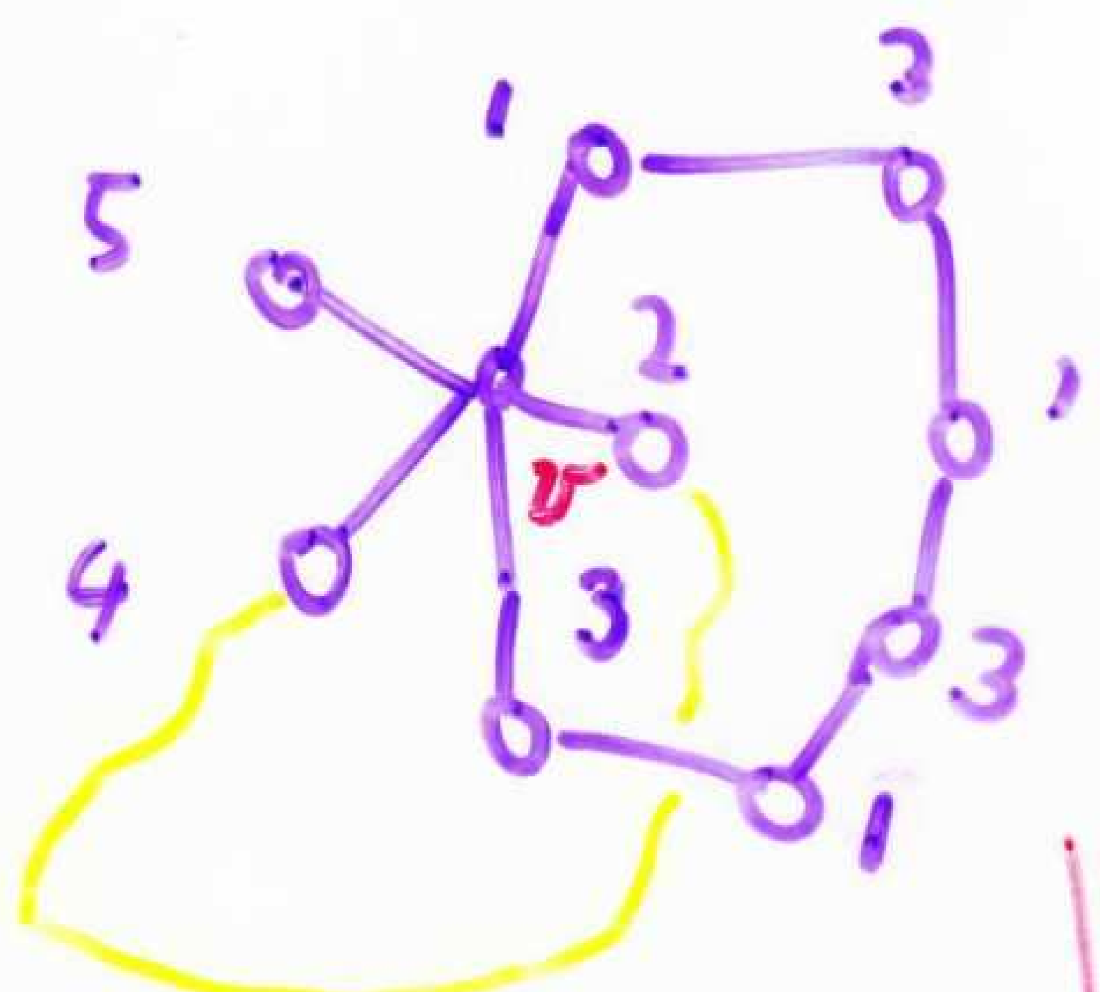
Observe that within any connected component of this induced graph, we can reverse the role of the two colors and still have a 5-coloring of $G - \{v\}$.



Thus if v_1 & v_3 do not belong to the same component, we can reverse colors on the component with v_1, leaving color 1 for v!



However, if there is a $v_1 - 3 - 1 - 3 \cdots v_3$ path, alternating them does <u>not</u> free a color.



But now observe that, since this is a planar embedding, there can be no $v_2 - 4 - 2 \cdots 2 - v_4$ path because it would cross the $v_1 - v_3$ path!

Thus we can reverse the roles of 2 & 4 in connected component with v_2 in the induced subgraph of 2 & 4 colored vertices, permitting us to color v with 2!

The proof does not suffice for four colors!

