C^∞ Smooth Freeform Surfaces Over Hyperbolic Domains

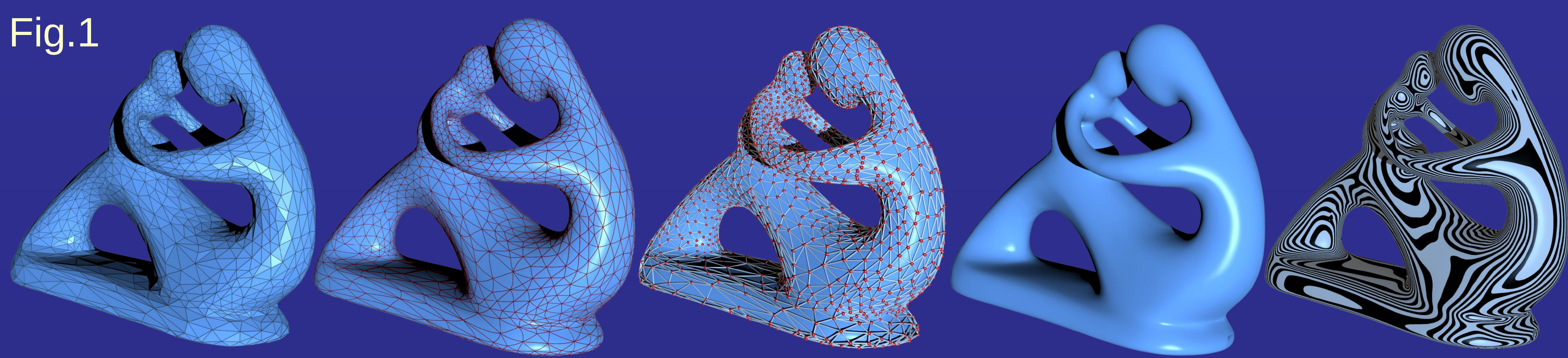
Wei Zeng [1] Ying He [2] Jiazhi Xia [2] Xianfeng Gu [1] Hong Qin [1]

[1] *Stony Brook University* [2] *Nanyang Technological University*

Highlights & Contributions

- A novel method to construct C^∞ *smooth surfaces* with negative Euler numbers ($\chi < 0$) based on hyperbolic geometry and discrete curvature flow.

Fig.1



- A true meshless method for modeling freeform surfaces with greatest flexibilities, without worrying about control point connectivity.

Advantages

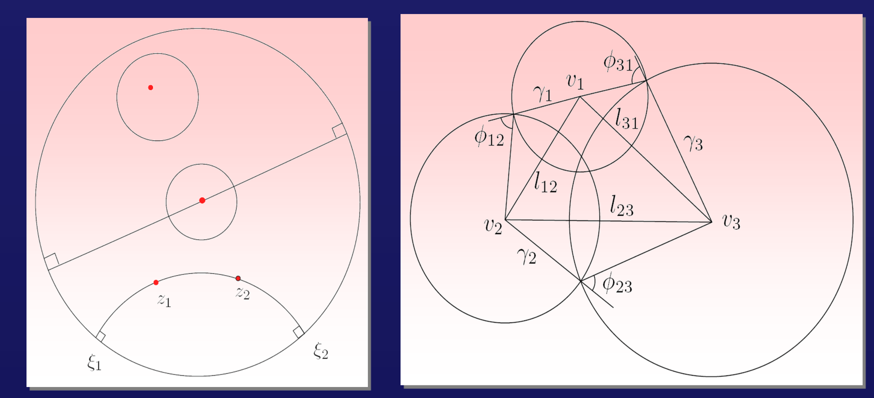
- General for arbitrary $\chi < 0$ surfaces.
- Independent of combinatorial structure (i.e. triangulation), and intrinsic.
- Freeform surface with C^∞ continuity and without any singularity.
- Associated with hyperbolic geometry and conformal structure of surface.

