

Generating Functions

Generating functions provide an alternate representation of an infinite sequence as a power series.

$$\langle a_0, a_1, a_2, \dots \rangle$$

$$A(z) = a_0 + a_1 z + a_2 z^2 + \dots = \sum_{k \geq 0} a_k z^k$$

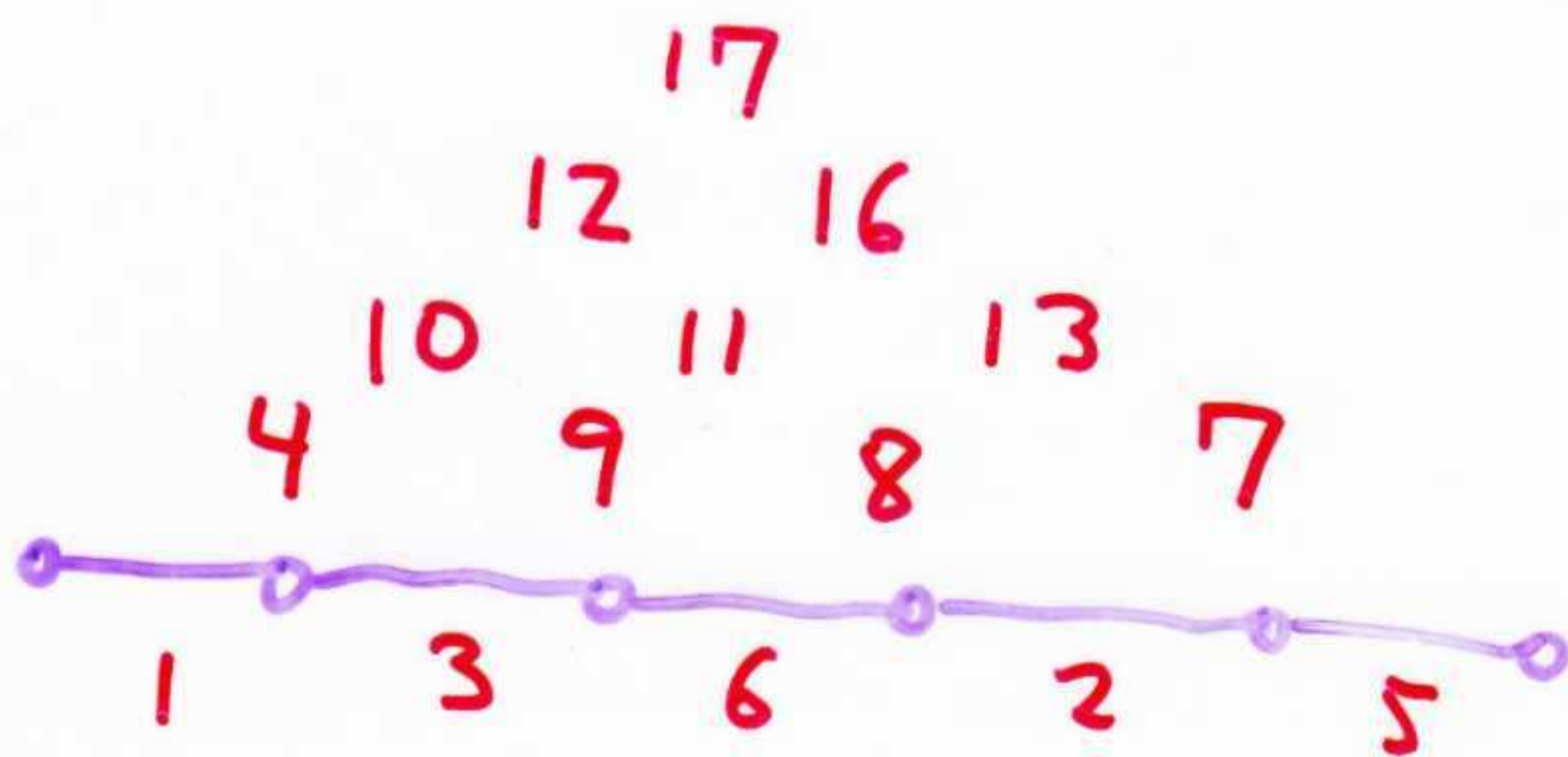
It is vitally important to see that generating functions are simply an alternate notation for representing a sequence.

