Combinatorics

There are a few basic counting problems which everyone should know how to solve!

How many ways are there to pick k things out of N <u>with replacement</u> when <u>order matters</u>

$$\underbrace{N \cdot N \cdots N}_{k} = N^k$$

How many ways are there to pick k things out of N <u>without replacement</u> when <u>order matters</u>

$$N(N-1)\cdots(N-k+1) = N!/(N-k)!$$

How many ways are there to pick k things out of N <u>without replacement</u> when <u>order doesn't matter</u>

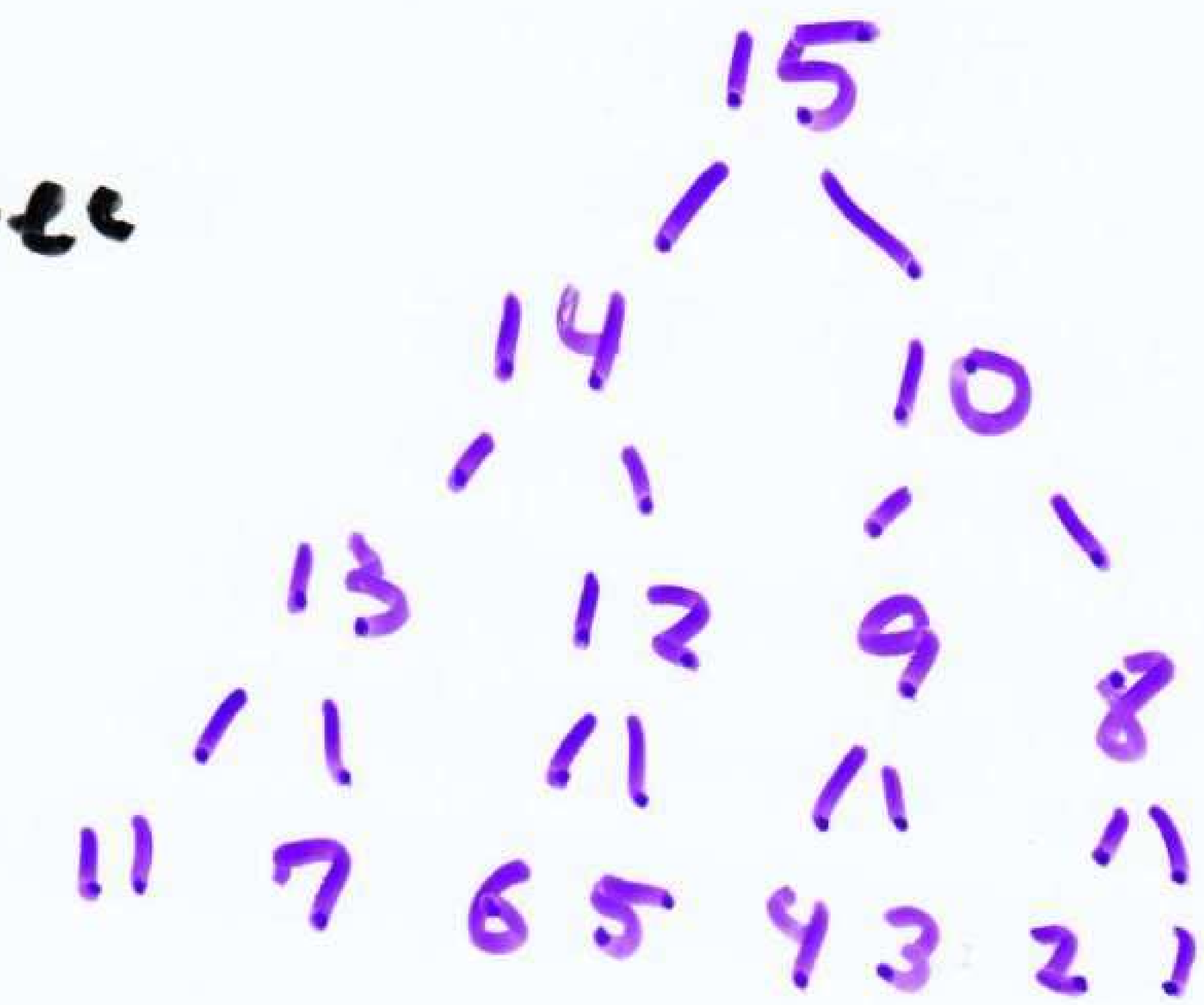
$$\frac{N!/k!}{(N-k)!} = \frac{N!}{k!(N-k)!} = \binom{N}{k}$$

$\boxed{\text{the binomial coefficients}}$

Binomial Coefficients are useful because they occur in a variety of combinatorial problems.