Previous work

- Conventional polynomial-based approach [Seidel 1993]: Manifold Spline with one singularity [He et al. 2006] [Gu et al. 2008] for $\chi < 0$ surfaces.
- Non-polynomial approach by smooth (e.g., *exponential*) functions: C^k-continuous [Grimm & Hughes 95], [Navau & Garcia 2000; Ying &Zorin 2004; Gallier et al. 2009; Vecchia et al. 2008; Siqueira et al. 2009] …
- Conformal structure: discrete Ricci flow [Jin et al. 2008]

Hyperbolic Ricci flow

- Universal Cover; Hyperbolic Geometry
- Ricci Energy for Circle Packing Metric
- Yamabe Equation



$$\overline{K} = e^{-2u}(-\Delta_g u + K) \qquad \frac{du_i(t)}{dt} = -K_i \qquad E(u) = \int_{u_0}^{u} \sum_{i=1}^{n} (\overline{K}_i - K_i) du_i$$

Proof for C∞ smoothness

- **Suppose S is a surface with negative Euler number. Then the hyperbolic freeform surface $F : S \to R^3$ is with C^∞ smoothness.**

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Algorithm

1. **Compute hyperbolic structure**: According to Riemann uniformization theorem, every surface with negative Euler number has a unique conformal Riemannian metric, which induces -1 Gaussian curvature everywhere.

Fig.2

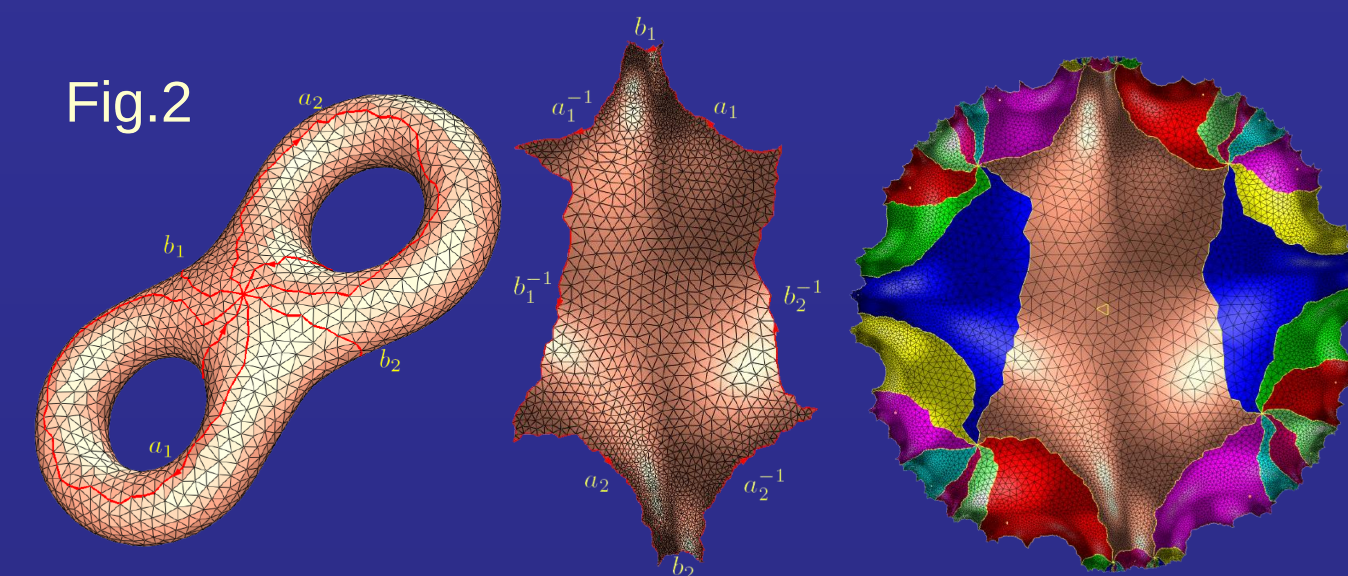
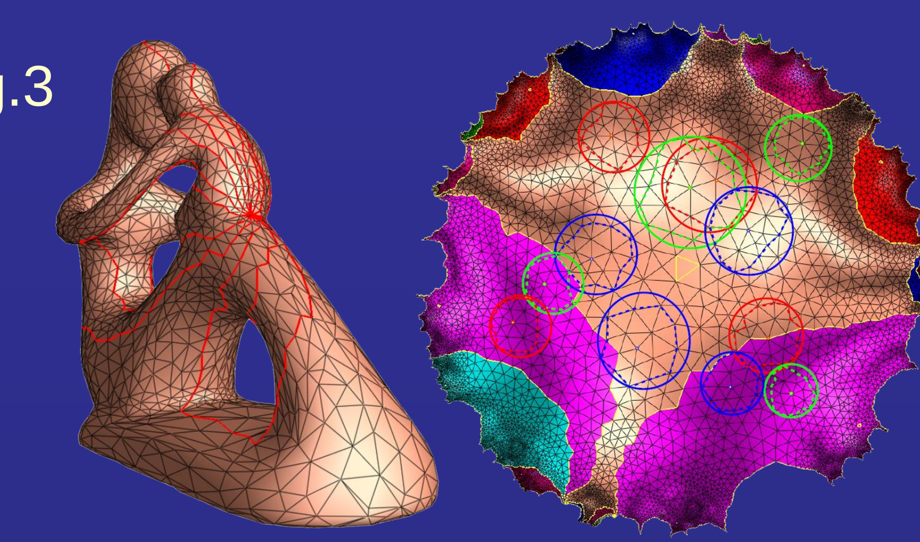


