Shape Analysis with Conformal Invariants for Multiply Connected Domains and its Application to Analyzing Brain Morphology

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Abstract

All surfaces can be classified by the conformal equivalence relation. Conformal invariants, which are shape indices that can be defined intrinsically on a surface, may be used to identify which surfaces are conformally equivalent, and they can also be used to measure surface deformation. Here we propose to compute a conformal invariant, or shape index, that is associated with the perimeter of the inner concentric circle in the hyperbolic parameter plane. With the surface Ricci flow method, we can conformally map a multiply connected domain to a multi-hole disk and this conformal map can preserve the values of the conformal invariant. Our algorithm provides a stable method to map the values of this shape index in the 2D (hyperbolic space) parameter domain. We also applied this new shape index for analyzing abnormalities in brain morphology in Alzheimer's disease (AD) and Williams syndrome (WS). After cutting along various landmark curves on surface models of the cerebral cortex or hippocampus, we obtained multiple connected domains. We conformally projected the surfaces to hyperbolic plane with surface Ricci flow method, accurately computed the proposed conformal invariant for each selected landmark curve, and assembled these into a feature vector. We also detected group differences in brain structure based on multivariate analysis of the surface deformation tensors induced by these Ricci flow mappings. Experimental results with 3D MRI data from 80 subjects demonstrate that our method powerfully detects brain surface abnormalities when combined with a constrained harmonic map based surface registration method.

1. Introduction

Shape analysis is a key research topic in anatomical modeling, statistical comparisons of anatomy and medical image registration. In research studies that analyze brain morphology, many shape analysis methods have been proposed, such as spherical harmonic analysis (SPHARM) [9, 6], medial representations (M-reps) [19], and minimum description length approaches [7], etc.; these methods may be applied to analyze shape changes or abnormalities in subcortical brain structures. Even so, a stable method to compute transformation-invariant shape descriptors would be highly advantageous in this research field. Here we propose a novel and intrinsic method to compute conformal invariants (shape indices) on multiply connected domains and we apply it to study brain morphology in Alzheimer's disease and Williams syndrome. Our conformal invariants are based on the surface conformal structure and can be accurately computed using the surface Ricci flow method.

All oriented surfaces have conformal structures. If there exists a conformal map between two surfaces, they are conformally equivalent. All surfaces may be classified by the conformal equivalence relation. Any two conformally equivalent surfaces have the same conformal invariants. By computing and studying conformal invariants and their statistical behavior, we can provide a promising approach for describing local changes or abnormalities in anatomical morphology due to disease or development.

By solving the Yamabe equation with the discrete surface Ricci flow method, we can conformally parameterize a multiple boundary surface by a multi-hole disk. The resulting parameterizations do not have any singularities and they are intrinsic and stable. For applications in brain morphology research, first, we convert a closed 3D surface model of the cerebral cortex into a multiple boundary surface by cutting it along selected anatomical landmark curves. Secondly, we conformally parameterize each cortical surface by the Ricci flow method. Next, we accurately compute a conformal invariant, the perimeter of the inner concentric circle on the hyperbolic space. This measure is invariant in the hyperbolic plane under conformal transformations of the original surface, differing at most by a rigid motion.

We tested our algorithm on cortical and hippocampal surfaces extracted from 3D anatomical brain MRI scans. The proposed algorithm can map the profile of differences

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in surface morphology between healthy controls and AD or WS subjects. When combined with a harmonic map based surface registration method, the proposed algorithm provides a systematic way to identify a reasonable canonical surface to which other surfaces can be registered. In cortical surface data from a group of 40 WS individuals and a group of 40 matched healthy control subjects, the proposed algorithm significantly improved the detection of statistical significant differences in regional cortical surface areas. In this group study, we created maps of statistically significant differences in surface morphology in hemispheric cortical surfaces of patients with William syndrome versus normal controls.