Ex: How many complete binary heaps can be formed with $2^{\ell}-1$ elements?

A heap is a binary tree ~~with~~ where each node is greater than its children.



$$S[1] = 1$$

$$S[\ell] = \binom{2^{\ell}-2}{2^{\ell-1}-1} S[\ell-1]^2$$

The position of the largest element is determined, pick the elements which go on the left and then make heaps of the children.

Binomial Coefficients are fun because they have lots of structure + hence identities.

- Gould gives over 500 different identities
- Wilf has a program which automatically proves millions of binomial coefficient identities.

If you are brave enough, you can simplify ugly looking sums fairly easily.

Pascal's Triangle

Most of the interesting identities with binomial coefficents appear somewhere in Pascal's triangle

$\binom{N}{0}$ $\binom{N}{N}$ $\binom{N}{N-1}$

$\binom{N}{1}$

$\binom{N}{2}$ $\binom{N}{N-2}$

$\binom{N}{3}$ $\binom{N}{N-3}$

1 → 1

1 1 → 2

1 2 1 → 4

1 3 3 1 → 8

1 4 6 4 1 → 16

1 5 10 10 5 1 → 32

1 6 15 20 15 6 1 → 64

1 7 21 35 35 21 7 1 → 128

1 8 28 56 70 56 28 8 1 → 256

1 9 36 84 126 126 84 36 9 1 → 512

Sucker bet: Let's flip 4 coins - will they be two-heads / two tails or not?

Every entry in Pascals triangle is a binomial coefficent and found by adding the two numbers above it.

For this construction to work, we must show: $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$ "addition formula"

With binomial coefficients there are three kinds of proofs:

1. Gory but straightforward:

$$\binom{r-1}{k} + \binom{r-1}{k-1} = \frac{(r-1)!}{k!(r-1-k)!} + \frac{(r-1)!}{(k-1)!(r-k)!}$$

$$= \frac{(r-1)\cdots(r-k)}{k!} + \frac{(r-1)\cdots(r-k-1)}{(k-1)!} =$$

$$= \frac{(r-1)\cdots(r-k-1)}{(k-1)!}\left[\frac{r-k}{k} + \frac{k}{k}\right] = \frac{r\cdots(r-k-1)}{k!} = \binom{r}{k}$$

The mess of manipulating factorials & the desire to generalize to real & negative numbers give the definition in terms of falling factorials:

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1} = \dfrac{r^{\underline{k}}}{k!} & \text{integer } k \geq 0 \\ 0 & \text{integer } k < 0 \end{cases}$$

2. Combinatorial Interpretation

How many ways are there to pick k things out of r? Consider the ugliest looking of the r things. Either we pick it or (hopefully - if it is REAL ugly) we don't.

If we do, there are $\binom{r-1}{k-1}$ ways to pick the rest.

If we don't, we must pick k of $r-1$ things left, so $\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}$

Combinatorial Proofs are <u>best</u>, because they provide insight, but they require inspiration

3. Use Binomial Identities

$$r\binom{r}{k} = (r-k+k)\binom{r}{k} = (r-k)\binom{r}{k} + k\binom{r}{k}$$

using the identity $\boxed{\binom{N}{k} = \binom{N}{N-k}}$

$$= (r-k)\binom{r}{r-k} + k\binom{r}{k}$$

using the identity $\boxed{k\binom{r}{k} = r\binom{r-1}{k-1}}$

$$= r\binom{r-1}{r-k-1} + r\binom{r-1}{k-1}$$

$$= r\binom{r-1}{k} + r\binom{r-1}{k-1}$$

Dividing both sides by r gives the result.

This kind of proof is usually the easiest way to go.

