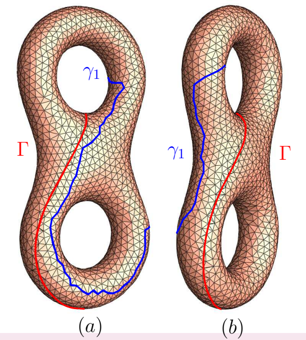
Canonical Homotopy Class Representative Using Hyperbolic Structure

Wei Zeng[1], Miao Jing[2], Feng Luo[3], Xianfeng David Gu[1]

[1]Department of Computer Science
SUNY at Stony Brook
[2]Center for Advanced Computer Studies
University of Louisiana at Lafayette
[3]Mathematics Department
Rutgers University

Tsinghua University

Canonical Homotopy Class Representative



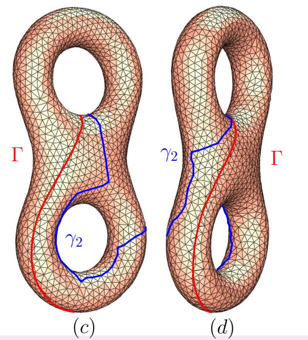


Figure: Two homotopic loops γ_1 and γ_2 are given. The canonical representative of their homotopy class is computed as the unique closed geodesic under the uniformization metric, shown as Γ.

Problem Statement

Definition (Loop)

Let S be a topological space, and let p_0 be a point of S. A loop with base point p_0 is a continuous function $\gamma : [0,1] \to S$, such that

$$\gamma(0) = p_0 = \gamma_1.$$

Definition (Homotopy)

Two loops γ_1, γ_2 are homotopic equivalent, if there exits a continuous map $h : [0,1] \times [0,1] \to S$, such that

$$h(t,0) = \gamma_0(t), h(t,1) = \gamma_1(t).$$

and

$$h(0,t) = p_0 = h(1,t).$$

h is called a homotopy from γ_0 to γ_1, and the corresponding equivalence class is called the homotopy class.

Problem Statement

Definition (product of loops)

The product $\gamma_0 \times \gamma_1$ of two loops γ_0 and γ_1 is defined by setting

$$(\gamma_0 \times \gamma_1)(t) := \begin{cases} \gamma_0(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_1(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Definition (inverse of a loop)

The inverse of a loop γ is the loop γ^{-1} defined by

$$\gamma^{-1}(t) = \gamma(1-t).$$

Problem Statement

The product of two homotopy classes of loops $[\gamma_0]$ and $[\gamma_1]$ is then defined as $[\gamma_0 \times \gamma_1]$, and this product does not depend on the choice of representatives.

Definition (Fundamental Group)

With the above product, the set of all homotopy classes of loops with base point p_0 forms the fundamental group of S at the point p_0 and is denoted $\pi_1(S,p_0)$. The identity element is the constant map at the basepoint.

If S is path-connected, fundamental groups with different base points are isomorphic. Therefore, we can write $\pi(S)$ instead of $\pi(S,p_0)$ without ambiguity whenever we care about the isomorphism class only.

Problem Statement

Homotopy Class Representative

Given a high genus metric surface S, with genus $g > 1$ and a Riemannian metric $\mathbf{g}$, define and compute the unique representative for each homotopy class.

Comparison

Comparison to Handle and Tunnel Loops

Geometry-aware handle loop and tunnel loop are the unique representatives for the corresponding **homology** classes. our method is for **homotopy** class. Each homology class has **infinite** number of homotopy classes, therefore our method is much more refiner.

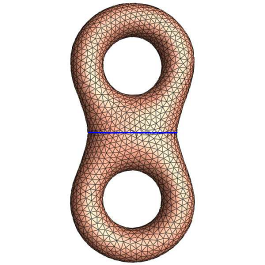


Figure: γ is homologous to zero, but homotopic nontrivial.