Our major contributions in this work include: a way to compute a new conformal invariant, the perimeter of the inner concentric circle in hyperbolic space, on the Poincaré disk in the parameter domain. Our way to compute conformal parameterization is equivalent to solve an elliptic partial differential equation on surfaces. Numerically, it is equivalent to optimize a convex energy so that the global optimum exists and is unique. Thus our method offers a stable way to calculate conformal invariants in 2D parametric coordinates. To the best of our knowledge, it is the first work to apply conformal invariants to brain morphology research. In our experiments, by modeling conformal invariants as a random vector and calculating the Mahalanobis distance from any vector to individual members of two groups of subjects, our method can find a canonical surface that can be used as a target to which other surfaces can be registered. We show that this novel and simple method can dramatically improve the statistical detection power of a harmonic map based surface registration method, for detecting abnormalities in brain surface morphology.

1.1. Related Work

In the computational analysis of brain anatomy, volumetric measures of structures identified on 3D MRI have been used to study group differences in brain structure and also to predict diagnosis [3]. Recent work has also used shape-based features [17, 15], analyzing surface changes using pointwise displacements of surface meshes, local deformation tensors, or surface expansion factors, such as the Jacobian determinant of a surface based mapping. For closed surfaces homotopic to a sphere, spherical harmonics have commonly been used for shape analysis, as have their generalizations, e.g., eigenfunctions of the Laplace-Beltrami operator in a system of spherical coordinates. These shape indices are also rotation invariant, i.e., their values do not depend on the orientation of the surface in space. Shape analysis based on spherical harmonic basis functions (SPHARM) are usually conducted in three steps, based on a pre-computed spherical parameterization of the surface: (1) estimating SH coefficients of x, y and z-components with a least-squares procedure, (2) normalizing the orientation of the first-order ellipsoid and (3) reconstructing the surface at regularly spaced points on the sphere [22, 9, 21]. Chung et al. [6] proposed a weighted spherical harmonic representation. For a specific choice of weights, the weighted SPHARM is shown to be the least squares approximation to the solution of an anisotropic heat diffusion on the unit sphere. Davies et al. performed a study of anatomical shape abnormalities in schizophrenia, using the minimal distance length approach to statistically align hippocampal parameterizations [7]. For classification, Linear Discriminant Analysis (LDA) or principal geodesic analysis can be used to find the discriminant vector in the feature space for distinguishing diseased subjects from controls. Tosun et al. [28] proposed the use of three different shape measures to quantify cortical gyrification and complexity. Gorczowski [10] presented a framework for discriminant analysis of populations of 3D multi-object sets. In addition to a sampled medial mesh representation, m-rep [19], they also considered pose differences as an additional statistical feature to improve the shape classification results.

For brain surface flattening [8] and brain surface parameterization research, Schwartz et al. [20], and Timsari and Leahy [27] computed quasi-isometric flat maps of the cerebral cortex. Hurdal and Stephenson [14] reported a discrete mapping approach that uses circle packings to produce "flattened" images of cortical surfaces on the sphere, the Euclidean plane, and the hyperbolic plane. The maps obtained are quasi-conformal approximations of classical conformal maps. Haker et al. [1] implemented a finite element approximation for parameterizing brain surfaces via conformal mappings. Gu et al. [11] proposed a method to find a unique conformal mapping between any two genus zero manifolds by minimizing the harmonic energy of the map. The holomorphic 1-form based conformal parameterization [31] can conformally parameterize high genus surfaces with boundaries but the resulting mappings have singularities. Other brain surface conformal parametrization methods, the Ricci flow method [29] and slit map method [30] can handle surfaces with complicated topologies (boundaries and landmarks) without singularities. Wang et al. [32] applied the Ricci flow method to study statistical group differences in a group of 21 healthy controls and 21 subjects with Williams syndrome, showing the potential of these surface-based descriptors for localizing cortical shape abnormalities in genetic disorders of brain development.

2. Surface Ricci Flow Method

In this section, we introduce the theory of Ricci flow in the continuous setting, which will be generalized to the discrete setting in Section 3.

Let S be a surface embedded in $\mathbb{R}^3$. S has a Riemannian

metric induced by the Euclidean metric of $\mathbb{R}^3$, denoted by $\mathbf{g}$. Suppose $u : S \to \mathbb{R}$ is a scalar function defined on S. It can be verified that $\bar{\mathbf{g}} = e^{2u}\mathbf{g}$ is also a Riemannian metric on S. Furthermore, angles measured by $\mathbf{g}$ are equal to those measured by $\bar{\mathbf{g}}$. Therefore, we say $\bar{\mathbf{g}}$ is *conformal* to $\mathbf{g}$, e^{2u} is called the *conformal factor*.

