

Oh, Domino!

How many ways are there to tile a $k \times N$ region with 2×1 tiles or dominos?

Consider a $2 \times N$ region first.

$N=0$ 1 way, no dominos \emptyset

$N=1$ 1 way, \square

$N=2$ 2 ways, $\square\square, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$

$N=3$ 3 ways, $\square\square\square, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ \hline \end{array}$

We gain considerable insight by thinking in terms of a generating function for all tilings:

$$T = \emptyset + \square + \square\square + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \square\square\square + \begin{array}{|c|c|} \hline \square & \begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \begin{array}{|c|} \hline \square \\ \hline \end{array} & \square \\ \hline \end{array} + \dots$$

If we think of this as a sum, we can manipulate it as a sum - if we are careful.

The two possible prefixes for tilings lets us for a nice & familiar recurrence:

$$\begin{aligned} T &= \square(\emptyset + \square + \square\square + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \dots) + \begin{array}{|c|c|} \hline \begin{array}{|c|} \hline \square \\ \hline \end{array} & (\emptyset + \square + \dots) + \emptyset \\ &= \emptyset + \square T + \begin{array}{|c|c|} \hline \begin{array}{|c|} \hline \square \\ \hline \end{array} & T \end{array} \end{aligned}$$

Fibonacci recurrence on length!

Rewriting our generating function...

$$T = \frac{b}{1 - (a + c)}$$

which may be expanded like any other geometric series:

$$T = b + b(a + c) + b(a + c)^2$$

$$= b + (a + c) + (aa + ac + ca + cc) + \dots$$

which permutes the order of the tilings.

The k^{th} term $(a + c)^k$ describes all tilings of k basic blocks!

This generating function describes each distinct tiling. To get a generating function which counts the tilings, we can relax constraints so the tilings representing the same count get matched together.

For example, suppose we want to consider the number of distinct tilings from k \square + l \square tiles:

$$T' = 1 + \square + \square^2 + \square^2 + \square^3 + 2\square\square^2 \dots$$

The exponents describe the multiplicity of the type of domino and the coefficient the number of legal tilings on those type of dominoes $2\square\square^2 \Rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$

Since each tiling consists of k blocks of either \square or $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$, geometric series

$$T' = 1 + (\square + \square^2) + (\square + \square^2)^2 + (\square + \square^2)^3 \dots = \frac{1}{1 - (\square + \square^2)}$$

$$\frac{1}{1 - (\square + \square^2)} = \sum_{k \geq 0} (\square + \square^2)^k$$

$$= \sum_{j, k \geq 0} \binom{k}{j} \square^j \square^{2k-2j}$$

} binomial theorem

$$= \sum_{j, m} \binom{j+m}{j} \square^j \square^{2m}$$

} variable substitution

coefficient

Thus there are $\binom{j+m}{j}$ tiling on $2m\square + j \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$, which makes sense in terms of picking the j positions where \square goes.

The problem of counting the total number of tilings can be solved by setting z^k to be the placeholder for a tiling of $2 \times k$:


$$T = \frac{1}{1 - (0 + z^2)} \Rightarrow \frac{1}{1 - (z + z^2)}$$

which is the same generating function we saw for the Fibonacci numbers, less a shift of z .

What about $3 \times N$ tilings?

$N=0$ 1 tiling, no tiles

$N=2k+1$ 0 tilings - covering an odd area with even areas

$N=2$ 3 tilings - 

$N=4$ gets complicated \rightarrow fractured tilings!



Thus there are 3 possible prefixes:



This is uglier than before.

Thus if: $U = \text{tile} + \text{tile} + \text{tile} \dots$ is all $3 \times N$ tilings, $N \rightarrow \infty$,

$$U = \text{tile} + \text{tile} V + \text{tile} \Lambda + \text{tile} U,$$

where $V = \text{tile} + \text{tile} + \text{tile} + \dots$
 $\Lambda = \text{tile} + \text{tile} + \text{tile} + \dots$

These series for $V + \Lambda$ have a nice recursive construction:

$$V = \text{tile} U + \text{tile} V, \quad \Lambda = \text{tile} U + \text{tile} \Lambda$$

Solving for V, Λ and inserting into U

we get:

$$U = \text{tile} + \text{tile} \cdot \left(\frac{1}{1 - \text{tile}} \right) U + \text{tile} \left(\frac{1}{1 - \text{tile}} \right) U + \text{tile} U$$

and solving for U :

$$U = \frac{\text{tile}}{\text{tile} - \text{tile} (1 - \text{tile}) - \text{tile} (1 - \text{tile}) - \text{tile}}$$

$$V = (1 - \Theta)^{-1} \circ U, \quad \Lambda = (1 - \Theta)^{-1} \circ U;$$

$$U = 1 + \beta(1 - \Theta)^{-1} \circ U + \rho(1 - \Theta)^{-1} \circ U + \Theta U.$$

And the final equation can be solved for U, giving the compact formula

$$U = \frac{1}{1 - \beta(1 - \Theta)^{-1} \circ - \rho(1 - \Theta)^{-1} \circ - \Theta}. \quad (7.8)$$

This expression defines the infinite sum U, just as (7.4) defines T.

