Discrete Laplace-Beltrami Operator Determines Discrete Riemannian Metric

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Abstract

The Laplace-Beltrami operator of a smooth Riemannian manifold is determined by the Riemannian metric. Conversely, the heat kernel constructed from its eigenvalues and eigenfunctions determines the Riemannian metric. This work proves the analogy on Euclidean polyhedral surfaces (triangle meshes), that the discrete Laplace-Beltrami operator and the discrete Riemannian metric (unique up to a scaling) are mutually determined by each other.

Given an Euclidean polyhedral surface, its Riemannian metric is represented as edge lengths, satisfying triangle inequalities on all faces. The Laplace-Beltrami operator is formulated using the cotangent formula, where the edge weight is defined as the sum of the cotangent of angles against the edge. We prove that the edge lengths can be determined by the edge weights unique up to a scaling using the variational approach.

First, we show that the space of all possible metrics of a polyhedral surface is convex. Then, we construct a special energy defined on the metric space, such that the gradient of the energy equals to the edge weights. Third, we show the Hessian matrix of the energy is positive definite, restricted on the tangent space of the metric space, therefore the energy is convex. Finally, by the fact that the parameter on a convex domain and the gradient of a convex function defined on the domain have one-to-one correspondence, we show the edge weights determines the polyhedral metric unique up to a scaling.

The constructive proof leads to a computational algorithm that finds the unique metric on a topological triangle mesh from a discrete Laplace-Beltrami operator matrix.

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1 Introduction

Laplace-Beltrami operator plays a fundamental role in Riemannian geometry [10]. Discrete Laplace-Beltrami operators on triangulated surface meshes span the entire spectrum of geometry processing applications, including mesh parameterization, segmentation, reconstruction, compression, re-meshing and so on [5, 9, 14].

Laplace-Beltrami operator is determined by the Riemannian metric. The heat kernel can be constructed from the eigenvalues and eigenfunctions of the Laplace-Beltrami operator, conversely, it fully determines the Riemannian metric (unique up to a scaling). In this work, we prove the discrete analogy to this fundamental fact, that the discrete Laplace-Beltrami operator and the discrete Riemannian metric are mutually determined by each other.

Related Works In real applications, a smooth metric surface is usually represented as a triangulated mesh. The manifold heat kernel is estimated from the discrete Laplace operator. The most well-known and widely-used discrete formulation of Laplace operator over triangulated meshes is the so-called *cotangent scheme*, which was originally introduced in [3, 7]. Xu [13] proposed several simple discretization schemes of Laplace operators over triangulated surfaces, and established the theoretical analysis on convergence. Wardetzky et al.[12] proved the theoretical limitation that the discrete Laplacians cannot satisfy all natural properties, thus, explained the diversity of existing discrete Laplace operators. A family of operations were presented by extending more natural properties into the existing operators. Reuter et al.[8] computed a discrete Laplace operator using the finite element method, and exploited the isometry invariance of the Laplace operator as shape fingerprint for object comparison. Belkin et al.[1] proposed the first discrete Laplacian that pointwise converges to the true Laplacian as the input mesh approximates a smooth manifold better. Tamal et al.[2] employed this mesh Laplacian and provided the first convergence to relate the discrete spectrum with the true spectrum, and studied the stability and robustness of the discrete approximation of Laplace spectra. The eigenfunctions of Laplace-Beltrami operator have been applied for global intrinsic symmetry detection in [6]. Heat Kernel Signature was proposed in [11], which is concise and characterizes the shape up to isometry.

Our Results In this work, we prove that the discrete Laplace-Beltrami operator based on the cotangent scheme [3, 7] is determined by the discrete Riemannian metric, and also determines the metric unique up to a scaling. The proof is using the variational approach, which leads to a practical algorithm to compute a Riemannian metric from a prescribed Laplace-Beltrami operator.

Paper Outline In Section 2, we briefly overview the fundamental theorem of smooth heat kernel and our theoretical claims of discrete case. We clarify the simplest case, one triangle mesh, in Section 3 first; then turn to the more general Euclidean polyhedral surfaces in Section 4. Finally, in Section 5, we present a variational algorithm to compute the unique Riemannian metric from from a Laplace-Beltrami matrix. The numerical experiments on different topological triangle meshes support the theoretic results.