Fig.3



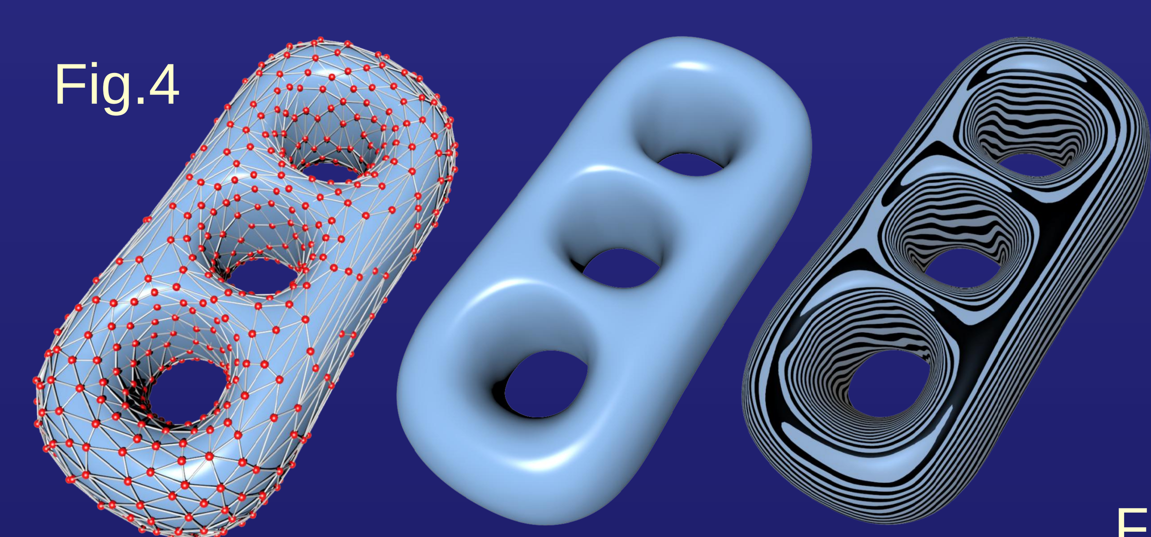
2. **Compute functional basis** f_i (exponential), Basis function B_i for control point, on *hyperbolic* (***H***²) *disk*, not on ***R***², ***C***². Freeform surface is built over hyperbolic domains while having C^∞ property, through the use of *partition-of-unity*.

