Multiple Sums

When the algorithm we are trying to analyze has nested loops, the time complexity is determined by a multiple summation

The sums get evaluated from inside out:

$$\sum_{1 \le j,k \le 3} a_j b_k = \sum_{j=1}^{3} \sum_{k=1}^{3} a_j b_k = \begin{array}{l}(a_1 b_1 + a_1 b_2 + a_1 b_3) + \\ (a_2 b_1 + a_2 b_3 + a_2 b_3) + \\ (a_3 b_1 + a_3 b_2 + a_3 b_3)\end{array}$$

The key operation we can perform is interchanging the order of summation

$$\sum_{1 \le j,k \le 3} a_j b_k = \sum_j \sum_k a_j b_k \, (1 \le j \le 3)(1 \le k \le 3)$$

$$= \sum_j (1 \le j \le 3)\, a_j \left(\sum_k b_k \, (1 \le k \le 3) \right)$$

$$= \sum_{j=1}^{3} a_j \sum_{k=1}^{3} b_k$$

Interchanging summations is trivial if and only if the indices are independent!

If the two indices are not independent, careful manipulation of the Iversonian notation provides a way to interchange sums:

$$\sum_{j=1}^{N} \sum_{k=j}^{N} a_{j,k} = \sum_{j,k} a_{jk} (1 \le j \le N)(j \le k \le N)$$

Once you are convinced

$$\boxed{(1 \le j \le N)(j \le k \le N) = (1 \le k \le N)(1 \le j \le k)}$$

We can now sum on j first:

$$\sum_{j=1}^{N} \sum_{k=j}^{N} a_{jk} = \sum_{k=1}^{N} \sum_{j=1}^{k} a_{jk}$$

An example: $S = \sum_{1 \le j < k \le N} (a_k - a_j)(b_k - b_j)$

observe: $$S = \underbrace{\sum_{1 \le k < j \le N} (a_j - a_k)(b_j - b_k)}_{\text{simple variable substitution}} = \underbrace{\sum_{1 \le k < j \le N} (a_k - a_j)(b_k - b_j)}_{-1 \cdot -1 = 1\ !}$$

thus $$2S = \sum_{j,k} (a_k - a_j)(b_k - b_j)\Big[(1 \le j < k \le N) + (1 \le k < j \le N)\Big]$$

Manipulating the Iversonian Notation:

$$\boxed{(1 \le j < k \le N) + (1 \le k < j \le N) = (1 \le j,k \le N) - (1 \le j = k \le N)}$$

All values of $i + j$ are included, except the N diagonal terms where $i = j$!

$$2S = \sum_{1 \le j,k \le N} (a_j - a_k)(b_j - b_k) - \cancel{\sum_{1 \le k \le N} (a_k - a_k)(b_k - b_k)}$$

each term is 0

$$= \sum_{1 \le j,k \le N} (a_j b_j + a_k b_k) - \sum_{1 \le j,k \le N} (a_j b_k + a_k b_j)$$

multiply out terms and regroup. Note that both terms in each group are identical!