It is also vitally important to see that z is just a placeholder, which permits us to manipulate our sequence like a polynomial.

In general, we do not care what z is and will not solve for it!!

Reconstructing Sets from Interpoint Distances

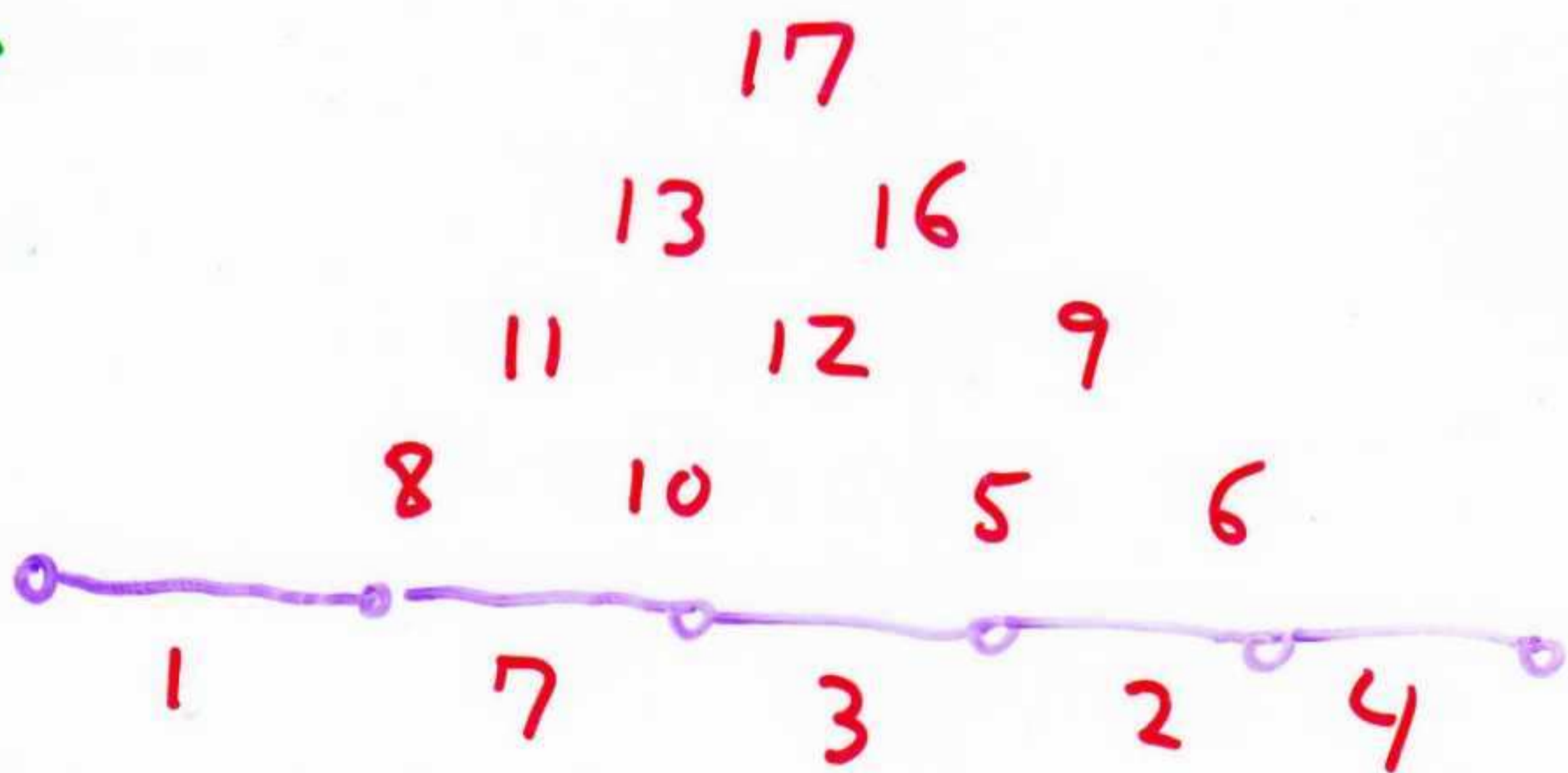
Given n points on a line, they define $\binom{n}{2}$ interpoint distances between them:



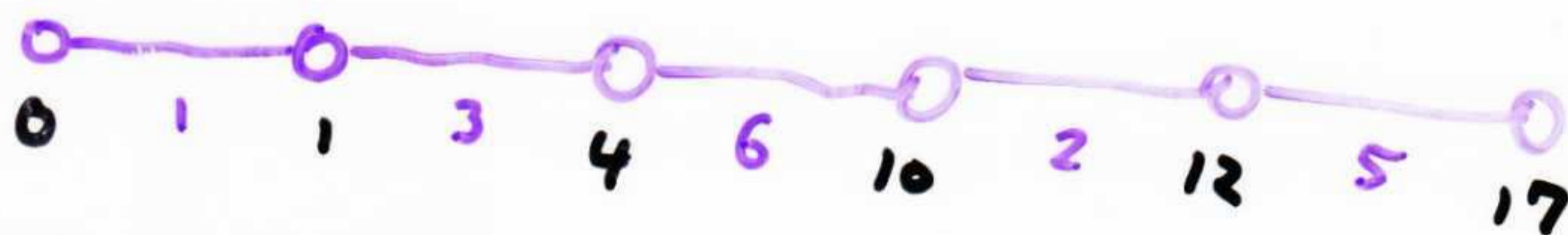
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17

Now, given $\binom{n}{2}$ distances, can you find a set of points which generate them?

The answer may not be unique:



It is useful to represent each point set on a line by a polynomial. If the leftmost point is set at $x=0$, the base distances are defined by point positions:



The exponents of each term of our polynomial will represent the position of one point:

$$P(z) = 1 + z^1 + z^4 + z^{10} + z^{12} + z^{17}$$

From this, let's compute the polynomial $P(z) \cdot P(1/z)$

$$Q(z) = P(z)P(1/z) = (1 + z^1 + z^4 + z^{10} + z^{12} + z^{17}) * (1 + z^{-1} + z^{-4} + z^{-10} + z^{-12} + z^{-17})$$

$$= z^{17} + z^{16} + z^{13} + z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 6 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7} + z^{-8} + z^{-9} + z^{-10} + z^{-11} + z^{-12} + z^{-13} + z^{-16} + z^{-17}$$

The exponents of this polynomial are the pairwise distances and the coefficients are the distance multiplicities!!!

Why does this work? Multiplying an N -term polynomial by an M -term one implicitly enumerates all NM pairs. Combining terms with the same exponent groups pairs which share some property.

But what good does this do us?
After all, a generating function is just an **alternate representation**

Well, given the $\binom{N}{2}$ distances we can construct the polynomial $Q(z)$ and factor it:

$$\begin{aligned} Q(z) &= (z^6 + z + 1)(z^{-6} + z^{-1} + 1) * \\ &\quad (z^{11} - z^5 + z^4 + 1)(z^{-11} - z^{-5} + z^{-4} + 1) \\ &= P_1(z) P_1(1/z) P_2(z) P_2(1/z) \end{aligned}$$

All point sets are defined by factorizations $P(z)P(1/z)$, which are

$$P_1(z) P_2(z) = 1 + z + z^4 + z^{10} + z^{12} + z^{17}$$

$$P_1(z) P_2(1/z) \cdot z^{11} = 1 + z + z^8 + z^{11} + z^{13} + z^{17}$$

Thus finding an appropriate point set is equivalent to factoring polynomials, we can use fast factoring algorithms to find such a point set in time polynomial in the size of the largest exponent.