2 Preliminaries and Proof Overview

2.1 Smooth Case

Suppose (M,g) is a complete Riemannian manifold, g is the Riemannian metric. Δ is the Laplace-Beltrami operator. The eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{\phi_n\}$ of Δ are

$$\Delta\phi_n = -\lambda_n\phi_n,$$

where ϕ_n is normalized to be orthonormal in $L^2(M)$. The spectrum is given by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_n \to \infty.$$

Then there is a heat kernel $K(x,y,t) \in C^\infty(M \times M \times \mathbb{R}^+)$, such that

$$K(x,y,t) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y).$$

Heat kernel reflects all the information of the Riemannian metric $\eth$. The details of the following theorem can be found in [11].

Theorem 2.1. *Let $f : (M_1,g_1) \to (M_2,g_2)$ be a diffeomorphism between two Riemannian manifolds. If f is an isometry, then*

$$K_1(x,y,t) = K_2(f(x),f(y),t), \forall x,y \in M, t > 0. \tag{1}$$

Conversely, if f is a surjective map, and Eqn. (1) holds, then f is an isometry.

2.2 Discrete Case

In this work, we focus on discrete surfaces, namely polyhedral surface. For example, a triangle mesh is piecewise linearly embedded in $\mathbb{R}^3$.

Definition 2.1 (Polyhedral Surface). *An Euclidean polyhedral surface is a triple (S,T,d) where S is a closed surface, T is a triangulation of S and d is a metric on S whose restriction to each triangle is isometric to an Euclidean triangle.*

The well-known cotangent edge weight [3, 7] on an Euclidean polyhedral surface is defined as follows:

Definition 2.2 (Cotangent Edge Weight). *Suppose $[v_i,v_j]$ is a boundary edge of M, $[v_i,v_j] \in \partial M$, then $[v_i,v_j]$ is associated with one triangle $[v_i,v_j,v_k]$, the angle against $[v_i,v_j]$ at the vertex v_k is α, then the weight of $[v_i,v_j]$ is given by $w_{ij} = \frac{1}{2}\cot\alpha$. Otherwise, if $[v_i,v_j]$ is an interior edge, the two angles against it are α,β, then the weight is $w_{ij} = \frac{1}{2}(\cot\alpha + \cot\beta)$.*

The discrete Laplace-Beltrami operator is constructed from the cotangent edge weight.

Definition 2.3 (Discrete Laplace Matrix). *The discrete Laplace matrix $L = (L_{ij})$ for an Euclidean polyhedral surface is given by*

$$L_{ij} = \begin{cases} -w_{ij} & i \neq j \\ \sum_k w_{ik} & i = j \end{cases}$$

Because L is symmetric, it can be decomposed as

$$L = \Phi\Lambda\Phi^T \tag{2}$$

where $\Lambda = diag(\lambda_0, \lambda_1, \cdots, \lambda_n)$, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of L, and $\Phi = (\phi_0|\phi_1|\phi_2|\cdots|\phi_n)$, $L\phi_i = \lambda_i\phi_i$ are the orthonormal eigenvectors, such that $\phi_i^T\phi_j = \delta_{ij}$.

Definition 2.4 (Discrete Heat Kernel). *The discrete heat kernel is defined as follows:*

$$K(t) = \Phi exp(-\Lambda t)\Phi^T. \tag{3}$$

The Main Theorem in this work is

Theorem 2.2 (Global Rigidity). *Suppose two Euclidean polyhedral surfaces* $(S, T, \mathbf{d_1})$ *and* $(S, T, \mathbf{d_2})$ *are given,*

$$L_1 = L_2,$$

if and only if $\mathbf{d_1}$ *and* $\mathbf{d_2}$ *differ by a scaling.*

Corollary 2.3. *Suppose two Euclidean polyhedral surfaces* $(S, T, \mathbf{d_1})$ *and* $(S, T, \mathbf{d_2})$ *are given,*

$$K_1(t) = K_2(t), \forall t > 0,$$

if and only if $\mathbf{d_1}$ *and* $\mathbf{d_2}$ *differ by a scaling.*

Proof. Note that,

$$\frac{dK(t)}{dt}|_{t=0} = -L.$$

Therefore, the discrete Laplace matrix and the discrete heat kernel mutually determine each other. □

2.3 Proof Overview

The main idea for the proof is as follows. We fix the connectivity of the polyhedral surface (S, T). Suppose the edge set of (S, T) is sorted as $E = \{e_1, e_2, \cdots, e_m\}$, where $m = |E|$ number of edges, the face set is denoted as F. A triangle $[v_i, v_j, v_k] \in F$ is also denoted as $\{i, j, k\} \in F$.