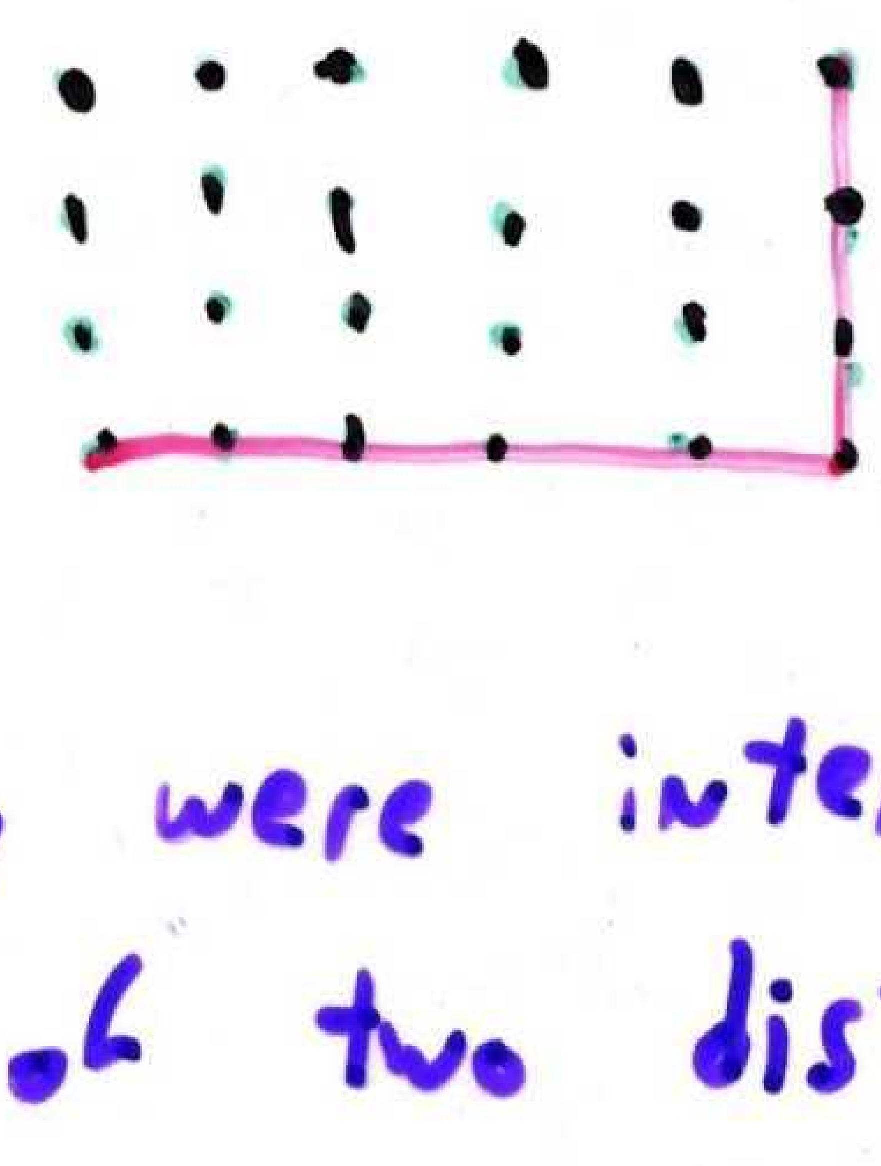
$$= 2 \sum_{1 \le k \le N} (a_k b_k) \sum_{1 \le j \le N} 1 - 2\left(\sum_{k=1}^{N} a_k\right)\left(\sum_{k=1}^{N} b_k\right)$$

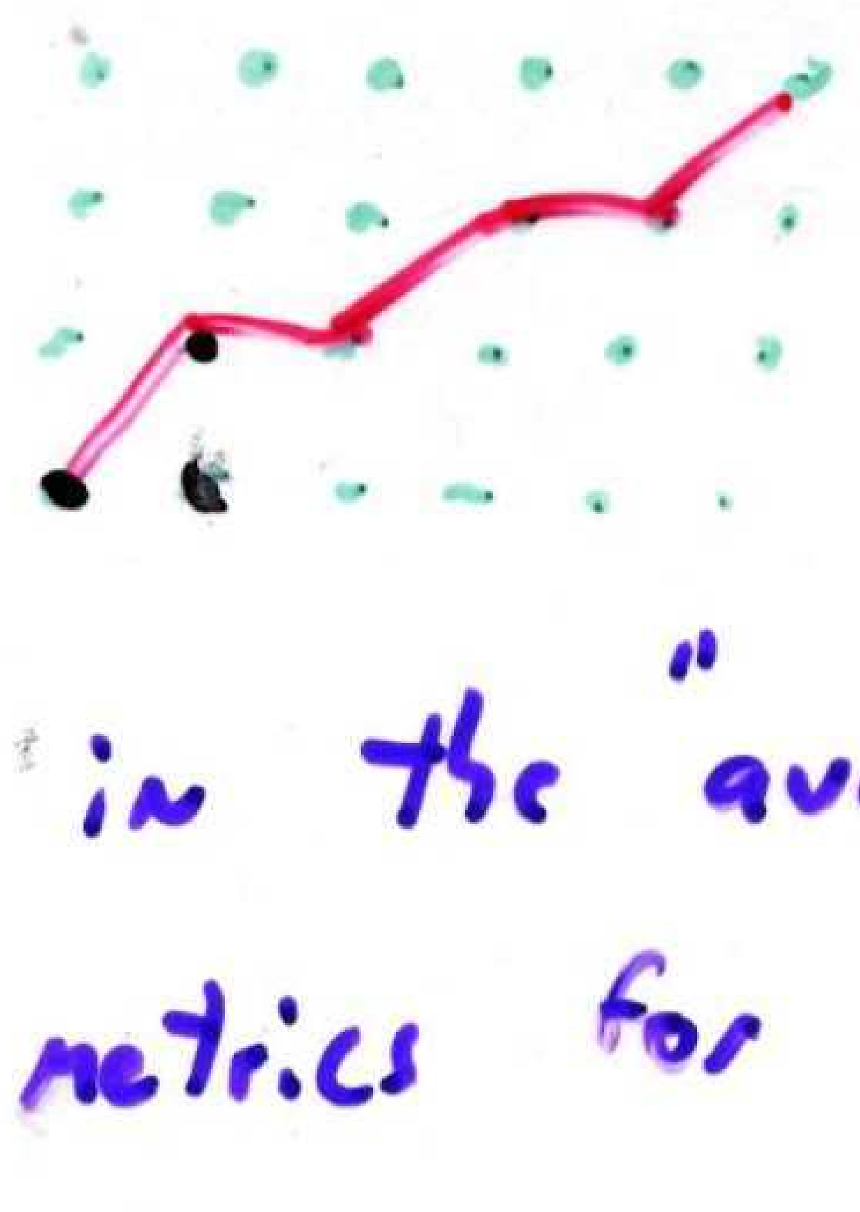
$$\boxed{2S = 2N \sum_{k=1}^{N} a_k b_k - 2\left(\sum_{k=1}^{N} a_k\right)\left(\sum_{k=1}^{N} b_k\right)}$$