When the Riemannian metric is conformally deformed, Gaussian curvatures will also be changed accordingly. The Gaussian curvature will become

$$\bar{K} = e^{-2u}(-\Delta_{\mathbf{g}} u + K), \tag{1}$$

where $\Delta_{\mathbf{g}}$ is the Laplacian-Beltrami operator under the original metric $\mathbf{g}$. The above equation is called the *Yamabe equation*. By solving the Yamabe equation, one can design a conformal metric $e^{2u}\mathbf{g}$ with a prescribed curvature $\bar{K}$.

The Yamabe equation can be solved using the *Ricci flow* method. The Ricci flow deforms the metric $\mathbf{g}(t)$ according to the Gaussian curvature $K(t)$ (induced by itself), where t is the time parameter

$$\frac{dg_{ij}(t)}{dt} = -2K(t)g_{ij}(t). \tag{2}$$

The Ricci flow defined in Eq. 2 is convergent and leads to a conformal metric with constant curvature; detailed proofs may be found in [13] for surfaces with non-positive Euler numbers, and in [4] for surfaces with positive Euler numbers.

The Ricci flow can be easily modified to compute a metric with a prescribed curvature $\bar{K}$ as the following,

$$\frac{du(t)}{dt} = 2(\bar{K} - K). \tag{3}$$

With this modification, the solution metric $\mathbf{g}(\infty)$ can be computed, which induces the curvature $\bar{K}$.

3. Theoretical Background on Discrete Surfaces

In engineering fields, smooth surfaces are often approximated by simplicial complexes (triangulated meshes). We denote a triangle mesh as Σ, a vertex set as V, an edge set as E, and a face set as F. e_{ij} represents the edge connecting vertices v_i and v_j, and f_{ijk} denotes the face formed by v_i, v_j, and v_k.

The edge lengths of a mesh Σ define the Riemannian metric, $l : E \to \mathbb{R}^+$. For each face f_{ijk}, the edge lengths satisfy the triangle inequality: $l_{ij} + l_{jk} > l_{ki}$.

The discrete Gaussian curvature K_i on a vertex $v_i \in \Sigma$ can be computed from the angle deficit,

$$K_i = \begin{cases} 2\pi - \sum_{f_{ijk} \in F} \theta_i^{jk}, & v_i \notin \partial\Sigma \\ \pi - \sum_{f_{ijk} \in F} \theta_i^{jk}, & v_i \in \partial\Sigma \end{cases} \tag{4}$$

where θ_i^{jk} represents the corner angle attached to vertex v_i in the face f_{ijk}, and $\partial\Sigma$ represents the boundary of the mesh.

The concept of the circle packing metric was introduced by Thurston in [26] for approximating conformal metric deformation. Conformal metric deformations, in the smooth case, preserve infinitesimal circles and the intersection angles among them. The discrete conformal deformation of metrics uses circles with finite radii to approximate these infinitesimal circles. Let Γ be a function defined on the vertices, $\Gamma : V \to \mathbb{R}^+$, which assigns a radius γ_i to the vertex v_i. Similarly, let Φ be a function defined on the edges, $\Phi : E \to [0, \frac{\pi}{2}]$, which assigns an acute angle $\Phi(e_{ij})$ to each edge e_{ij} and is called a *weight* function on the edges. The ordered pair consisting of a vertex radius function and an edge weight function on a mesh Σ, (Γ, Φ), is called a *circle packing metric* of Σ.

Two circle packing metrics (Γ_1, Φ_1) and (Γ_2, Φ_2) on the same mesh are *conformally equivalent* if $\Phi_1 \equiv \Phi_2$. A *conformal deformation* of a circle packing metric only modifies the vertex radii and preserves the intersection angles on the edges. The discrete Ricci flow is defined as follows:

$$\frac{du_i(t)}{dt} = (\bar{K}_i - K_i), \tag{5}$$

where $\bar{\mathbf{k}} = (\bar{K}_1, \bar{K}_2, \cdots, \bar{K}_n)^T$ is the user defined target curvature, $u_i = \log \gamma_i$. The discrete Ricci flow can be formulated in a variational setting, as it can be represented as the negative gradient flow of a special energy form:

$$f(\mathbf{u}) = \int_{\mathbf{u_0}}^{\mathbf{u}} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i, \tag{6}$$

where $\mathbf{u_0}$ is an arbitrary initial metric. Therefore, the above integral is well-defined. It is the so-called *discrete Ricci energy*. The discrete Ricci flow is the negative gradient flow of the discrete Ricci energy. The discrete metric which induces $\bar{\mathbf{k}}$ is the minimizer of the energy.