By definition, an Euclidean polyhedral metric on (S, T) is given by its edge length function $d : E \to \mathbb{R}^+$. We denote a metric as $d = (d_1, d_2, \cdots, d_m)$, where $d_i = d(e_i)$ is the length of edge e_i. Let

$$E_d(2) = \{(d_1, d_2, d_3)|d_i + d_j > d_k\}$$

be the space of all Euclidean triangles parameterized by the edge lengths, where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. In this work, for convenience, we use $u = (u_1, u_2, \cdots, u_m)$ to represent the metric, where $u_k = \frac{1}{2}d_k^2$.

Definition 2.5 (Admissible Metric Space). *Given a triangulated surface* (S, K)*, the admissible metric space is defined as*

$$\Omega_u = \{(u_1, u_2, u_3 \cdots, u_m)| \sum_{k=1}^{m} u_k = m, (\sqrt{u_i}, \sqrt{u_j}, \sqrt{u_k}) \in E_d(2), \forall\{i, j, k\} \in F\}$$

We show that Ω_u is a convex domain in $\mathbb{R}^m$.

Definition 2.6 (Energy). *An energy $E:\Omega_u\to\mathbb{R}$ is defined as:*

$$E(u_1,u_2\cdots,u_m)=\int_{(1,1,\cdots,1)}^{(u_1,u_2\cdots,u_m)}\sum_{k=1}^{m}w_k(\mu)d\mu_k, \tag{4}$$

where $w_k(\mu)$ is the cotangent weight on the edge e_k determined by the metric μ.

Next we show this energy is convex in Lemma 3.5. According to the following lemma, the gradient of the energy $\nabla E(d):\Omega\to\mathbb{R}^m$

$$\nabla E:(u_1,u_2\cdots,u_m)\to(w_1,w_2,\cdots w_m)$$

is an embedding. Namely the metric is determined by the edge weight unique up to a scaling.

Lemma 2.4. *Suppose $\Omega\subset\mathbb{R}^n$ is an open convex domain in $\mathbb{R}^n$, $E:\Omega\to\mathbb{R}$ is a strictly convex function with positive definite Hessian matrix, then $\nabla E:\Omega\to\mathbb{R}^n$ is a smooth embedding.*

Proof. If $\mathbf{p}\neq\mathbf{q}$ in Ω, let $\gamma(t)=(1-t)\mathbf{p}+t\mathbf{q}\in\Omega$ for all $t\in[0,1]$. Then $f(t)=E(\gamma(t)):[0,1]\to\mathbb{R}$ is a strictly convex function, so that

$$\frac{df(t)}{dt}=\nabla E|_{\gamma(t)}\cdot(\mathbf{q}-\mathbf{p}).$$

Because

$$\frac{d^2f(t)}{dt^2}=(\mathbf{q}-\mathbf{p})^TH|_{\gamma(t)}(\mathbf{q}-\mathbf{p})>0,$$

$\frac{df(0)}{dt}\neq\frac{df(1)}{dt}$, therefore

$$\nabla E(\mathbf{p})\cdot(\mathbf{q}-\mathbf{p})\neq\nabla E(\mathbf{q})\cdot(\mathbf{q}-\mathbf{p}).$$

This means $\nabla E(\mathbf{p})\neq\nabla E(\mathbf{q})$, therefore ∇E is injective.

On the other hand, the Jacobi matrix of ∇E is the Hessian matrix of E, which is positive definite. It follows that $\nabla E:\Omega\to\mathbb{R}^n$ is a smooth embedding. □

From the discrete Laplace-Beltrami operator (Eqn. (2)) or the heat kernel (Eqn. (3)), we can compute all the cotangent edge weights, then because the edge weight determines the metric, we attain the Main Theorem 2.2.

3 Euclidean Triangle

In this section, we show the proof for the simplest case, a Euclidean triangle; in the next section, we generalize the proof to all types of triangle meshes.