Observe that we manipulated the indices so the sums were independent, then did the summation in the order we chose.

Length Ratios for Discrete Distance Metrics

In Prof. Kaufman's voxel graphics, a common operation is drawing a line across a cubic grid. Different line drawing algorithms trade off distance for various topological properties

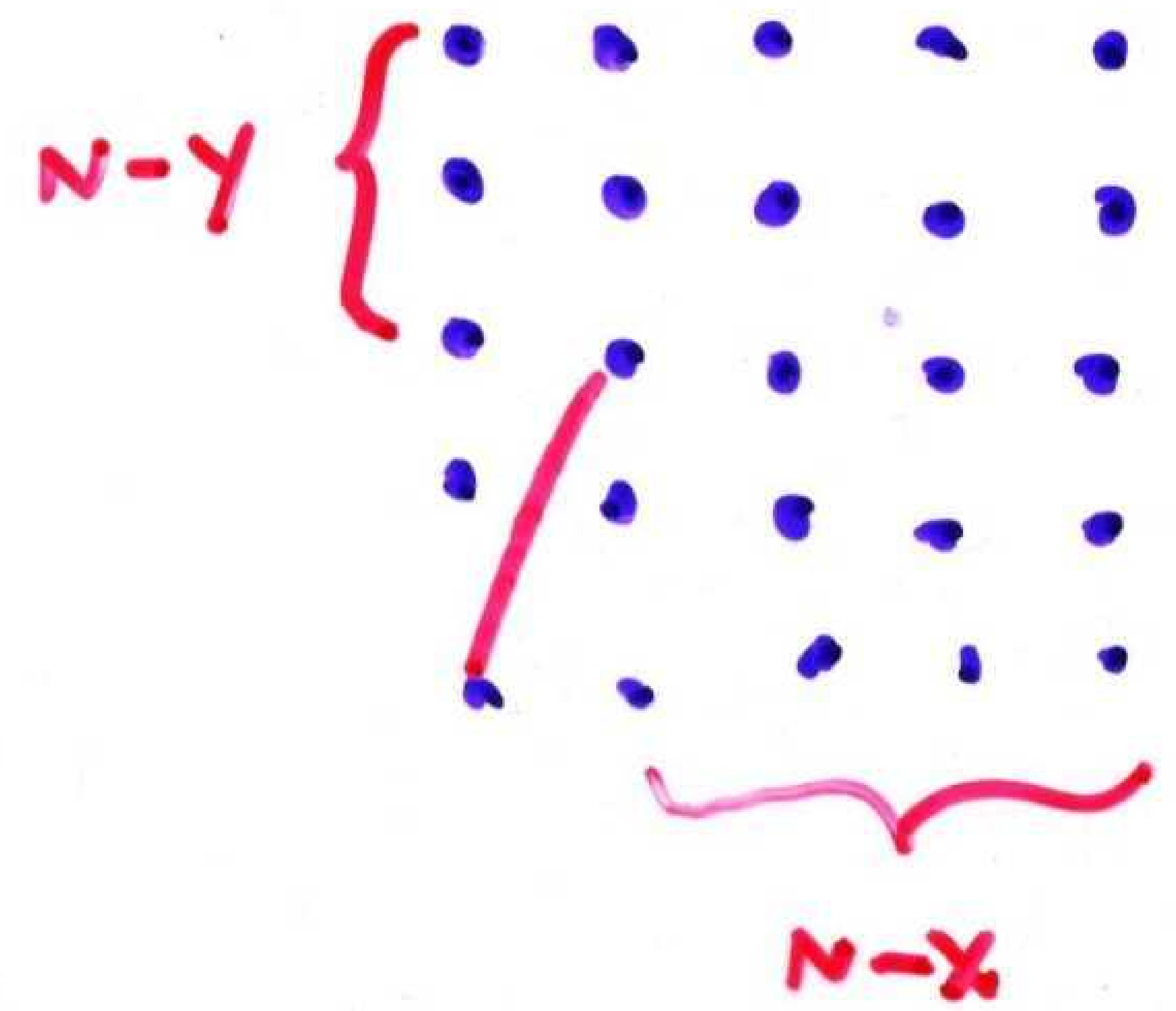




They were interested in the "average" ratio of two distance metrics for random pairs of points in Z^3

$$S = \sum_{p \in Z^3} \sum_{q \in Z^3} \frac{d_A(p,q)}{d_B(p,q)}$$

The expected ratio is $\lim_{N \to \infty} S / N^6$



Any pair of points, by translation + reflection, can have one of the points be the origin.

The number of translations of $\Delta X, \Delta Y$ is $(N-\Delta X)(N-\Delta Y)$ in an $N \times N$ grid

Therefore, we want to sum up over each point in the grid the number of pairs it counts for times its distance ratio.

$$\frac{d_6}{d_{26}} = \frac{\Delta X + \Delta Y + \Delta Z}{\Delta X}$$

can be broken into 3 separate sums

$$\frac{d_{26}}{d_6} = \frac{\Delta X}{\Delta X + \Delta Y + \Delta Z}$$

by substitution, can be left with Harmonic numbers.

For their 1991 CSE 547 project, Sridhar Balakrishnan & Chang Xu worked out the exact values for these limits for the important distance metrics!

The 6-distance $d_6 = |\Delta x| + |\Delta y| + |\Delta z|$

The path must share faces between adjacent cubes, also known as the Manhattan or L1 distance

The 26-distance $d_{26} = Max(\Delta x, \Delta y, \Delta z)$

The path can go through any neighboring cubes - this gives the shortest path.