Blue-Green chain 2-4

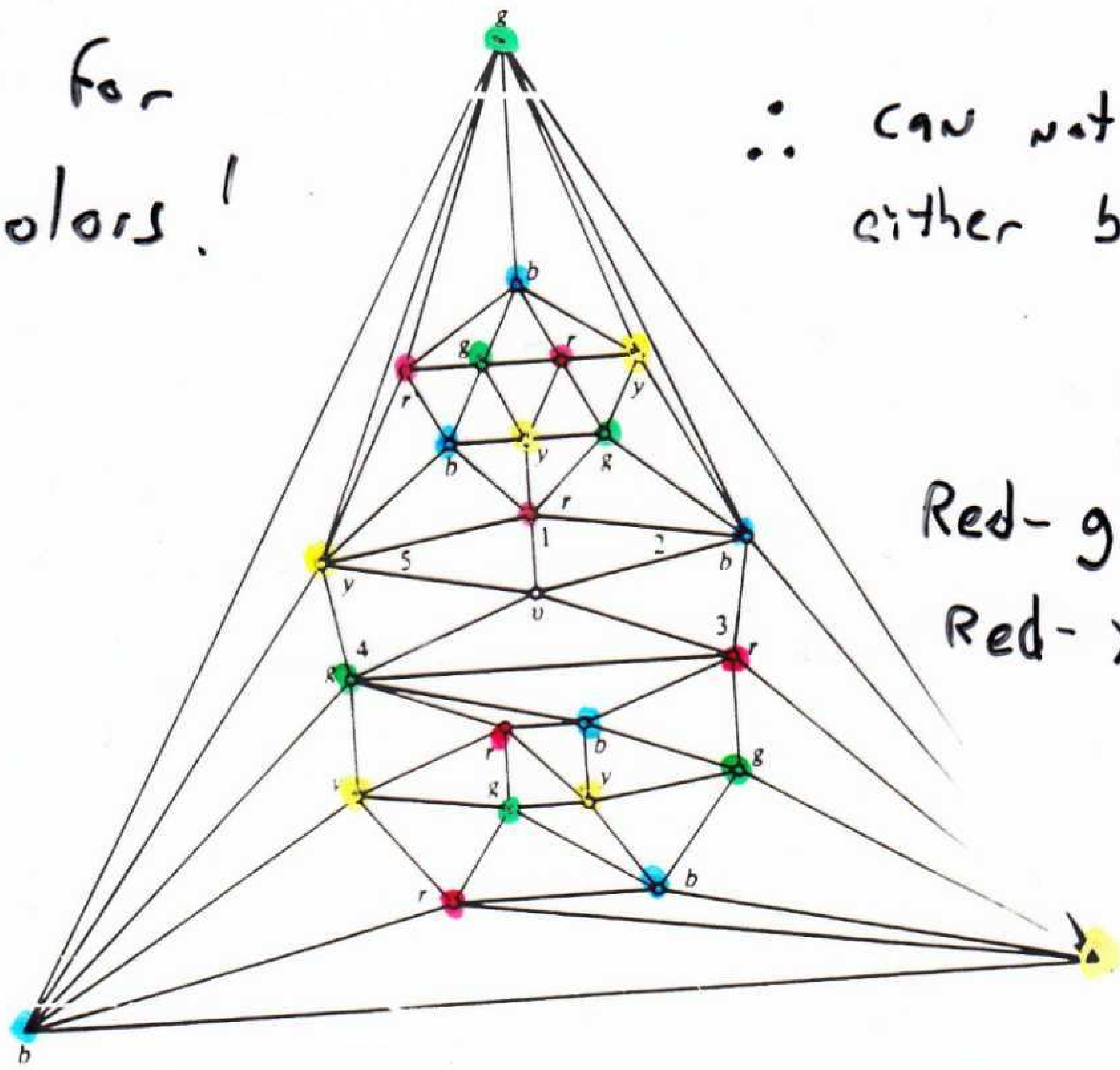
Blue-Yellow chain 2-5

∴ can not free up either blue, green, yellow

<u>but NO</u>

Red-green 1-4

Red-yellow 3-5

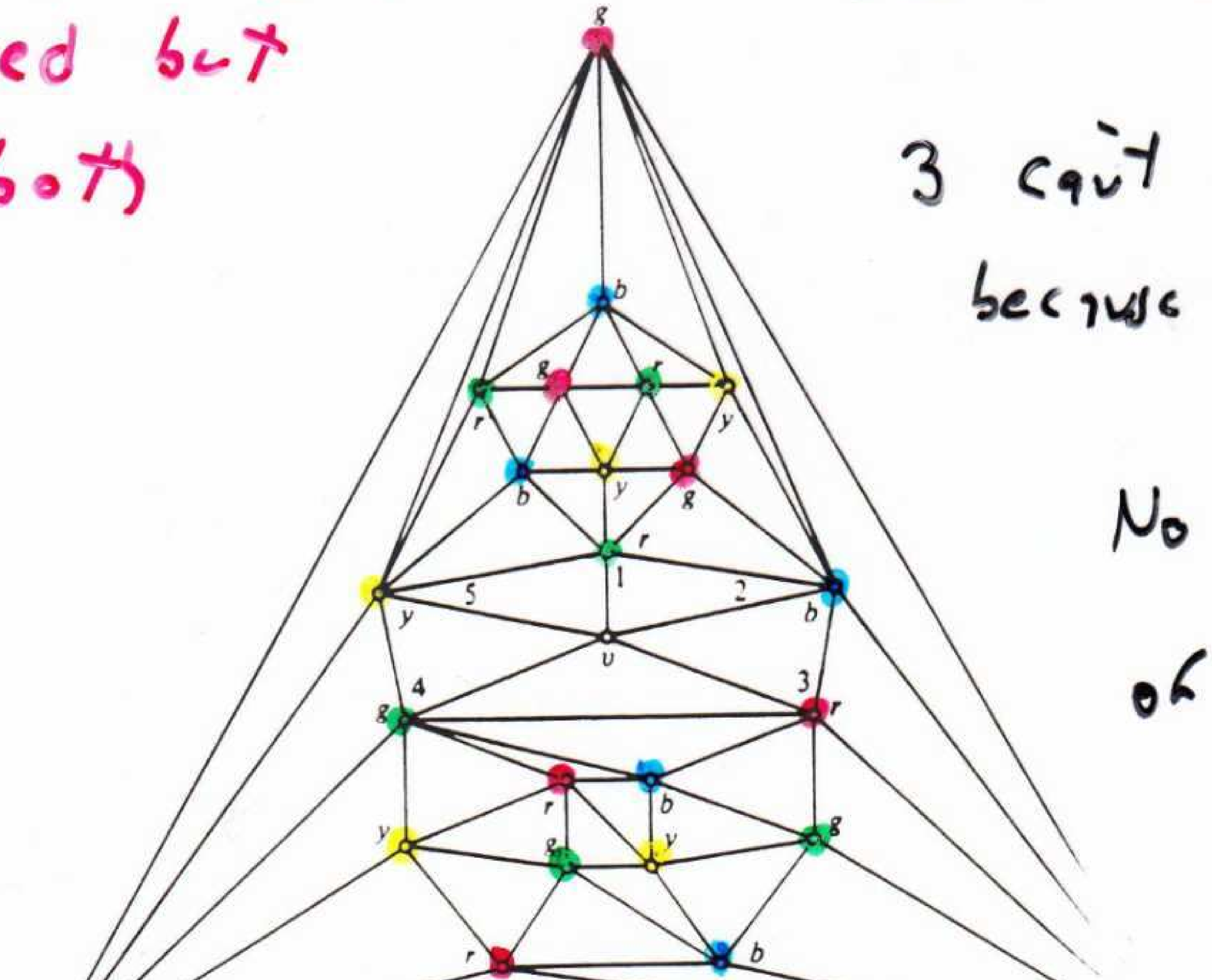


Suppose we flip the Red at 1 to Green!

You can remove <u>ONE</u> red but not both

3 can't flip to Green because of 4

No <u>simple</u> series of flips suffice



Chromatic Polynomials

Suppose I could compute a function $f(G,z)$ on any graph G which would count how many ways G can be colored with z colors.

$f(G,1)=0$

$f(G,2)=2$

1	2	1
2	1	2

1 2 3, 1 3 2
2 1 3, 2 3 1
3 2 1, 3 2 1

1 2 1, 2 1 2
1 3 1, 3 1 3
2 3 2, 3 2 3

$f(G,3)=12$

In fact, such a function always exists and is called the chromatic polynomial of G.

For the path on 3 vertices, $f(G,z)=z(z-1)^2$

Observe that the chromatic polynomial of an empty graph on n vertices is z^n

each vertex can be colored in z ways independently.

We can compute the chromatic polynomial of any graph with the following observation:

Suppose G contains edge $\{x, y\}$. The graph $G - (x,y)$ can be colored every way G can, plus all the ways x & y can have the same color.