The next step is to go commutative. Everything simplifies beautifully when we detach all the dominoes and use only powers of \square and \circ :

$$U = \frac{1}{1 - \square^2 \circ (1 - \circ^3)^{-1} - \square^2 \circ (1 - \circ^3)^{-1} - \circ^3}$$

$$= \frac{1 - \circ^3}{(1 - \circ^3)^2 - 2\square^2 \circ}$$

$$= \frac{(1 - \circ^3)^{-1}}{1 - 2\square^2 \circ (1 - \circ^3)^{-2}}$$

$$= \frac{1}{1 - \circ^3} + \frac{2\square^2 \circ}{(1 - \circ^3)^3} + \frac{4\square^4 \circ^2}{(1 - \circ^3)^5} + \frac{8\square^6 \circ^3}{(1 - \circ^3)^7} + \dots$$

$$= \sum_{k \geq 0} \frac{2^k \square^{2k} \circ^k}{(1 - \circ^3)^{2k+1}}$$

$$= \sum_{k, m \geq 0} \binom{m + 2k}{m} 2^k \square^{2k} \circ^{k+3m}.$$

Multiply by $(1 - \square^2)$ to get rid of bottom:

Multiply by $\frac{1}{(1 - \circ^3)^2}$ to get sum of series:

Expand bottom, multiply by top.

Invoke Binomial Theorem, but not obvious

$$\frac{1}{(1 - z)^{n+1}} = \sum_{k \geq 0} \binom{n+k}{n} z^k$$

Nice move - at some point we must expand

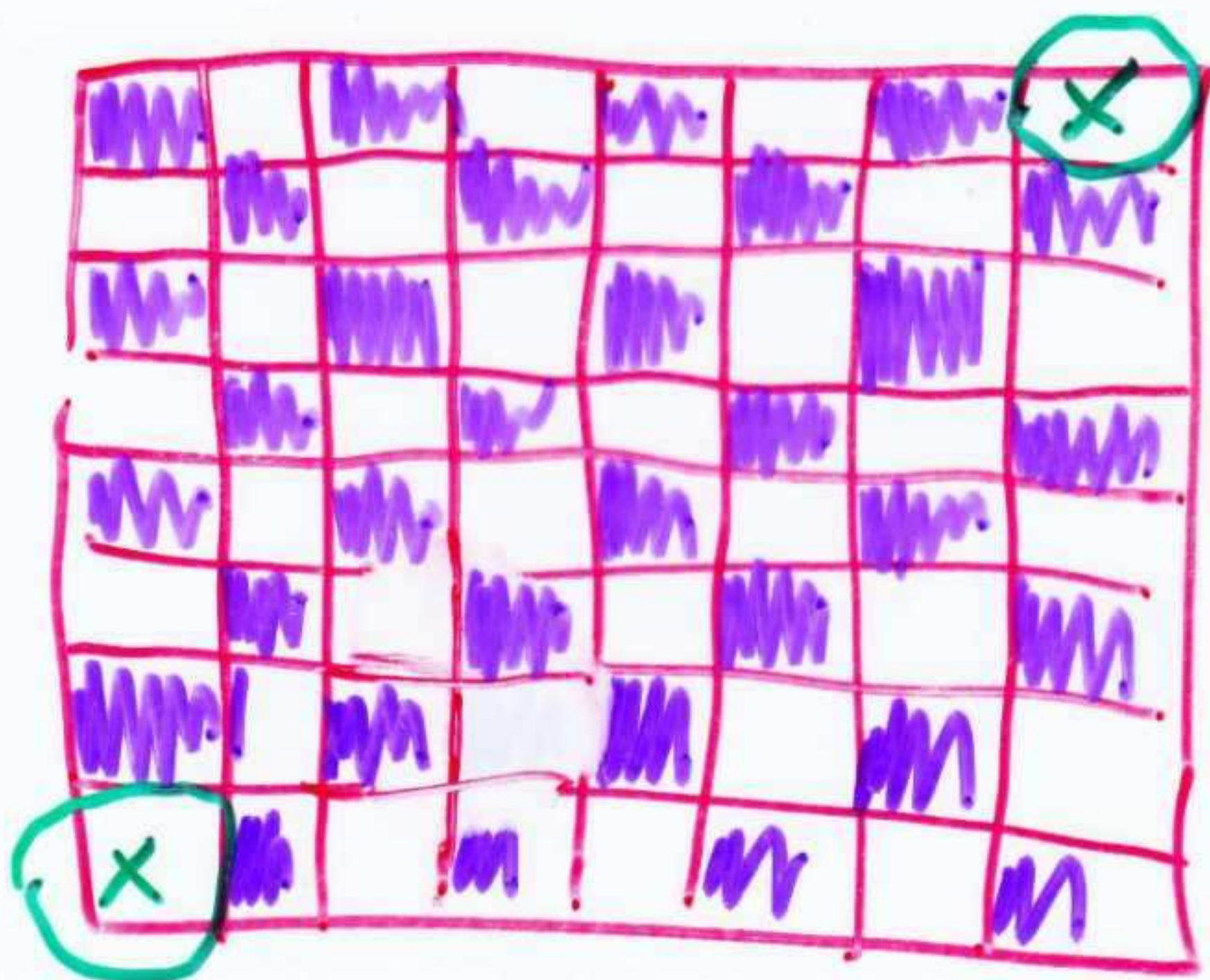
Thus the number of ways to build a $3 \times N$ tiling from a fixed number of $\square + \square$ is given by the coefficients:

$$\sum_{k, n \geq 0} \binom{2k+n}{n} 2^k \square^{2k} \square^{k+3n}$$

To get the generating function for the total number of $3 \times N$ tilings, once again take the generating function for \square and replace $\square^3 \rightarrow z^3$ and $\square \rightarrow z$.

The number of tilings of $3 \times N$ will be the # of tilings on $3N/2$ dominoes.

Classic Quiz question: How many ways are there to cover a chessboard without corners by dominoes:



Answer: 0 \rightarrow more purple squares than white ones, yet each domino covers one of each!