The 18-distance $d_{18} = Max(d_{26}, \lceil d_6/2 \rceil)$

Neighboring cubes must share an edge.

To handle these sums, we must reduce the problem from 6 dimensions to 3 via translation.

$$
\begin{aligned}
S_{d_i/d_j} &\approx 24 \sum_{\Delta x=1}^{n} \sum_{\Delta y=1}^{\Delta x} \sum_{\Delta z=1}^{\Delta x-\Delta y} (n-\Delta x)(n-\Delta y)(n-\Delta z) L_{(i,j)} \\
&+ 24 \sum_{\Delta x=1}^{n} \sum_{\Delta y=1}^{\Delta x} \sum_{\Delta z=\Delta x-\Delta y}^{\Delta x} (n-\Delta x)(n-\Delta y)(n-\Delta z) M_{(i,j)}
\end{aligned} \tag{1}
$$

$$
L_{(i,j)} = \begin{cases} 1 & \text{if } (i,j) \in ((6,6),(18,18),(18,26),(26,18),(26,26)) \\ \frac{\Delta x+\Delta y+\Delta z}{\Delta x} & \text{if } i \in ((6,18),(6,26)) \\ \frac{\Delta x}{\Delta x+\Delta y+\Delta z} & \text{if } i \in ((18,6),(26,6)) \end{cases}
$$

$$
M_{(i,j)} = \begin{cases} 1 & \text{if } (i,j) \in (6,6),(18,18),(26,26) \\ 2 & \text{if } (i,j) = (6,18) \\ \frac{\Delta x+\Delta y+\Delta z}{\Delta x} & \text{if } (i,j) = (6,26) \\ \frac{1}{2} & \text{if } (i,j) = (18,6) \\ \frac{1}{2}\frac{\Delta x+\Delta y+\Delta z}{\Delta x} & \text{if } (i,j) = (18,26) \\ \frac{\Delta x}{\Delta x+\Delta y+\Delta z} & \text{if } (i,j) = (26,6) \\ 2\frac{\Delta x}{\Delta x+\Delta y+\Delta z} & \text{if } (i,j) = (26,18) \end{cases}
$$

The limiting length ratios for a n^3 grid between 6-lines, 18-lines, and 26-lines are presented in Table 2. In Table 3 we compare the randomly computed Cohen-Kaufman Ratios with the actual analytical results. The results are quite consistant for all ratios.