Atlas {Ui, φi}, Control points C_i
Functional basis f_i: $\varphi_i(U_i) \to R$,
Basis function $B_i(p) = f_i(p)/\sum_j f_j(p)$
Function $F(p) = \sum_i C_i B_i(p)$

$$D_i(c_i, r_i) := \{p \in M \mid d_g(c_i, p) < r_i\}$$

$$f_{D_i}(p) = \begin{cases} \exp(r_i^{-2} - 1/(d_g(c_i,p) - r_i)^2 & p \in D_i(c_i, r_i) \\ 0 & p \notin D_i(c_i, r_i) \end{cases}$$

Fig.4



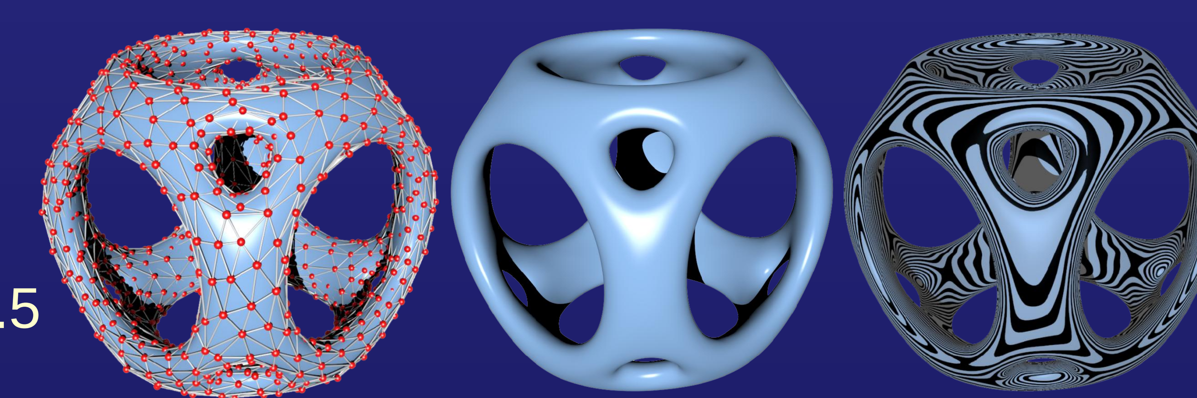


Fig.5

Experiments

- Comparison. Left: Loop subdivision [Loop 1987], right: C∞ freeform surface.

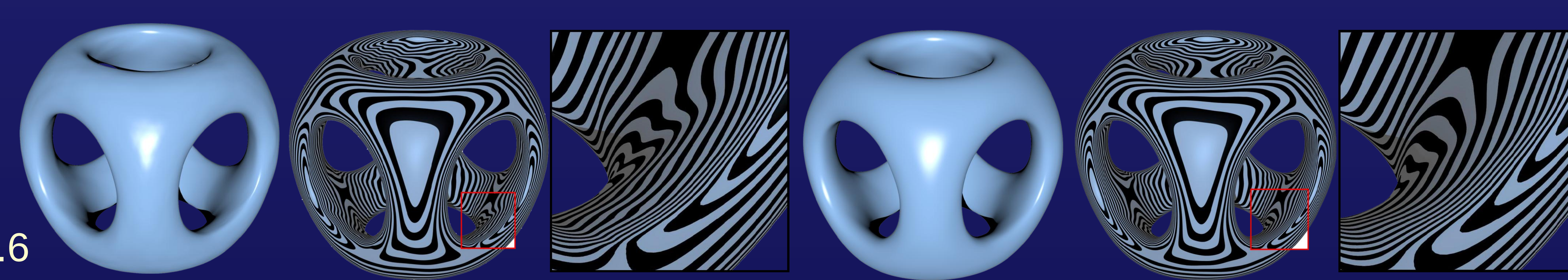


Fig.6

- Deformation of control mesh.
- General high genus surfaces.
- Statistics: 3GRAM, 2.66GCPU.

Fig.7

Genus	N_{ctrl}	N_{face}	T_{hyp} (s)	$T_{e\,eb}$(s)
3 Fig.4	746	1500	7	4
4 Fig.1	2108	4228	22	20
5 Fig. 6	492	1000	5	101
13 Fig.5	1234	2516	12	138

Fig.8