Computing the desired metric with user-defined curvature $\bar{\mathbf{k}}$ is equivalent to minimizing the discrete Ricci energy. The discrete Ricci energy (see Eq. 6) has been proven to be strictly convex (namely, its Hessian is positive definite) in [5]. The global minimum uniquely exists, corresponding to the metric $\bar{\mathbf{u}}$, which induces $\bar{\mathbf{k}}$. The discrete Ricci flow converges to this global minimum.

3.1. Conformal Invariant for Multiply Connected Domains

A genus 0 surface with multiple boundaries is called a *multiply connected domain*. Let $\phi : \mathbb{D} \to \mathbb{D}$ be a conformal map from the unit disk $\mathbb{D}$ to itself. Here ϕ is a Möbius transformation. Let $z_0 \in \mathbb{D}$ be a point inside $\mathbb{D}$, then a

Möbius transformation that maps z_0 to the origin is given by

$$z \rightarrow e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Möbius transformations preserve circles. All Möbius transformations form a group.

The *hyperbolic metric* on the unit disk is defined as

$$ds^2 = \frac{dz d\bar{z}}{(1 - \bar{z}z)^2}, \quad (7)$$

which induces a constant Gaussian curvature, with the value -1 everywhere. Möbius transformations are isometric transformations for the hyperbolic metric.

The following theorem on the canonical conformal representation of multiply connected domains is fundamentally important for our current work.

Theorem 3.1 (Ahlfors) *Let S be a multiply connected domain with a Riemannian metric. Then there exists a conformal map, which maps S to the unit disk with circular holes. All such conformal maps, of this kind, differ by Möbius transformations.*

Let S be a multiply connected domain, and let the boundary of S consist of n connected components, $\partial S = c_1 \cup c_2 \cdots c_n$. The canonical conformal representation may be computed using the Ricci flow. The target curvature is set to be:

$$\bar{K}_i = \begin{cases} 0 & v_i \notin \partial S \\ 0 & v_i \in c_1 \cup c_2 \\ \frac{-2\pi}{|c_k|} & v_i \in c_k, k > 2 \end{cases}$$

where $|c_k|$ represents the number of vertices in c_k. After obtaining the desired metric, the target curvatures for vertices on $c_k, k > 2$ are updated as follows. Assume the edges on c_k are ordered, in a counter-clockwise fashion, as $\{e_1, e_2, \cdots, e_n\}$, and the two edges adjacent to $v_i \in c_k$ are e_i, e_{i+1}. Then the target curvature for v_i is updated as

$$\bar{K}_i = \frac{-\pi(|e_i| + |e_{i+1}|)}{\sum_j |e_j|}, \quad (8)$$

where $|e_j|$ represents the edge length of e_j under the new metric. After updating the target curvature, we use the Ricci flow to compute the desired metric again. After several iterations of using the Ricci flow and updating the target curvature, the process will converge, and this finally leads to a metric $\bar{g}$.

Then we compute the shortest path from c_1 to c_2, denoted by γ. The surface is sliced open along γ to form another surface $\tilde{S}$. Then we isometrically flatten $\tilde{S}$ onto a planar rectangle with holes using the new metric $\bar{g}$. c_1 and c_2 are mapped to two parallel edges of a rectangle, $c_k, k > 2$ are mapped to circular holes. Using a combination of rotation, scaling and the exponential map $z \rightarrow e^z$, we can map

the rectangle to an annulus, and therefore we can map the original surface S to a disk with circular holes, such that c_1 and c_2 are mapped to two concentric circles.

The perimeters of inner circles of the canonical conformal representation of S under the hyperbolic metric are invariant under Möbius transformations. Therefore, they are conformal invariants of the surface S.