Further, we can use results on the maximum number of factors to determine bounds on the number of distinct point sets realizing the same distances!

Our alternate representation, as generating functions, gave us access to new tools, which let us solve our problem!

Manipulating Generating Functions

Since they are polynomials, we can treat them like they are:

$$A(z) = a_0 + a_1 z + a_2 z^2 \dots$$

$$B(z) = b_0 + b_1 z + b_2 z^2 \dots$$

The product of two polynomials is another polynomial,

So:

$$A(z)B(z) = a_0 b_0 + (a_0 b_1 + b_0 a_1)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 \dots$$

$$C(z) = \sum_{N \geq 0} z^N \left(\sum_{k=0}^N a_k b_{N-k} \right)$$

Thus we get a new sequence whose N^{th} term is the convolution of the first N terms of the sequences A + B .

Thus multiplying two generating functions is equivalent to convolving two sequences.

The N^{th} term of C is simply the coefficient of z^N .

Generating Vandermonde's Convolution

Last class, we proved $\binom{r+s}{n} = \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k}$ using a combinatorial argument. Since it is a convolution, we can do the job neatly with generating functions.

By the binomial theorem

$$(x+y)^r = \sum_{k=0}^r \binom{r}{k} x^k y^{r-k}$$

Setting $x=z$, $y=1$ gives

$$(1+z)^r = \sum_{k=0}^r \binom{r}{k} z^k$$

Multiplying two such series together gives us

$$(1+z)^r (1+z)^s = (1+z)^{r+s} = \sum_{k=0}^{r+s} \binom{r+s}{k} z^k$$

This gives us a generating function whose coefficients are the left side of our identity

But what is the expansion of the left side?

$$(1+z)^r (1+z)^s = \sum_{n \geq 0} z^n \left(\sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} \right)$$

Why? Because the product of two generating functions gives a generating function whose n th term is the convolution of the first n terms of the sequences

Since

$$\sum_{n \geq 0} z^n \binom{r+s}{n} = \sum_{n \geq 0} z^n \left(\sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} \right)$$

for each value of n ,

$$\binom{r+s}{n} = \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k}$$

Shifty - Looking Moves

What happens when we take a generating function and multiply it by z^N ?

$$\{a_0, a_1, a_2, \dots\} \Leftrightarrow f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

$$z^N f(z) = a_0 z^N + a_1 z^{N+1} + a_2 z^{N+2} + \dots$$

$$\Rightarrow \underbrace{\{0, 0, 0, \dots\}}_N, a_0, a_1, a_2, \dots$$

By just reading off the coefficients, it is clear multiplying by z^N is a right-shift N places.

What about multiplying by z^{-N} ?

$$\begin{aligned} z^{-N} (a_0 z^N + a_1 z^{N+1} + a_2 z^{N+2} + \dots) \\ = a_0 + a_1 z + a_2 z^2 + \dots \Rightarrow \{a_0, a_1, a_2, \dots\} \end{aligned}$$

Multiplying by z^{-N} is a left-shift N places.

Geometric Series

What is the generating function for an infinite sequence of ones: $\{1, 1, 1, 1, 1, \dots\}$

$$f(z) = 1 + z^1 + z^2 + \dots = \sum_{k \geq 0} z^k$$

It is a sum of an infinite geometric series, so

$$\frac{1}{1-z} = \sum_{k \geq 0} z^k$$

if the series converges. But we have been using z as a formal parameter, and we only have to worry about convergence if we evaluate the polynomial!

With generating functions, we can ignore questions of convergence!