Solution

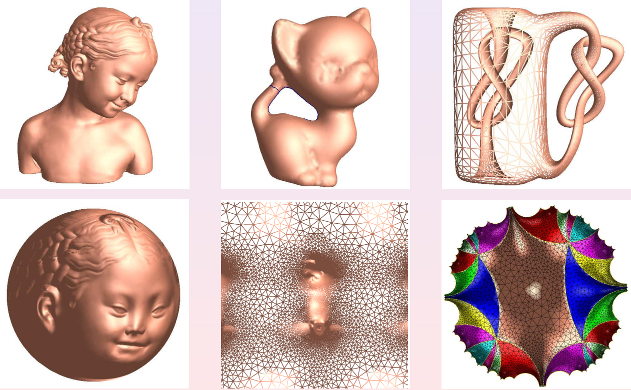
Solution to Homotopy Class Representative Problem

1. Compute the canonical uniformization metric $\tilde{\mathbf{g}}$, such that
   1. $\tilde{\mathbf{g}}$ is conformal to the original metric $\mathbf{g}$.
   2. $\tilde{\mathbf{g}}$ induces -1 constant Gaussian curvature everywhere.
2. Compute the unique geodesic loop Γ, which is homotopic to the input loop γ.

Theoretic Foundation - Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

Theoretic Foundation - Geodesic Uniqueness

Theorem (Gauss-Bonnet Theorem)

Let $(S,\mathbf{g})$ be a 2-dimensional Riemannian manifold with boundaries, then

$$\int_S K dA + \int_{\partial S} k_g ds = 2\pi\chi(S),$$

where K is the Gaussian curvature, k_g is the geodesic curvature $\chi(S)$ is the Euler number of S.

Corollary (Uniqueness of Geodesic Loop)

Let $(S,\mathbf{g})$ be a 2-dimensional Riemannian manifold with negative Gaussian curvature, then each homotopy class has a unique geodesic.

Application 1

Problem : Homotopy Detection

Given two loops γ_1 and γ_2 on a surface, verify if they are homotopic to each other.

Solution

Compute the unique representative Γ_1 of $[\gamma_1]$, Γ_2 of $[\gamma_2]$. If Γ_1 coincides with Γ_2, then γ_1 and γ_2 are homotopic.

Application 2

Problem: Shortest Word

Given a high genus surface S, and the generators of $\pi_1(S, p_0)$, a loop γ. Find the shortest word of $[\gamma]$ in $\pi_1(S, p_0)$.

Solution

Compute the unique representative Γ of $[\gamma]$, lift Γ in the universal covering space of S isometrically embedded in the hyperbolic space with the uniformization metric. Compute the word in the hyperbolic space.

How to compute the metric? Ricci flow!

Definition (Hamilton's Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -Kg_{ij}.$$

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant every where.

Ricci Flow

Theorem (Hamilton 1982)

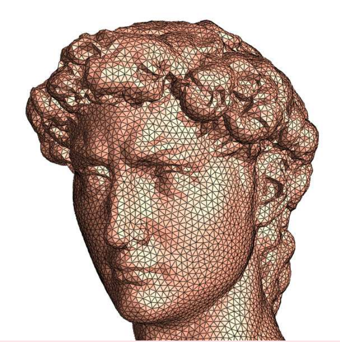
For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

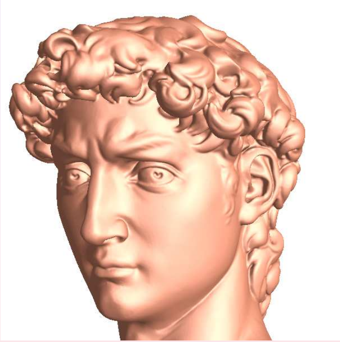
Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{H}^2$.