4. Experimental Results

4.1. Conformal Invariant Computation

We applied our shape analysis to various anatomical surfaces extracted from 3D MRI scans of the brain. In this paper, the segmentations are regarded as given, and result from automated and manual segmentations detailed in other prior works, e.g. Thompson et al. [23] and Morra et al. [18], et al. We tested our algorithm on the left hippocampal surface, a key structure in the medial temporal lobe of the brain, for which parametric shape models are commonly developed for tracking shape differences or longitudinal atrophy in disease. Prior studies [23, 18] have shown that there is atrophy (volume loss and shape change) as the disease progresses. Figure 1 illustrates our testing results on the left hippocampal surfaces of a control subject and an AD patient. We leave two holes on the front and back of the hippocampal surface, representing its anterior junction with the amygdala, and its posterior limit as it turns into the white matter of the fornix. The surface model can be logically represented as an open boundary genus-one surface. The conformal maps of these two surfaces to a 1-hole disk are illustrated in the second column. Each of the two boundaries are mapped to a circle in the parameter domain. From the parameter domain, the computed conformal invariants are 0.009 (control) and 0.123 (AD), respectively. Although multi-subject studies are clearly necessary, this demonstrates our conformal invariants may potentially be useful as a shape index to classify and compare different hippocampal surfaces.

We also applied our algorithm to surface models of the cerebral cortex. The cerebral cortex and landmark data are the same ones used in [24]. The landmark data set included 10 selected landmark curves per hemisphere: Central Sulcus, Superior Temporal Sulcus Main Body, Inferior Frontal Sulcus, Middle Frontal Sulcus, Inferior Temporal Sulcus, Secondary Intermediate Sulcus, Transverse Occipital Sulcus, Inferior Callosal Outline Segment, Superior Rostral Sulcus, and Subparietal Sulcus. The definitions of these anatomical lines are reported in [25]. After we cut the cortical surface along each of these landmark curves, the cortical surface becomes topologically equivalent to an open boundary genus-9 surface. So the cortical surface can be conformally mapped to a multi-hole disk with 10 boundaries. The center and radius of each circle is determined by
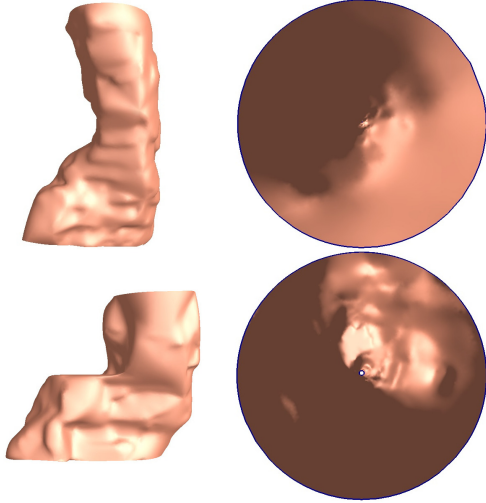
Figure 1. Conformal invariant illustration for hippocampal surfaces. There are two left hippocampal surfaces with their conformal parameterization results. The surface on the first row is from a control subject and the second row is from an AD patient. We leave two holes on the front and back of the hippocampal surface, representing its anterior junction with the amygdala and its posterior limit as it turns into the white matter of the fornix. Thus the surface is an open boundary surface with two boundaries. The computed conformal invariants are 0.009 (for the control subject) and 0.123 (for the AD patient), respectively.
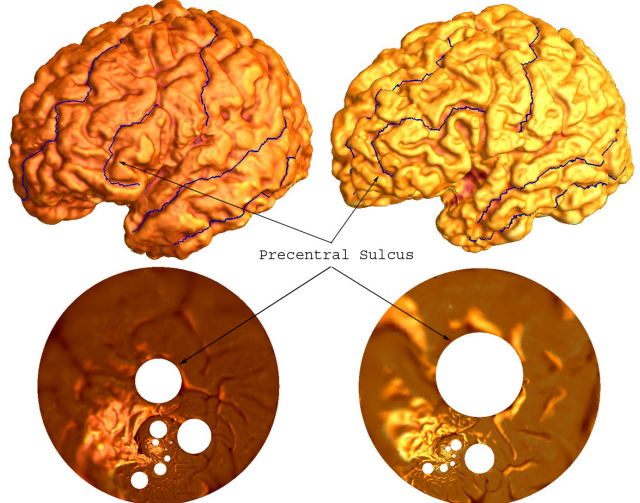