Since the generating function:

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

yields the sequence $\langle 1, 1, 1, 1, \dots \rangle$, and convolving any sequence with it yields the prefixes of the sums,

$$A(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

$$\frac{A(z)}{1-z} = a_0 + (a_0 + a_1)z + (a_0 + a_1 + a_2)z^2 \dots$$

$$= \sum_{N \geq 0} z^N \left(\sum_{k=0}^N a_k \right)$$

Multiplying a generating function by $\frac{1}{1-z}$ gives the generating function of the summation

$$\text{Ex: } A(z) = \sum_{k \geq 0} z^k = \frac{1}{1-z}$$

$$\frac{A(z)}{1-z} = \frac{1}{(1-z)^2} = \sum_{N \geq 0} z^N \left(\sum_{k=0}^N 1 \right) = \sum_{N \geq 0} (N+1)z^N$$

Thus $\frac{1}{(1-z)^2}$ is the generating function of $\langle 1, 2, 3, 4, \dots \rangle$

Infinite Series is where we get the
CON from CONCRETE MATH!

Remember Taylor Series? They were power series
expansions for approximating functions

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \dots \quad -1 < z \leq 1$$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} \dots$$

Some of the generating functions we will
encounter will make use of such expansions.

This was intended to be a quick
introduction to generating functions. You
should get comfortable with the notation
and idea. We will take a more detailed
look later.

Hypergeometric Functions

The general hypergeometric function F is:

$$F\left(\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix} \middle| z\right) = \sum_{k \geq 0} \frac{a_1^{\bar{k}} \dots a_m^{\bar{k}}}{b_1^{\bar{k}} \dots b_n^{\bar{k}}} \frac{z^k}{k!}$$

Note that each term is a rising factorial.

$$a^{\bar{k}} = a(a+1)\dots(a+k-1)$$

Each term belongs in the numerator or the denominator!

Interesting special cases include:

$$F\left(\begin{matrix} 1 \\ 1 \end{matrix} \middle| z\right) = \sum \frac{z^k}{k!} = e^z$$

$$F\left(\begin{matrix} 1 \\ 1 \end{matrix} \middle| z\right) = \sum \frac{k! k!}{k! k!} z^k = 1 + z + z^2 + \dots = \frac{1}{1-z}$$

$$F\left(\begin{matrix} 1 \\ 2 \end{matrix} \middle| -z\right) = \sum \frac{k! k! (-z)^k}{(k+1)! k!} = \sum \frac{(-1)^k}{k+1} z^k$$

must be shifted by 2.

$$= \frac{\ln(1+z)}{z}$$

Thus hypergeometric functions cover a lot of ground. In fact, any sequence whose first term is 1 and where the ratio of successive terms is a rational function of k is hypergeometric:

$$F\left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_n \end{matrix} \middle| z\right) = \sum_{k \geq 0} t_k$$

$$\begin{aligned} \frac{t_{k+1}}{t_k} &= \frac{a_1^{k+1} \dots a_n^{k+1}}{a_1^k \dots a_n^k} \cdot \frac{b_1^k \dots b_n^k}{b_1^{k+1} \dots b_n^{k+1}} \cdot \frac{k! \cdot z^{k+1}}{(k+1)! \cdot z^k} \\ &= \frac{(k+a_1) \dots (k+a_n) z}{(k+b_1) \dots (k+b_n)(k+1)} \end{aligned}$$

Thus we can convert many functions to hypergeometric representation.

The Moral: if we can work with hypergeometries, and we can convert our function to a hypergeometric, we are in business.

There has been a tremendous amount of work on manipulating hypergeometrics, which we will not discuss, but which may be of interest if you like systematic, general approaches to summations.

Gosper's algorithm for indefinite summation works by translating your function into a hypergeometric and using properties of hypergeometrics to simplify it. Thus it can be said to give you the answer if one exists in terms of hypergeometrics.

This was a quick introduction to gamma functions. You should get comfortable with the notation and the idea - we will take another look at them later.