Given a triangle $\{i,j,k\}$, three corner angles denoted by $\{\theta_i,\theta_j,\theta_k\}$, three edge lengths denoted by $\{d_i,d_j,d_k\}$, as shown in Figure 1. In this case, the problem is trivial. Given $(w_i,w_j,w_k)=(\cot\theta_i,\cot\theta_j,\cot\theta_k)$, we can compute $(\theta_i,\theta_j,\theta_k)$ by taking the arctan function. Then the normalized edge lengths are given by

$$(d_i,d_j,d_k)=\frac{3}{\sin\theta_i+\sin\theta_j+\sin\theta_k}(\sin\theta_i,\sin\theta_j,\sin\theta_k).$$

Although this approach is direct and simple, it can not be generalized to more complicated polyhedral surfaces. In the following, we use a different approach, which can be generalized to all polyhedral surfaces.

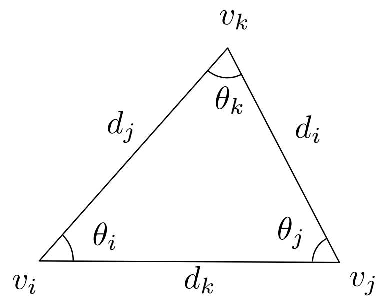

Figure 1: An Euclidean triangle.

Lemma 3.1. *Suppose an Euclidean triangle is with angles $\{\theta_i, \theta_j, \theta_k\}$ and edge lengths $\{d_i, d_j, d_k\}$, angles are treated as the functions of the edge lengths, $\theta_i(d_i, d_j, d_k)$ then*

$$\frac{\partial \theta_i}{\partial d_i} = \frac{d_i}{2A} \tag{5}$$

and

$$\frac{\partial \theta_i}{\partial d_j} = -\frac{d_i}{2A}\cos\theta_k, \tag{6}$$

where A is the area of the triangle.

Proof. According to Euclidean cosine law,

$$\cos\theta_i = \frac{d_j^2 + d_k^2 - d_i^2}{2d_j d_k} \tag{7}$$

we take derivative on both sides with respective to d_i

$$-\sin\theta_i \frac{\partial \theta_i}{\partial d_i} = \frac{-2d_i}{2d_j d_k}$$

$$\frac{\partial \theta_i}{\partial d_i} = \frac{d_i}{d_j d_k \sin\theta_i} = \frac{d_i}{2A} \tag{8}$$

where $A = \frac{1}{2} d_j d_k \sin\theta_i$ is the area of the triangle. Similarly,

$$\frac{\partial}{\partial d_j}(d_j^2 + d_k^2 - d_i^2) = \frac{\partial}{\partial d_j}(2d_j d_k \cos\theta_i)$$

$$2d_j = 2d_k \cos\theta_i - 2d_j d_k \sin\theta_i \frac{\partial \theta_i}{\partial d_j}$$

$$2A\frac{\partial \theta_i}{\partial d_j} = d_k \cos\theta_i - d_j = -d_i \cos\theta_k$$

We get

$$\frac{\partial \theta_i}{\partial d_j} = -\frac{d_i \cos\theta_k}{2A}.$$

□

Lemma 3.2. *In an Euclidean triangle, let $u_i = \frac{1}{2}d_i^2$ and $u_j = \frac{1}{2}d_j^2$ then*

$$\frac{\partial \cot\theta_i}{\partial u_j} = \frac{\partial \cot\theta_j}{\partial u_i} \tag{9}$$

Proof.

$$\frac{\partial \cot\theta_i}{\partial u_j} = \frac{1}{d_j}\frac{\partial \cot\theta_i}{\partial d_j} = -\frac{1}{d_j}\frac{1}{\sin^2\theta_i}\frac{\partial\theta_i}{\partial d_j} = \frac{1}{d_j}\frac{1}{\sin^2\theta_i}\frac{d_i\cos\theta_k}{2A} = \frac{d_i^2}{\sin^2\theta_i}\frac{\cos\theta_k}{2Ad_id_j}$$
$$= \frac{4R^2}{2A}\frac{\cos\theta_k}{d_id_j} \tag{10}$$

where R is the radius of the circum circle of the triangle. The righthand side of Eqn. (10) is symmetric with respect to the indices i and j. □

Corollary 3.3. *The differential form*

$$\omega = \cot\theta_i du_i + \cot\theta_j du_j + \cot\theta_k du_k \tag{11}$$

is a closed 1-form.