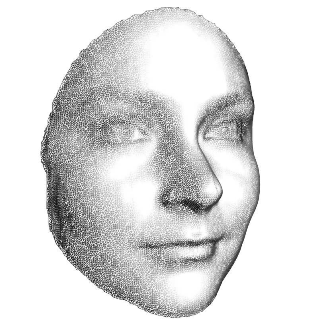


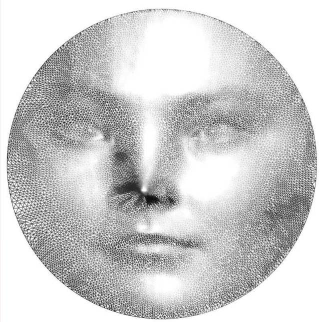
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \rightarrow \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





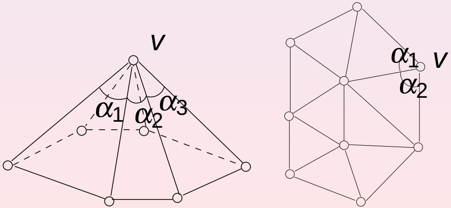
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

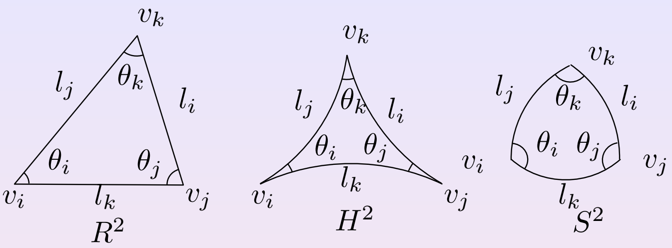
$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics Determines the Curvatures



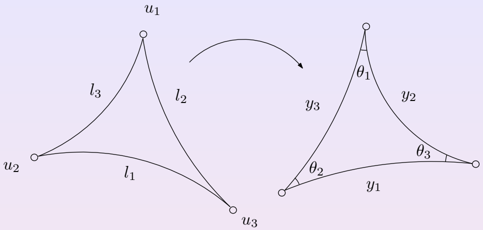
cosine laws

$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \quad (1)$$

$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} \quad (2)$$

$$1 = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \quad (3)$$

Discrete Conformal Factor for Yamabe Flow



conformal factor

The following formula is given in [25] Bobenko, Springborn and Pinkall "Discrete conformal equivalence and ideal hyperbolic polyhedra".

$$\sinh \frac{y_k}{2} = e^{u_i} \sinh \frac{l_k}{2} e^{u_j}$$

Properties: $\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$ and $d\mathbf{K} = \Delta d\mathbf{u}$.

Discrete Curvature Flow

Analogy

- Curvature flow
$$\frac{du}{dt} = \bar{K} - K,$$
- Energy
$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$
- Hessian of E denoted as Δ,
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

Discrete Curvature Flow

Theorem (25 Bobenko, Springborn, Pinkall)

The discrete hyperbic Yamabe energy is convex.

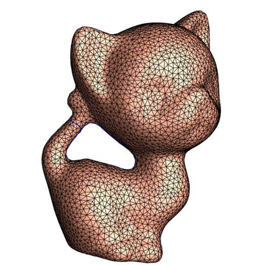
1. If solution exits, it is unique.
2. No theoretic proof for the existence yet.
3. The u-domain is not convex, the step length need to be carefully controlled during the optimization.

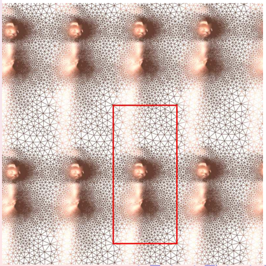
How to compute the geodesic? Axis of the Möbius transformation!

Universal Covering Space

Definition

Universal Cover A *covering space* of S is a space $\tilde{S}$ together with a continuous surjective map $h : \tilde{S} \to S$, such that for every $p \in S$ there exists an open neighborhood U of p such that $h^{-1}(U)$ is a disjoint union of open sets in $\tilde{S}$ each of which is mapped homeomorphically onto U by h. The map h is called the *covering map*. A simply connected covering space is a *universal cover*.