Although ugly, these sums "obviously" could be solved!

$$
\begin{aligned}
S_{d_4/d_8} &= 8 \sum_{\Delta y=1}^{n-2} \sum_{\Delta x=\Delta y+1}^{n-1} (n-\Delta x)(n-\Delta y)\frac{\Delta x+\Delta y}{\Delta x} + \\
&\quad 8 \sum_{\Delta x=1}^{n-1} (n-\Delta x)^2 + 4 \sum_{\Delta x=1}^{n-1} n(n-\Delta x) \\
&= \frac{(13n^4 - 19n^2 + 6n)}{9}
\end{aligned}
$$

$$
\begin{aligned}
S_{d_8/d_4} &= 8 \sum_{\Delta y=1}^{n-2} \sum_{\Delta x=\Delta y+1}^{n-1} (n-\Delta x)(n-\Delta y)\frac{\Delta x}{\Delta x+\Delta y} + \\
&\quad 2 \sum_{\Delta x=1}^{n-1} (n-\Delta x)^2 + 4 \sum_{\Delta x=1}^{n-1} n(n-\Delta x) \\
&= \frac{-n(16n^3 - 12n^2 + 11n + 12)}{24} - \frac{32n^4 - 24n^2 + 1}{16} H_n + \frac{(4n^2-1)^2}{8} H_{2n} \\
&\approx (2\ln 2 - \frac{2}{3})n^4
\end{aligned}
$$

$$
\begin{aligned}
S_{d_8/d_{26}} &= 48 \sum_{\Delta y=1}^{n-3} \sum_{\Delta z=\Delta y+1}^{n-2} \sum_{\Delta x=\Delta z+1}^{n-1} (n-\Delta x)(n-\Delta y)(n-\Delta z)\frac{\Delta x+\Delta y+\Delta z}{\Delta x} + \\
&\quad 24 \sum_{\Delta z=1}^{n-2} \sum_{\Delta x=\Delta z+1}^{n-1} (n-\Delta x)(n-\Delta z)^2 \frac{\Delta x+2\Delta z}{\Delta x} + 24 \sum_{\Delta x=1}^{n-1} (n-\Delta x)^3 + \\
&\quad 24 \sum_{\Delta z=1}^{n-2} \sum_{\Delta x=\Delta z+1}^{n-1} (n-\Delta x)^2(n-\Delta z)\frac{2\Delta x+\Delta z}{\Delta x} + S_{d_4/d_8} \\
&= \frac{84n^6 - 195n^5 + 185n^4 + 15n^3 - 119n^2 + 30n}{45}
\end{aligned}
$$

$$
\begin{aligned}
S_{d_{26}/d_8} &= 48 \sum_{\Delta y=1}^{n-3} \sum_{\Delta z=\Delta y+1}^{n-2} \sum_{\Delta x=\Delta z+1}^{n-1} (n-\Delta x)(n-\Delta y)(n-\Delta z)\frac{\Delta x}{\Delta x+\Delta y+\Delta z} + \\
&\quad 24 \sum_{\Delta z=1}^{n-2} \sum_{\Delta x=\Delta z+1}^{n-1} (n-\Delta x)(n-\Delta z)^2 \frac{\Delta x}{\Delta x+2\Delta z} + \frac{8}{3} \sum_{\Delta x=1}^{n-1} (n-\Delta x)^3 + \\
&\quad 24 \sum_{\Delta z=1}^{n-2} \sum_{\Delta x=\Delta z+1}^{n-1} (n-\Delta x)^2(n-\Delta z)\frac{\Delta x}{2\Delta x+\Delta z} + S_{d_8/d_4} \\
&\approx (\frac{42}{5}\ln 3 - \frac{176}{15}\ln 2 - \frac{8}{15})n^6
\end{aligned}
$$

	d_6	d_{18}	d_{26}
d_6	1	$\frac{44}{25}$	$\frac{28}{15}$
d_{18}	$\frac{521}{180} - \frac{10}{3}\ln 2$	1	$\frac{79}{75}$
d_{26}	$\frac{42}{5}\ln 3 - \frac{176}{15}\ln 2 - \frac{8}{15}$	$\frac{84}{5}\ln 3 - \frac{84}{5}\ln 2 - \frac{527}{90}$	1