Proof. By the above Lemma 3.2 regarding symmetry,

$$\begin{aligned} d\omega = &(\frac{\partial \cot\theta_j}{\partial u_i} - \frac{\partial \cot\theta_i}{\partial u_j})du_i\wedge du_j + (\frac{\partial \cot\theta_k}{\partial u_j} - \frac{\partial \cot\theta_j}{\partial u_k})du_j\wedge du_k \\ &+ (\frac{\partial \cot\theta_i}{\partial u_k} - \frac{\partial \cot\theta_k}{\partial u_i})du_k\wedge du_i \\ &= 0. \end{aligned}$$

□

Definition 3.1 (Admissible Metric Space). *Let $u_i = \frac{1}{2}d_i^2$, the admissible metric space is defined as*

$$\Omega_u := \{(u_i,u_j,u_k)|(\sqrt{u_i},\sqrt{u_j},\sqrt{u_k}) \in E_d(2), u_i+u_j+u_k = 3\}$$

Lemma 3.4. *The admissible metric space Ω_u is a convex domain in $\mathbb{R}^3$.*

Proof. Suppose $(u_i,u_j,u_k) \in \Omega_u$ and $(\tilde{u}_i,\tilde{u}_j,\tilde{u}_k) \in \Omega_u$, then from $\sqrt{u_i}+\sqrt{u_j} > \sqrt{u_k}$, we get $u_i+u_j+2\sqrt{u_iu_j} > u_k$. Define $(u_i^\lambda,u_j^\lambda,u_k^\lambda) = \lambda(u_i,u_j,u_k)+(1-\lambda)(\tilde{u}_i,\tilde{u}_j,\tilde{u}_k)$, where $0<\lambda<1$. Then

$$\begin{aligned} u_i^\lambda u_j^\lambda &= (\lambda u_i + (1-\lambda)\tilde{u}_i)(\lambda u_j + (1-\lambda)\tilde{u}_j) \\ &= \lambda^2 u_iu_j + (1-\lambda)^2\tilde{u}_i\tilde{u}_j + \lambda(1-\lambda)(u_i\tilde{u}_j + u_j\tilde{u}_i) \\ &\geq \lambda^2 u_iu_j + (1-\lambda)^2\tilde{u}_i\tilde{u}_j + 2\lambda(1-\lambda)\sqrt{u_iu_j\tilde{u}_i\tilde{u}_j} \\ &= (\lambda\sqrt{u_iu_j} + (1-\lambda)\sqrt{\tilde{u}_i\tilde{u}_j})^2 \end{aligned}$$

It follows

$$\begin{aligned} &u_i^\lambda + u_j^\lambda + 2\sqrt{u_i^\lambda u_j^\lambda} \geq \lambda(u_i+u_j+2\sqrt{u_iu_j}) + (1-\lambda)(\tilde{u}_i+\tilde{u}_j+2\sqrt{\tilde{u}_i\tilde{u}_j}) \\ &> \lambda u_k + (1-\lambda)\tilde{u}_k = u_k^\lambda \end{aligned}$$

This shows $(u_i^\lambda,u_j^\lambda,u_k^\lambda) \in \Omega_u$. □

Similarly, we define the edge weight space as follows.

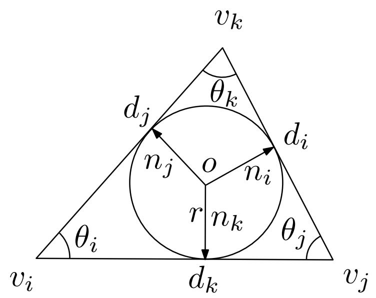

Figure 2: The geometric interpretation of the Hessian matrix. The incircle of the triangle is centered at O, with radius r. The perpendiculars n_i, n_j and n_k are from the incenter of the triangle and orthogonal to the edge e_i, e_j and e_k respectively.

Definition 3.2 (Edge Weight Space). *The edge weights of an Euclidean triangle form the edge weight space*

$$\Omega_\theta = \{(\cot\theta_i, \cot\theta_j, \cot\theta_k) | 0 < \theta_i, \theta_j, \theta_k < \pi, \theta_i + \theta_j + \theta_k = \pi\}$$

Note that,

$$\cot\theta_k = -\cot(\theta_i + \theta_j) = \frac{1 - \cot\theta_i \cot\theta_j}{\cot\theta_i + \cot\theta_j}$$

Lemma 3.5. *The energy $E : \Omega_u \to \mathbb{R}$*

$$E(u_i, u_j, u_k) = \int_{(1,1,1)}^{(u_i,u_j,u_k)} \cot\theta_i d\tau_i + \cot\theta_j d\tau_j + \cot\theta_k d\tau_k \tag{12}$$

is well defined on the admissible metric space Ω_u and is convex.