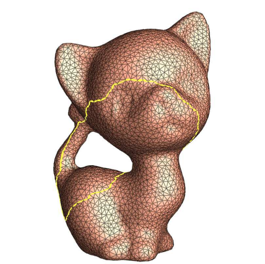


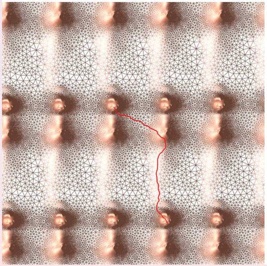


Lift

Definition (Lift)

Suppose $\gamma \subset S$ is a loop through the base point p on S. Let $\tilde{p}_0 \in \tilde{S}$ be a preimage of the base point p, $\tilde{p}_0 \in h^{-1}(p)$, then there exists a unique path $\tilde{\gamma} \subset \tilde{S}$ lying over γ (i.e. $h(\tilde{\gamma}) = \gamma$) and $\tilde{\gamma}(0) = \tilde{p}_0$. $\tilde{\gamma}$ is a *lift* of γ.





Deck Transformation

Definition (Deck Transformation)

A *deck transformation* of a cover $h : \tilde{S} \to S$ is a homeomorphism $f : \tilde{S} \to \tilde{S}$ such that $h \circ f = h$. All deck transformations form a group, the so-called *deck transformation group*.

Deck transformation group is isomorphic to the fundamental group.

Poincaré disk model

Definition (Poincaré disk model)

Poincaré disk is to model the hyperbolic space $\mathbb{H}^2$, which is the unit disk $|z| < 1$ with the metric $ds^2 = \frac{4dzd\bar{z}}{(1-z\bar{z})^2}$.

The rigid motion is the Möbius transformation

$$z \to e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 z},$$

where θ and z_0 are parameters. The geodesic of Poincaré disk is a Euclidean circular arc, which is perpendicular to the unit circle.

Funchsian Group

Definition (Fuchsian Group)

Suppose S is a high genus closed surface with the hyperbolic uniformization metric $\tilde{\mathbf{g}}$. Its universal covering space $(\tilde{S},\tilde{\mathbf{g}})$ can be isometrically embedded in $\mathbb{H}^2$. Any deck transformation of $\tilde{S}$ is a Möbius transformation, and called a *Fuchsian transformation*. The deck transformation group is called the *Fuchsian group* of S.

Axis of a Fuchsian Transformation

Definition (Poincaré disk model)

Let ϕ be a Fuchsian transformation, let $z \in \mathbb{H}^2$, the *attractor* and *repulser* of ϕ are $\lim_{n\to\infty} \phi^n(z)$ and $\lim_{n\to\infty} \phi^{-n}(z)$ respectively. The *axis* of ϕ is the unique geodesic through its attractor and repulser.

Geodesic Representative

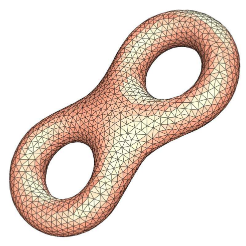
Theorem (Geodesic Representative)

Suppose a high genus surface S is with the uniformization metric. γ is a loop on S, $[\gamma] \in \pi_1(S)$, there exists a unique Fuchsian transformation $\phi \in Fuchs(S)$, then the unique geodesic loop in $[\gamma]$ is the axis of ϕ.

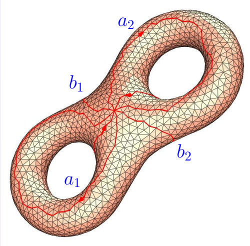
Given γ, we can lift it to the universal covering space, this gives the Fuchsian transformation ϕ.