Table 2: The average length ratios between discrete distances in 3D

		d_6	d_{18}	d_{26}
d_6	random	1	1.76	1.84
	actual	1	1.76	1.86667
d_{18}	random	0.57	1	1.05
	actual	0.58395	1	1.05333
d_{26}	random	0.54	0.96	1
	actual	0.52608	0.95626	1

Table 3: Random vs Actual Average Length Ratios Between Discrete Distances

ANOTHER EXAMPLE

$$S_n = \sum_{1\leq k\leq n} \sum_{1\leq j<k} \frac{1}{k-j}$$ summing first on j

$$= \sum_{1\leq k\leq n} \sum_{1\leq k-j<k} \frac{1}{j}$$ replacing j by $k-j$

simple substitution

$$= \sum_{1\leq k\leq n} \sum_{0<j\leq k-1} \frac{1}{j}$$ simplifying the bounds on j

$$= \sum_{1\leq k\leq n} H_{k-1}$$ by (2.13), the definition of H_{k-1}

$$= \sum_{1\leq k+1\leq n} H_k$$ replacing k by $k+1$

$$= \sum_{0\leq k<n} H_k .$$ simplifying the bounds on k

Alas! We don't know how to get a sum of harmonic numbers into closed form.

If we try summing first the other way, we get

index change a variation on our previous identity

$$S_n = \sum_{1\leq j\leq n} \sum_{j<k\leq n} \frac{1}{k-j}$$ summing first on k

$$= \sum_{1\leq j\leq n} \sum_{j<k+j\leq n} \frac{1}{k}$$ replacing k by $k+j$

$$= \sum_{1\leq j\leq n} \sum_{0<k\leq n-j} \frac{1}{k}$$ simplifying the bounds on k

do ~~this~~ carefully

$$= \sum_{1\leq j\leq n} H_{n-j}$$ by (2.13), the definition of H_{n-j}

$$= \sum_{1\leq n-j\leq n} H_j$$ replacing j by $n-j$

$$= \sum_{0\leq j<n} H_j .$$ simplifying the bounds on j

But there's *another* way to proceed, if we replace k by $k + j$ deciding to reduce S_n to a sum of sums:

$$
\begin{aligned}
S_n &= \sum_{1\le j<k\le n} \frac{1}{k-j} && \text{recopying the given sum}\\
&= \sum_{1\le j<k+j\le n} \frac{1}{k} && \text{replacing } k \text{ by } k+j\\
&= \sum_{1\le k\le n}\ \sum_{1\le j\le n-k} \frac{1}{k} && \text{summing first on } j \text{ because the expression is independent of } j\\
&= \sum_{1\le k\le n} \frac{n-k}{k} && \text{the sum on } j \text{ is trivial}\\
&= \sum_{1\le k\le n} \frac{n}{k} - \sum_{1\le k\le n} 1 && \text{by the associative law}\\
&= n\Bigl(\sum_{1\le k\le n} \frac{1}{k}\Bigr) - n && \text{by gosh}\\
&= \boxed{nH_n - n.} && \text{by (2.13), the definition of } H_n
\end{aligned}
$$

key step

Aha! We've found S_n. Combining this with the false starts we made gives us a further identity as a bonus:

$$\boxed{\sum_{0\le k<n} H_k = nH_n - n.} \tag{2.36}$$

What $\underline{\text{is}}$ $S_N = \sum_{1 \le j < k+j \le N} 1/k$?

What values can k take on?

$J = 1 \rightarrow 1 < k+1 \le N \rightarrow 1 \le k \le N-1$

$J = 2 \rightarrow 2 < k+2 \le N \rightarrow 1 \le k \le N-2$

$\vdots$

$J = N-1 \rightarrow N-1 < k+N-1 \le N \rightarrow 1 \le k \le 1$

What values can j take on?

$J \ge 1$ ✓ $\qquad 1 \le j \le N-k$

$k+j \le N \rightarrow j \le N-k$

$$\sum_{1 \le k \le N-1} \sum_{1 \le j \le N-k} 1/k = \sum_{1 \le k \le N-1} 1/k \sum_{1 \le j \le N-k} 1$$

$$= \sum_{1 \le k \le N-1} \frac{N-k}{k} = N \sum_{1 \le k \le N-1} 1/k - (N-1)$$

$$= \underline{N H_{N-1} - (N-1)}$$ But $H_{N-1} = H_N - 1/N$

so $S_N = N(H_N - 1/N) - (N-1) = \boxed{N H_N - N}$

Infinite Sums

Infinite sums are a different kind of beast than finite sums, for what does it mean to sum up an infinite number of terms?