Figure 2. Conformal invariant computation for the Precentral Sulcus. The first column shows the left hemisphere cortical surface model and its parameterization for a control subject and the second column shows the same models for a WS patient. The second row shows their conformal parameterization results where the origin is the center of a circle to which the Precentral Sulcus landmark has been mapped. The computed conformal invariants are 0.472 (for the control subject) and 1.513 (for the WS patient), respectively.

the conformal structure of the original cortical surface.

Figure 2 illustrates the conformal parameterization of two left cortical surfaces, one is from a healthy control subject and one from a WS patient. The selected landmark curves are labeled in blue on the cortical surfaces. The second row is their conformal parameterization to a multi-hole unit disk. The perimeters of the inner circles are the conformal invariants. In Figure 2, we select a landmark curve, Precentral Sulcus, and have it as the concentric circle with the outer boundary. The computed conformal invariants are 0.47 (control) and 1.513 (WS), respectively. This suggests that the surface geometries surrounding these landmark curves have very different structures. This can be verified by observing the areas surrounding the curved landmark features in the first row of the figure.

4.2. Application of Conformal Invariants to Multivariate Statistics on Tensor Based Morphometry

A conformal invariant is computed for each landmark curve. The conformal invariant is associated with the surface conformal structure in the areas surrounding each landmark. When the cortical surface being analyzed has 10 landmark features lying in it, we can compute 9 conformal invariants per surface. We can then build a 9-element

feature vector as a compact description of the shape information in the cortical surface. The obtained feature vector can then be used to statistically analyze anatomical shapes and compare 3D anatomical models across subjects. Here we propose to use these conformal invariants to improve the detection power of multivariate statistics that are commonly applied to the deformation tensor in research studies of brain morphometry.

Our data set consists of cortical surface models extracted from 3D brain MRI scans of a group of 40 WS individuals and a group of 40 healthy control subjects [24]. We selected a group of 10 landmark curves (as explained in the previous section). After we cut a cortical surface open along the selected landmark curves, a cortical surface becomes topologically equivalent to an open boundary genus-9 surface. This surface can be conformally mapped to a multi-hole disk with 10 boundaries.

Because of the shape differences between different cortices, the centers and the radii of the inner circles are different. To register these surfaces to each other, we need to apply a constrained harmonic map from each individual conformal map to a canonical multi-hole disk in the parameter domain. The constrained harmonic map can be computed as follows [29].

Given two surfaces S_1 and S_2, their punctured disk parameterizations are $\tau_1 : S_1 \rightarrow R^2$ and $\tau_2 : S_2 \rightarrow R^2$, we want to compute a map, $\phi : S_1 \rightarrow S_2$. Instead of di-

rectly computing ϕ, we can easily find a harmonic map between the parameter domains. We look for a harmonic map, $\tau : R^2 \to R^2$, such that $\tau \circ \tau_1(S_1) = \tau_2(S_2), \tau \circ \tau_1(\partial S_1) = \tau_2(\partial S_2), \Delta \tau = 0$. Then the map ϕ can be obtained by $\phi = \tau_1 \circ \tau \circ \tau_2^{-1}$. Since τ is a harmonic map while τ_1 and τ_2 are conformal map, the resulting ϕ is a harmonic map. For landmark curve matching, we guarantee the matching of both ends of the curves and match other parts based on unit speed parameterization on both curves.

We propose to use a multivariate statistics based on surface deformation tensors to study the cortical surface morphometry of the brain. Based on the surface matching results, the surface deformation tensors J were computed [32]. Using multivariate statistics on these deformation tensors, as in [16], we define the deformation tensors as $S = (J^T J)^{1/2}$. Instead of analyzing shape change based on the eigenvalues of the deformation tensor, we consider a new family of metrics, the "Log-Euclidean metrics" [2] and compute distances in log-Euclidean space based on Hotelling's T^2 test. For each point on the cortical surface, we ran permutation test with 5000 random assignments of subjects to groups to estimate the statistical significance of the areas with group differences in surface morphometry. Also, given a statistical threshold of p=0.05 at each surface point, we also applied permutation test to the overall rejection areas (i.e., using the suprathreshold area statistic) to evaluate the overall significance of the experimental results [23].