Algorithm

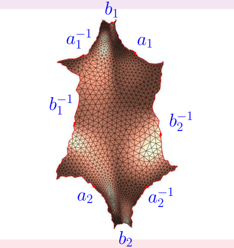
Algorithm Pipeline - Stage One



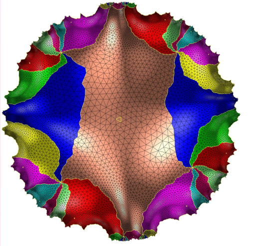
(a) Input genus two surface



(b) Canonical homotopy group basis

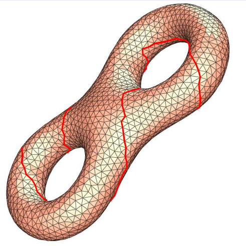


(c) Fundamental domain

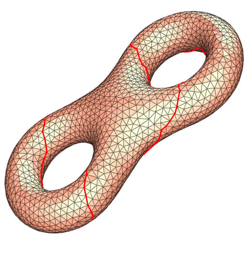


(d) Portion of universal covering space

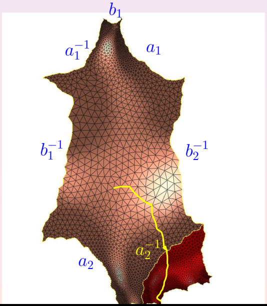
Algorithm Pipeline - Stage Two

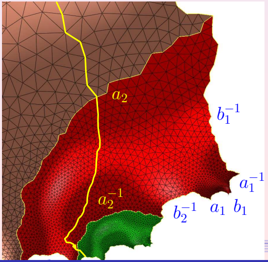


(a) Input loop front view

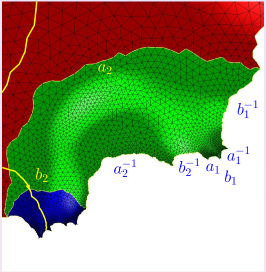


(b) Input loop back view

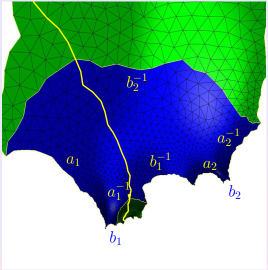




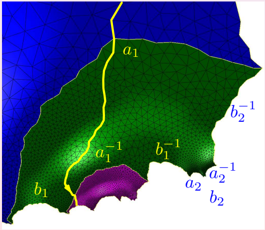
Algorithm Pipeline - Stage Two



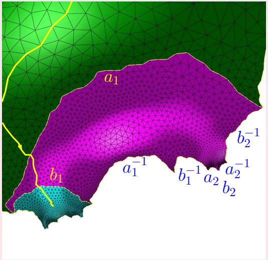
(e) 3. pass through b_2



(f) 4. pass through a_1^{-1}

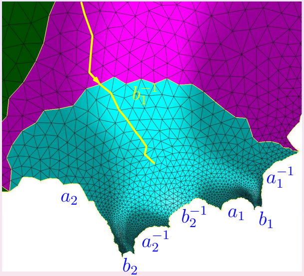


(g) 5. pass through a^{-1}

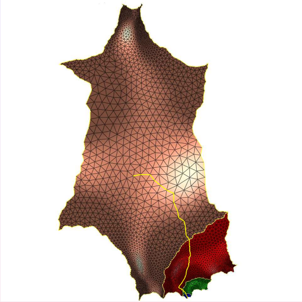


(h) 6. pass through b

Algorithm Pipeline - Stage Two

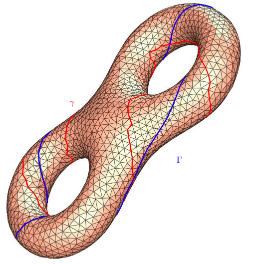


(i) Final ending

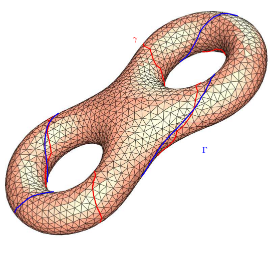


(j) Whole lift in $\mathbb{H}^2$

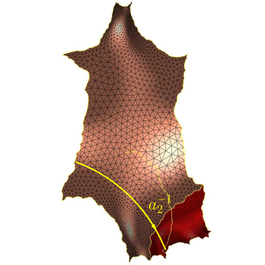
Algorithm Pipeline - Stage Two



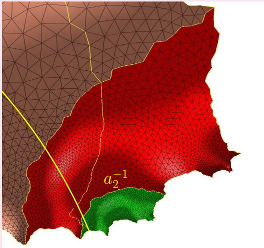
(a) Closed geodesic front view



(b) Closed geodesic back view

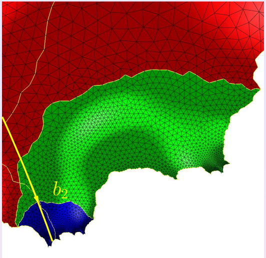


(c) 1 pass through a_2^{-1}