Proof. According to Corollary 3.3, the differential form is closed. Furthermore, the admissible metric space Ω_u is a simply connected domain. The differential form is exact, therefore, the integration is path independent, and the energy function is well defined.

Then we compute the Hessian matrix of the energy,

$$H = -\frac{2R^2}{A} \begin{bmatrix} \frac{1}{d_i^2} & -\frac{\cos\theta_k}{d_i d_j} & -\frac{\cos\theta_j}{d_i d_k} \\ -\frac{\cos\theta_k}{d_j d_i} & \frac{1}{d_j^2} & -\frac{\cos\theta_i}{d_j d_k} \\ -\frac{\cos\theta_j}{d_k d_i} & -\frac{\cos\theta_i}{d_k d_j} & \frac{1}{d_k^2} \end{bmatrix} = -\frac{2R^2}{A} \begin{bmatrix} (\eta_i, \eta_i) & (\eta_i, \eta_j) & (\eta_i, \eta_k) \\ (\eta_j, \eta_i) & (\eta_j, \eta_j) & (\eta_j, \eta_k) \\ (\eta_k, \eta_i) & (\eta_k, \eta_j) & (\eta_k, \eta_k) \end{bmatrix}$$

As shown in Figure 2, $d_i \mathbf{n}_i + d_j \mathbf{n}_j + d_k \mathbf{n}_k = 0$,

$$\eta_i = \frac{\mathbf{n}_i}{rd_i}, \eta_j = \frac{\mathbf{n}_j}{rd_j}, \eta_k = \frac{\mathbf{n}_k}{rd_k},$$

where r is the radius of the incircle of the triangle. Suppose $(x_i, x_j, x_k) \in \mathbb{R}^3$ is a vector in $\mathbb{R}^3$, then

$$[x_i, x_j, x_k] \begin{bmatrix} (\eta_i, \eta_i) & (\eta_i, \eta_j) & (\eta_i, \eta_k) \\ (\eta_j, \eta_i) & (\eta_j, \eta_j) & (\eta_j, \eta_k) \\ (\eta_k, \eta_i) & (\eta_k, \eta_j) & (\eta_k, \eta_k) \end{bmatrix} \begin{bmatrix} x_i \\ x_j \\ x_k \end{bmatrix} = \|x_i \eta_i + x_j \eta_j + x_k \eta_k\|^2 \geq 0$$

If the result is zero, then $(x_i, x_j, x_k) = \lambda(u_i, u_j, u_k), \lambda \in \mathbb{R}$. That is the null space of the Hessian matrix. In the admissible metric space Ω_u, $u_i + u_j + u_k = C (C = 3)$, then $du_i + du_j + du_k = 0$. If (du_i, du_j, du_k) belongs to the null space, then $(du_i, du_j, du_k) = \lambda(u_i, u_j, u_k)$, therefore, $\lambda(u_i + u_j + u_k) = 0$. Because u_i, u_j, u_k are positive, $\lambda = 0$. In summary, the energy on Ω_u is convex. $\square$

Theorem 3.6. *The mapping* $\nabla E : \Omega_u \to \Omega_\theta, (u_i, u_j, u_k) \to (\cot\theta_i, \cot\theta_j, \cot\theta_k)$ *is a diffeomorphism.*

Proof. The energy $E(u_i, u_j, u_k)$ is a convex function defined on the convex domain Ω_u, according to Lemma 2.4, $\nabla E : (u_i, u_j, u_k) \to (\cot\theta_i, \cot\theta_j, \cot\theta_k)$ is a diffeomorphism. □

4 Euclidean Polyhedral Surface

In this section, we consider the whole polyhedral surface.

4.1 Closed Surfaces

Given a polyhedral surface (S, T, d), the admissible metric space and the edge weight have been defined in Section 2.2 respectively.