Ex: $S = \sum_{k=0}^{\infty} 2^{-k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} \cdots = 2 + S$$

$$\underline{S = 2}$$

Ex: $T = \sum_{k=0}^{\infty} 2^{k} = 1 + 2 + 4 + 8 \cdots$

$$2T = 2 + 4 + 8 \cdots = T - 1$$

$$\underline{T = -1} \text{ ???}$$

The correct definition of what an infinite sum is must grant us the first and weed out the second.

For simplicity, lets talk first about sums of positive numbers.

If there is a bounding constant A such that $\sum_{k \in F} a_k \leq A$ for all finite subsets F of k, then $\sum_{k \in k} a_k$ is the least such A

This definition captures the idea of approaching but not exceeding a limit. There is no such finite constant in the second example, so $T = \infty$.

Things can get confusing when we sum up an infinite number of positive and negative things.

Ex: $\sum_{k \geq 0} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 \cdots$

$$\left.\begin{aligned} &= (1-1) + (1-1) \cdots = 0 \\ &= 1 - (1-1) - (1-1) \cdots = 1 \end{aligned}\right\} \text{but?}$$

Is ∞ even or odd?

Our previous definition rules out this case, since our subset can be $A+1$ ones for any A.

But how do we know when things go bad?

The techniques we have seen for finite sums will work correctly if all terms are positive and the sum does converge to a constant A. If the sum diverges, we are stuck.

More interesting is when the terms are either positive or negative. If we group them by sign, we get two (infinite) series which approach A^+ and A^-.

[Clearly if A^+ & A^- are both finite, the sum of them is finite & we are ok.

[If $A^+ \to \infty$ & A^- is finite, the sum of them approaches ∞

[If $A^- \to -\infty$ & A^+ is finite, the sum approaches $-\infty$

The only case which gives us trouble is when both of them diverge. The way to get intuition is to sum over a finite number of terms.

$$\sum_{k=1}^{N} (-1)^k = \begin{cases} 0 & \text{if } N \text{ is even} \\ -1 & \text{if } N \text{ is odd} \end{cases}$$

General Methods for Sums

The textbook works one example using all of these methods - quite instructive

Method 0: Look it up

Sloane's <u>Handbook of Integer Sequences</u> gives 2300 sequences sorted by their prefix. If you can't figure out what 1, 3, 7, 31, 211, 2311... are, look it up in Sloane - he gives names and references. http://www.research.att.com/~njas/sequences/

Using a computer algebra system like Mathematica running Gosper's algorithm for symbolic summation can help.

```
<< ~math/Packages/Algebra/GosperSum.m
GosperSum[2^k, {k,0,n}]
```

$-1+2\cdot 2^n$

```
GosperSum[k^2 (1-k), {k,1,n}]
```

$$\frac{-((1-n)\,n\,(-2-5n-3n^2))}{12}$$

Method 1: Prove it by Induction

Good for testing what you get back from Gosper's algorithm, Bad in that it provides no intuition about the problem

Method 2: Purturb the Sum

Name-and-conquer techniques are powerful

Method 3: Build a Repertoire

Convert the sum into a recurrence and solve the recurrence.

Method 4: Replace Sums by Integrals

Good for approximation - if you remember calculus.

Method 5: Expand + Contract

Fool around with the expression a while

Method 6: Use finite Calculus

very interesting discussion in book for the curious

Method 7: Use generating functions!

wait till next month (or two)

The Chebyshev Inequalities

Suppose we have two monotone non-decreasing sequences, say 1, 3, 5, 7 ... and 2, 4, 6, 8 ... .
Can we get a bound on the sum of all products of pairs without computing all N^2 terms?

Yes, with our previous sum! Rearranging it, we get

$$\left(\sum_{k=1}^{N} a_k\right)\left(\sum_{k=1}^{N} b_k\right) = N\sum_{k=1}^{N} a_k b_k - \sum_{1\le j<k\le N}(a_k-a_j)(b_k-b_j)$$

if $a_1 \le \cdots \le a_N$ and $b_1 \le \cdots \le b_N$, every term in the second sum is ≥ 0, so

$$\left(\sum_{k=1}^{N} a_k\right)\left(\sum_{k=1}^{N} b_k\right) \le N\sum_{k=1}^{N} a_k b_k$$

What if one sequence is non-increasing?
What if both sequences are non-increasing?

The Finite Calculus

The integral calculus is a well developed theory for performing summations over a continuous interval. The finite calculus is a less well developed theory for summing over discrete intervals.