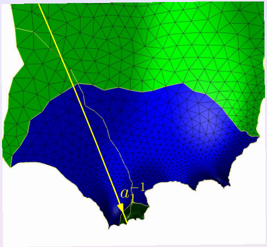


(d) 2. pass through a_2^{-1}

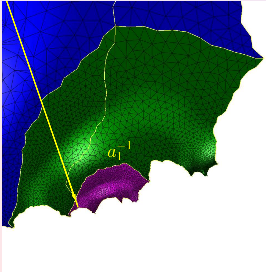
Algorithm Pipeline - Stage Two



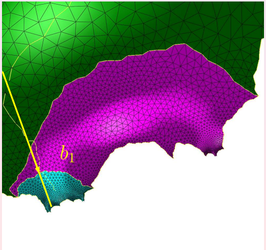
(e) 3.pass through b_2



(f) 4. pass through a_1^{-1}

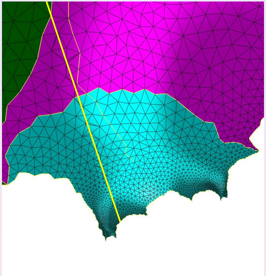


(g) 5.pass through a_1^{-1}

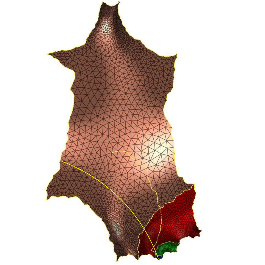


(h) 6. pass through b_1

Algorithm Pipeline - Stage Two



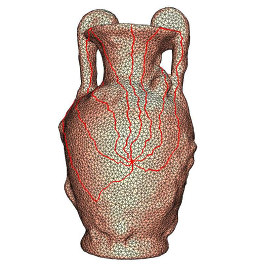
(i) Final ending



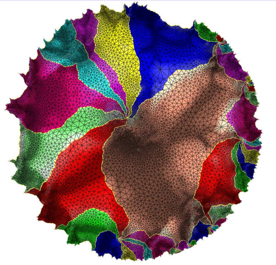
(j) Whole lift in $\mathbb{H}^2$

Experimental Results

Hyperbolic Fuchsian Group Generators

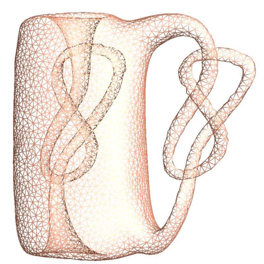


(a) Fundamental group generators

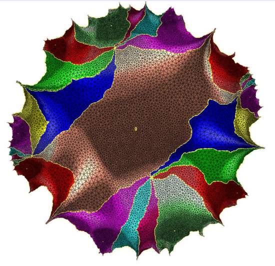


(b) Universal covering space

Figure: Hyperbolic metric and the Fuchsian group generators for the Amphora model.



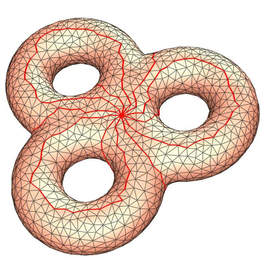
(a) Fundamental group generators



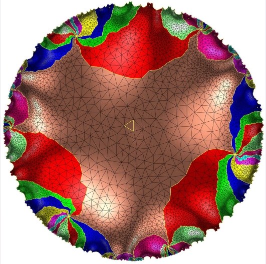
(b) Universal covering space

Figure: Hyperbolic metric and the Fuchsian group generators for the Knotty model.

Hyperbolic Fuchsian Group Generators



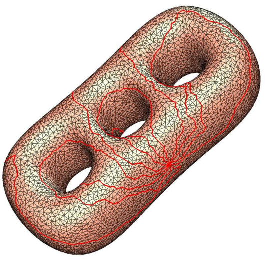
(a) Fundamental group generators



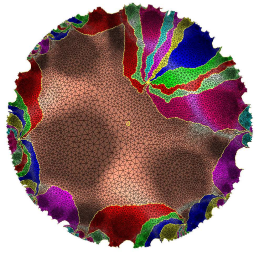
(b) Universal covering space

Figure: Hyperbolic metric and the Fuchsian group generators for the 3-hole model.