Mining Pascal's Triangle For Identities

$$\binom{N}{k} = \binom{N}{N-k}$$ integer $N \geq 0$, integer k "symmetry identity"

$\binom{N}{0}$ $\binom{N}{1}$ $\binom{N}{2}$ $\binom{N}{3}$ $\binom{N}{N}$ $\binom{N}{N-1}$ $\binom{N}{N-2}$ $\binom{N}{N-3}$

1	→ 1	0^{th}
1 1	→ 2	1^{st}
1 2 1	→ 4	2^{nd}
1 3 3 1	→ 8	3
1 4 6 4 1	→ 16	4
1 5 10 10 5 1	→ 32	5
1 6 15 20 15 6 1	→ 64	6
1 7 21 35 35 21 7 1	→ 128	7
1 8 28 56 70 56 28 8 1	→ 256	8
1 9 36 84 126 126 84 36 9 1	→ 512	9

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$$ integer $k \neq 0$ "absorbsion identity"

$$56 = \frac{8}{3} \cdot 21$$

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N$$ integer $N \geq 0$ special case of the binomial theorem

Two More identities

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k} \quad \text{integers } m, k$$

<u>proof:</u> $\binom{r}{m}\binom{m}{k} = \frac{r!}{m!(r-m)!} \cdot \frac{m!}{k!(m-k)!}$

$$= \frac{r!}{k!\,(m-k)!\,(r-m)!} \cdot \frac{(r-k)!}{(r-k)!}$$

$$= \frac{r!}{k!(r-k)!}\,\frac{(r-k)!}{(m-k)!\,(r-m)!} = \binom{r}{k}\binom{r-k}{m-k}$$

$\blacksquare$

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$$

<u>proof:</u> $\frac{r}{k}\binom{r-1}{k-1} = \frac{r}{k} \cdot \frac{(r-1)!}{(r-k)!(k-1)!}$

$$= \frac{r!}{(r-k)!\,k!}$$

$\blacksquare$

Summation Identities

$$\sum_{k \le n} \binom{r+k}{k} = \binom{r}{0} + \binom{r+1}{1} \cdots + \binom{r+n}{n} = \binom{r+n+1}{n}$$

$$\binom{7}{3} = \binom{6}{3} + \binom{5}{2} + \binom{4}{1} + \binom{3}{0}$$ (in green)

$\binom{n}{0}$ $\binom{n}{n}$ $\binom{n}{n-1}$ $\binom{n}{1}$ $\binom{n}{2}$ $\binom{n}{n-2}$ $\binom{n}{3}$ $\binom{n}{n-3}$

1 → 1
1 1 → 2
1 2 1 → 4
1 3 3 1 → 8
1 4 6 4 1 → 16
1 5 10 10 5 1 → 32
1 6 15 20 15 6 1 → 64
1 7 21 35 35 21 7 1 → 128
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1 9 36 84 126 126 84 36 9 1 → 512

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \cdots \binom{n}{m} = \binom{n+1}{m+1}$$

$$\binom{7}{3} = \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$

(in red)

Of course, these identities must be proven before we really believe them, but that is easy!

$$\sum_{0 \le k \le N} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \cdots \binom{N}{m} = \binom{N+1}{m+1}$$

<u>Combinatorial proof</u>: Suppose the $N+1$ things are labeled 1 to $N+1$. If the largest of the $m+1$ we pick is $k+1$, there are $\binom{k}{m}$ ways to finish the job.

$$\sum_{k \le N} \binom{r+k}{k} = \binom{r}{0} + \binom{r+1}{1} + \cdots \binom{r+N}{N} = \binom{r+N+1}{N}$$

<u>Identity flavored proof!</u>

$$\binom{r+N+1}{N} = \binom{r+N}{N} + \binom{r+N}{N-1}$$

$$= \binom{r+N}{N} + \binom{r+N-1}{N-1} + \binom{r+N-1}{N-2}$$

$$= \binom{r+N}{N} + \binom{r+N-1}{N-1} \cdots \binom{r}{0}$$

$$= \sum_{k \le N} \binom{r+k}{k}$$

The Binomial Theorem

$$(x+y)^0 = 1x^0y^0$$

$$(x+y)^1 = 1x^1y^0 + 1x^0y^1$$

$$(x+y)^2 = 1x^2y^0 + 2xy + 1x^0y^2$$

$$\vdots$$

$$(x+y)^r = \sum_k \binom{r}{k} x^k y^{r-k}$$

integer $r \geq 0$

or $\underbrace{|x/y| < 1}_{\text{necessary when we sum to } \infty \text{ + not } r.}$

The combinatorial interpretation follows from the factorization:

$$(x+y)^r = \underbrace{(x+y)(x+y)\cdots(x+y)}_{r \text{ factors}}$$

each term $x^k y^{r-k}$ is a yes or no for each of the r factors.