The derivative operator D in infinite calculus is

$$Df(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Since over the integers, the smallest gap is $h \to 1$, $\Delta f(x) = f(x+1) - f(x)$

Observe the D + Δ define new functions for any function f. Thus they are operators rather than functions.

These functions are interesting because of the inverse operator $\int$ or $\sum$

The Fundamental Theorem of Calculus:

$$g(x) = D f(x) \text{ iff } \int g(x)\,dx = f(x) + C$$

The Fundamental Theorem of Finite Calculus:

$$g(x) = \Delta f(x) \text{ iff } \underbrace{\sum g(x)\,dx}_{\text{"indefinite sum"}} = f(x) + C$$

The idea is that we can develope a class of functions whose indefinite sums we know and rules for manipulating them like integrals, we can systematically get closed forms for summations!

Ex: Polynomials in the infinite calculus

$$D(x^m) = m x^{m-1}$$

$$\int_0^n x^m\,dx = \frac{n^{m+1}}{(m+1)}$$

Having the same thing for the finite calculus would give us an easy way to compute $\sum_{i=0}^{n} x^m$. However, polynomials are not so nice in the finite calculus.

A function which $\underline{\text{is}}$ nice is the falling factorial or Pochhammer symbol:

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1)$$

$$x^{\underline{1}} = x$$

what is the difference function of $x^{\underline{m}}$

$$\Delta x^{\underline{m}} = (x+1)^{\underline{m}} - x^{\underline{m}}$$

$$= \underline{(x+1)}(x)(x-1)\cdots(x-m+2) - (x)(x-1)\cdots(x-m+2)\underline{(x-m+1)}$$

$$= m(x)(x-1)\cdots(x-m+2) = m\,x^{\underline{m-1}}$$

So:

$$\Delta x^{\underline{m}} = m\,x^{\underline{m-1}}$$

since

$$x^{\underline{m}} = \Delta \frac{x^{\underline{m+1}}}{m+1}$$

$$\sum_{0 \le x < n} x^{\underline{m}} = \left.\frac{x^{\underline{m+1}}}{m+1}\right|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

giving us a closed form for any m!

Our indefinite summation operation observes the following properties: $g(x) = \Delta f(x) = f(x+1) - f(x)$

$$\sum_a^a g(x)\,dx = f(a) - f(a) = 0$$

$$\sum_a^{a+1} g(x)\,dx = f(a+1) - f(a) = g(a)$$

$$\sum_a^b g(x)\,dx = \sum_{k=a}^{b-1} g(k) \rightarrow \begin{matrix} f(b) - f(a) \\ \text{(sum telescopes)} \end{matrix}$$

- summing up over $b-a+1$ terms is equivallent to summing over $b-a$ differences.

Although the falling factorial function might not be the best thing to have an identity for, we can use it to find other identities:

$$k^2 = k^{\underline{2}} + k^{\underline{1}} = k(k-1) + k$$

thus

$$\sum_{0 \le k < N} k^2 = \sum_{0 \le k < N} k^{\underline{2}} + k^{\underline{1}} = \frac{N^{\underline{3}}}{3} + \frac{N^{\underline{2}}}{2}$$

$$= \frac{N}{3}(N-1)(N-\tfrac{1}{2})$$

We can find other functions and there difference identities!

The falling factorials can be generalized to negative powers as follows:

$$x^{\underline{-m}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$$

$$x^{\underline{-1}} = \frac{1}{x+1}$$

Our previous identity is not defined when $n=-1$

The log function was defined so!

$$\int_a^b x^{-1}\,dx = \ln x \Big|_a^b$$

We need:

$$x^{\underline{-1}} = \frac{1}{x+1} = \Delta f(x) = f(x+1) - f(x)$$

A solution is

$$f(x) = \sum_{i=1}^{x} \frac{1}{x} = 1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{x} = H(x)$$

Thus the Harmonic numbers take the role of logarithms in finite calculus.

If you remember your calculus, $De^x = e^x$

what is the function so that $f(x) = \Delta f(x)$?

$$f(x) = f(x+1) - f(x)$$
$$f(x+1) = 2f(x)$$

Thus $\Delta 2^x = 2^x$.

There is an analogous notion of summation by parts to integration by parts

$$\Delta(uv) = u\Delta v + Ev\Delta u,$$

where $Ef(x) = f(x+1)$

The finite calculus is part of an attempt to produce a general theory of combinatorics (Rota ...) which to date hasn't really been successful.

This is a powerful technique for proving general results, but since in general we are interested in isolated sums ad-hoc methods are often just as appropriate.

we will see these again when we get to binomial coefficients.