Hyperbolic Fuchsian Group Generators



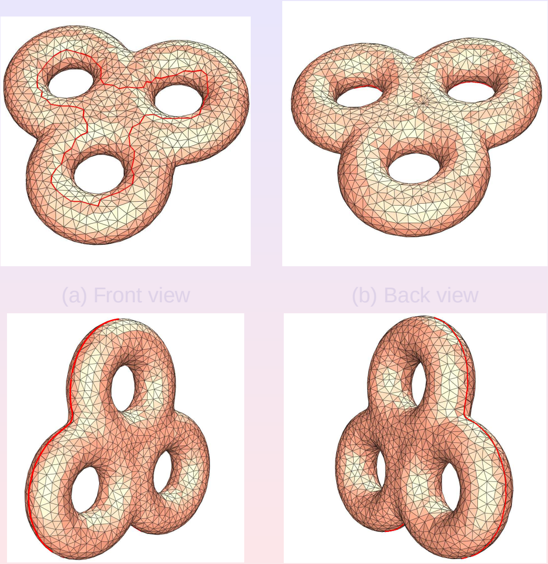
(a) Fundamental group generators



(b) Universal covering space

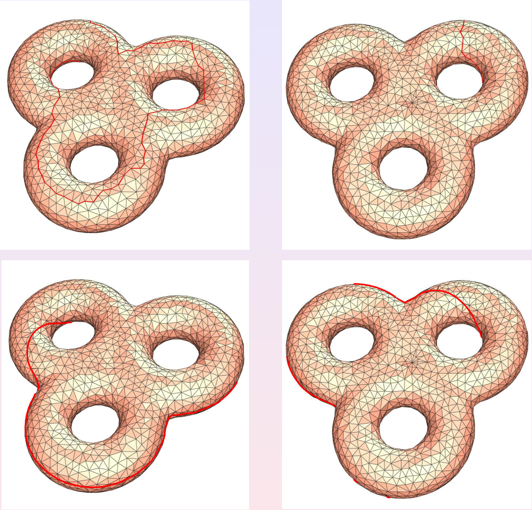
Figure: Hyperbolic metric and the Fuchsian group generators for the 3-torus model.

Hyperbolic Fuchsian Group Generators



(a) Front view (b) Back view (c) Left view (d) Right view

Hyperbolic Fuchsian Group Generators

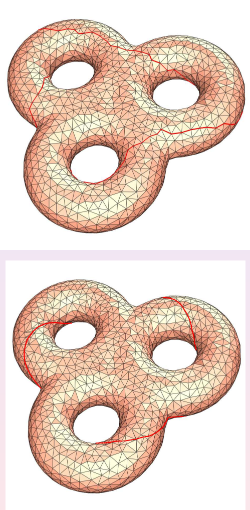


(a) Front view

(b) Back view

Figure: Homotopy geodesic on 3-hole torus 2.

Hyperbolic Fuchsian Group Generators


(a) Front view

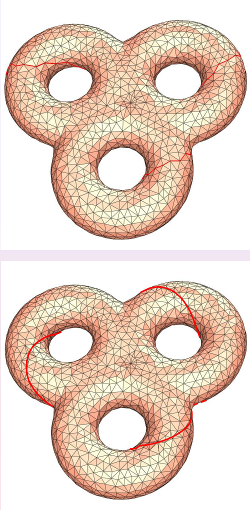
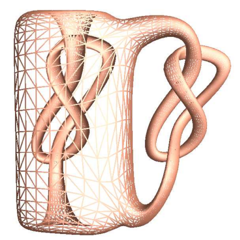
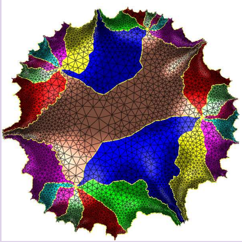

(b) Back view

Figure: Homotopy geodesic on 3-hole torus 3.

Thanks

For more information, please email to gu@cs.sunysb.edu.





Thank you!