– XY is adjacent the union of what x & y are adjacent to

If x & y **must** receive the same, it is the same as coloring the contraction of G with respect to x & y.

Thus $f(G, z) = f(G - \{x, y\}, z) - f(\text{contract } x \text{ \& } y \text{ of } G, z)$

This gives a recursive construction which eventually ends with an empty graph.

Example: $f(\circ\!-\!\circ\!-\!\circ, z)$

$$f(\circ\!-\!\circ\!-\!\circ, z) = f(\circ \quad \circ\!-\!\circ, z) - f(\circ\!-\!\circ, z)$$

$$f(\circ \quad \circ\!-\!\circ, z) = f(\circ \quad \circ \quad \circ, z) - f(\circ \quad \circ, z)$$

$$f(\circ\!-\!\circ, z) = f(\circ \quad \circ, z) - f(\circ, z).$$

$$f(\circ, z) = z$$

$$f(\circ \quad \circ, z) = z^2$$

$$f(\circ \quad \circ \quad \circ, z) = z^3$$

$$f(\circ\!-\!\circ, z) = z^2 - z$$

$$f(\circ \quad \circ\!-\!\circ, z) = z^3 - z^2$$

$$\begin{aligned} f(\circ\!-\!\circ\!-\!\circ, z) &= (z^3 - z^2) - (z^2 - z) \\ &= z^3 - 2z^2 + z \\ &= z(z^2 - 2z + 1) \\ &\boxed{= z(z-1)^2} \end{aligned}$$

Of course, this construction is exponential in the number of edges.