Lemma 4.1. *The admissible metric space* Ω_u *is convex.*

Proof. For a triangle $\{i, j, k\} \in F$, define

$$\Omega_u^{ijk} := \{(u_i, u_j, u_k) | (\sqrt{u_i}, \sqrt{u_j}, \sqrt{u_k}) \in E_d(2)\}.$$

Similar to the proof of Lemma 3.4, Ω_u^{ijk} is convex. The admissible metric space for the mesh is

$$\Omega_u = \bigcap_{\{i,j,k\}\in F} \Omega_u^{ijk} \bigcap \{(u_1, u_2, \cdots, u_m) | \sum_{k=1}^{m} u_k = m\},$$

the intersection Ω_u is still convex. □

Definition 4.1 (Differential Form). *The differential form* ω *defined on* Ω_u *is the summation of the differential form on each face,*

$$\omega = \sum_{\{i,j,k\}\in F} \omega_{ijk} = \sum_{i=1}^{m} 2w_i du_i,$$

where ω_{ijk} *is given in Eqn. (11) in Corollary 3.3.* w_i *is the edge weight on* e_i*.*

Lemma 4.2. *The differential form* ω *is a closed 1-form.*

Proof. According to Corollary 3.3,

$$d\omega = \sum_{\{i,j,k\}\in F} d\omega_{ijk} = 0.$$

□

Lemma 4.3. *The energy function*

$$E(u_1, u_2, \cdots, u_n) = \sum_{\{i,j,k\}\in F} E_{ijk}(u_1, u_2, \cdots, u_n) = \int_{(1,1,\cdots,1)}^{(u_1,u_2,\cdots,u_n)} \sum_{i=1}^{n} w_i du_i$$

is well defined and convex on Ω_u*, where* E_{ijk} *is the energy on the face, defined in Eqn. (12).*

Proof. For each face $\{i,j,k\} \in F$, the Hessian matrices of E_{ijk} is semi-positive definite, therefore, the Hessian matrix of the total energy E is semi-positive definite.

Similar to the proof of Lemma 3.5, the null space of the Hessian matrix H is

$$kerH = \{\lambda(d_1, d_2, \cdots, d_n), \lambda \in \mathbb{R}\}.$$

The tangent space of Ω_u at $u = (u_1, u_2, \cdots, u_n)$ is denoted by $T\Omega_u(u)$. Assume $(du_1, du_2, \cdots, du_n) \in T\Omega_u(u)$, then from $\sum_{i=1}^m u_i = m$, we get $\sum_{i=1}^m du_m = 0$. Therefore,

$$T\Omega_u(u) \cap KerH = \{0\},$$

hence H is positive definite restricted on $T\Omega_u(u)$. So the total energy E is convex on Ω_u. □

Theorem 4.4. *The mapping on a closed Euclidean polyhedral surface* $\nabla E : \Omega_u \to \mathbb{R}^m, (u_1, u_2, \cdots, u_n) \to (w_1, w_2, \cdots, w_n)$ *is a smooth embedding.*

Proof. The admissible metric space Ω_u is convex as shown in Lemma 4.1, the total energy is convex as shown in Lemma 4.3. According to Lemma 2.4, ∇E is a smooth embedding. □

4.2 Open Surfaces

By the double covering technique [4], we can convert a polyhedral surface with boundaries to a closed surface. First, let $(\bar{S}, \bar{T})$ be a copy of (S, T), then we reverse the orientation of each face in $\bar{M}$, and glue two surfaces S and $\bar{S}$ along their corresponding boundary edges, the resulting triangulated surface is a closed one. We get the following corollary

Corollary 4.5. *The mapping on an Euclidean polyhedral surface with boundaries* $\nabla E : \Omega_u \to \mathbb{R}^m, (u_1, u_2, \cdots, u_n) \to (w_1, w_2, \cdots, w_n)$ *is a smooth embedding.*

Surely, the cotangent edge weights can be uniquely obtained from the discrete heat kernel. By combining Theorem 4.4 and Corollary 4.5, we obtain the major Theorem 2.2, *Global Rigidity Theorem*, of this work.

5 Numerical Experiments

From above theoretic deduction, we can design the algorithm to compute discrete metric with user prescribed edge weights.

Problem 5.1. *Let* (S, T) *be a triangulated surface,* $\bar{\mathbf{w}}(\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_n)$ *are the user prescribed edge weights. The problem is to find a discrete metric* $\mathbf{u} = (u_1, u_2, \cdots, u_n)$*, such that this metric* $\bar{\mathbf{u}}$ *induces the desired edge weight* $\mathbf{w}$*.*

The algorithm is based on the following theorem.