The number of ways of choosing k things of r is $\binom{r}{k}$.

Products of Binomial Coefficents

Even these have nice identities associated with them:

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

integer n

Vandermonde's convolution

$\sum_{k=0}^{n} X_k X_{n-k}$ is a convolution

Combinatorial Proof:

How many ways are there to pick n people from r men & s women?

no men: $\binom{s}{n}\binom{r}{0}$

one man: $\binom{s}{n-1}\binom{r}{1}$

$\vdots$

k men: $+ \binom{s}{n-k}\binom{r}{k}$

$$\overline{}$$

$\binom{r+s}{n}$

Lot's of other identities come from this by simple tweaking

Formality and Combinatorial Proofs

What makes a proof "correct" is the validity of its arguments, not whether it is expressed in English or Mathematical notation:

Suppose $a=b$, then

$$ab = a^2$$

$$ab - b^2 = a^2 - b^2$$

$$b(a-b) = (a+b)(a-b)$$

$$b = a+b$$

$$b = 2b$$

$$\therefore 1 = 2$$

QED (?)

Middle Binomial Coefficients

An especially interesting set of binomial coefficients are in the middle of the even rows in Pascal's triangle, $\binom{2N}{N}$

$$\binom{2N}{N} = \binom{N+N}{N} = \sum_k \binom{N}{N-k}\binom{N}{k} \Big\} \text{Vandermonde's Convolution}$$

$$= \sum_k \binom{N}{k}^2$$

$$\binom{8}{4} = \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$$

$$70 = 1^2 + 4^2 + 6^2 + 4^2 + 1^2$$

That middle binomial coefficients are a good thing should be obvious from the fact that only one parameter is necessary to define them!

Problem 5.97

$$\binom{2N}{N} \equiv (-1)^N \bmod (2N+1)$$ if $2N+1$ is prime

Proof: $(2N)(2N-1)\cdots(N+1) \equiv (-1)(-2)\cdots(-N) \bmod (2N+1)$

$$\equiv (-1)^N (N!) \bmod (2N+1)$$

if $2N+1$ is prime, each integer $1 \ldots N$ is relatively prime to $2N+1$.

Thus we can divide both sides by $N!$

$$\frac{(2N)(2N-1)\cdots(N+1)}{N(N-1)\cdots 1} \equiv \binom{2N}{N} \equiv (-1)^N (\bmod\ 2N+1)$$

The case of $2N+1$ being composite is not so easy - the congruence holds for $N = 2953$ but nothing less than that!

We would expect it to hold but rarely under the roulette wheel interpretation of congruences.

If you are fluent with these identities, you will do Ok with Binomial Coefficents:

$\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$,	integers $n \geqslant k \geqslant 0$.	*factorial expansion*
$\binom{n}{k} = \binom{n}{n-k}$,	integer $n \geqslant 0$, integer k.	*symmetry*
$\binom{r}{k} = \dfrac{r}{k}\binom{r-1}{k-1}$,	integer $k \neq 0$.	*absorption/extraction*
$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$,	integer k.	*addition/induction*
$\binom{r}{k} = (-1)^k\binom{k-r-1}{k}$,	integer k.	*upper negation*
$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$,	integers m, k.	*trinomial revision*
$\sum_k \binom{r}{k} x^k y^{r-k} = (x+y)^r$,	integer $r \geqslant 0$, or $\lvert x/y \rvert < 1$.	*binomial theorem*
$\sum_{k \leqslant n} \binom{r+k}{k} = \binom{r+n+1}{n}$,	integer n.	*parallel summation*
$\sum_{0 \leqslant k \leqslant n} \binom{k}{m} = \binom{n+1}{m+1}$,	integers $m, n \geqslant 0$.	*upper summation*
$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,	integer n.	*Vandermonde convolution*