Theorem 5.2. *Suppose* (S, T) *is a triangulated surface. If there exists an* $\bar{\mathbf{u}} \in \Omega_u$*, which induces* $\bar{\mathbf{w}}$*, then* $\mathbf{u}$ *is the unique global minimum of the energy*

$$E(\mathbf{u}) = \int_{(1,1,\cdots,1)}^{(u_1,u_2,\cdots,u_n)} \sum_{i=1}^{n} (\bar{w}_i - w_i) d\mu_i. \tag{13}$$

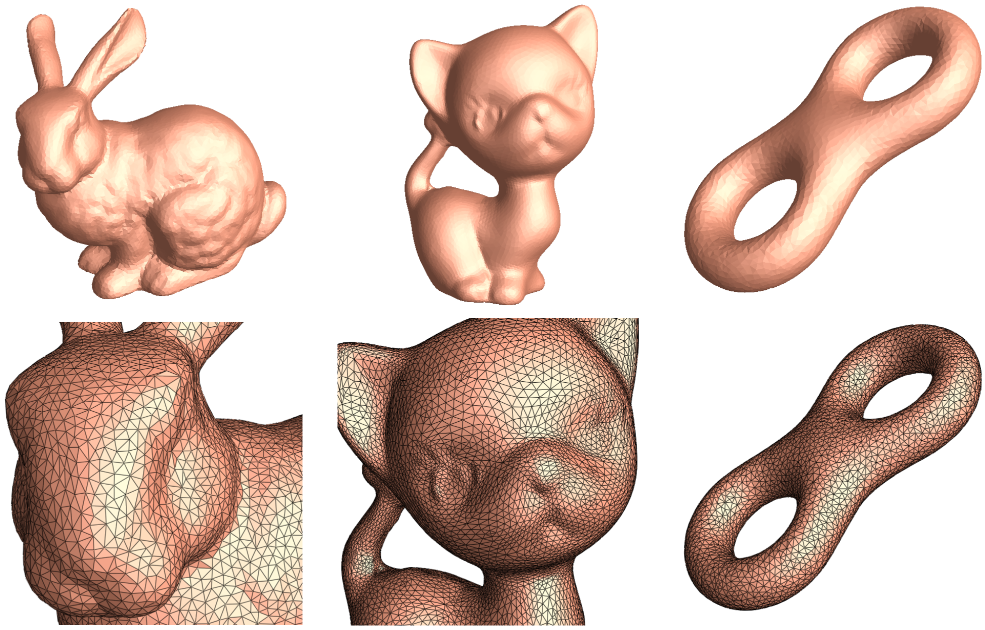

Figure 3: Euclidean polyhedral surfaces used in the experiments.

Proof. The gradient of the energy $\nabla E(\mathbf{u}) = \bar{\mathbf{w}} - \mathbf{w}$, and since $\nabla E(\bar{\mathbf{u}}) = 0$, therefore $\bar{\mathbf{u}}$ is a critical point. The Hessian matrix of $E(\mathbf{u})$ is positive definite, the domain Ω_u is convex, therefore $\bar{\mathbf{u}}$ is the unique global minimum of the energy. $\square$

In our numerical experiments, as shown in Figure 3, we tested surfaces with different topologies, with different genus, with or without boundaries. All discrete polyhedral surfaces are triangle meshes scanned from real objects. Because the meshes are embedded in $\mathbb{R}^3$, they have induced Euclidean metric, which are used as the desired metric $\bar{\mathbf{u}}$. From the induced Euclidean metric, the desired edge weight $\bar{\mathbf{w}}$ can be directly computed. Then we set the initial discrete metric to be the constant metric $(1,1,\cdots,1)$. By optimizing the energy in Eqn. (13), we can reach the global minimum, and recovered the desired metric, which differs from the induced Euclidean metric by a scaling.

6 Future Work

We conjecture that the Main Theorem 2.2 holds for arbitrary dimensional Euclidean polyhedral manifolds, that means discrete Laplace-Beltrami operator (or equivalently the discrete heat kernel) and the the discrete metric for any dimensional Euclidean polyhedral manifold are mutually determined by each other. On the other hand, we will explore the possibility to establish the same theorem for different types of discrete Laplace-Beltrami operators.

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