In prior work [32], the reference canonical multi-hole disk was arbitrarily chosen and statistics were subsequently
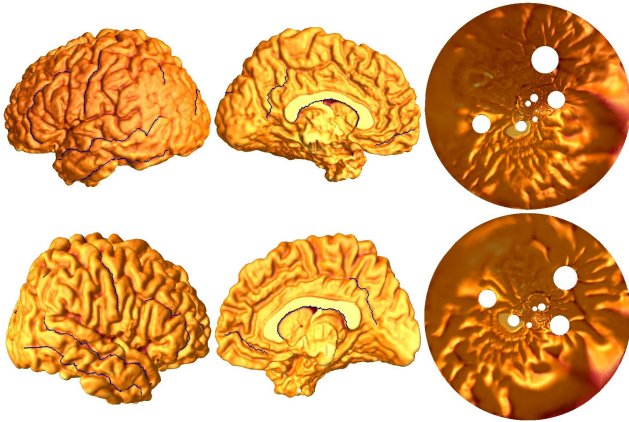

Figure 3. Illustration of selected surface templates for left (first row) and right (second row) hemispheres, for our multivariate statistical study of group differences in cortical morphometry. By computing the Mahalanobis distance from each individual cortical surface to each of the two groups, the ones with smallest sum of the two distances were selected. Their parameterization results were used as the template meshes to which all other cortical surfaces were registered.

plotted on this surface. For our studies of surface morphology, we need to develop more systematic way to select a parameterized surface as a target to which all other surfaces are registered. Since a brute force method that tries to use every surface in a data set would be extremely time consuming, we need some simple feature index to measure how well a particular disk might serve as the parameterization template. We propose to use the conformal invariant and Hotelling's T^2 test to find the "best" canonical parameterization.

After we compute the vector of conformal invariants, v, for each cortical surface, for a given hemisphere, we have two sets of feature vectors, $C = (x_1, x_2, ..., x_m)$ and $W = (y_1, y_2, ..., y_n)$, where m and n are the number of subjects in the control group and WS subject group, respectively (note that each subject has a feature vector). Based on the Mahalanobis distance, we can define distance as

$$
\begin{aligned}
d(v) \quad &= \sqrt{(v - \mu_C)^T \Sigma_C^{-1}(v - \mu_C)} \\
&+ \sqrt{(v - \mu_W)^T \Sigma_W^{-1}(v - \mu_W)}
\end{aligned}
$$

where μ_C, μ_W, Σ_C and Σ_W are the mean and covariance of two groups, respectively. The subject whose surface has the smallest distance to the others is selected as the canonical mesh for further statistical analysis of surface morphology. Figure 3 show the two selected template meshes and their parameterization results for left hemisphere (top row) and right hemisphere (bottom row) cortical surfaces with 10 landmark curves.

After fixing the template parameterization, we used Log-Euclidean metrics to establish a metric on the surface deformation tensors at each point, and conducted a permutation test on the suprathreshold area of the resulting Hotelling's T^2 statistics. The statistical maps are shown in Figure 4. The threshold for significance at each surface point was chosen to be p=0.05. The permutation-based overall significance p values, corrected for multiple comparisons, are $p = 0.001$ for the right hemisphere and 0.0006, for the left hemisphere, respectively.

To evaluate if our proposed conformal invariant vector really helped to improve the detection power, we also computed maps of statistically significant group differences based on using each of the other cortical surfaces as the template parameter mesh. For each statistically significant map, we measured the ratio of the statistically significant area over the whole surface area, and the overall corrected significance value, p. We found that the detected statistically significant areas were highly consistent, regardless of which surface was used as a registration target. We also found that for the two selected template meshes (i.e. with a conformal invariant vector that has least Mahalanobis distance to all the others), the significance area ratio was in the top of 5 of the whole data set and was 5% more than the median

of the whole data set. The overall statistical significance level p was also 60% less than the median p value of the whole data set. Although this deserves more validation on other data sets, these experimental results suggest that (1) our constrained harmonic surface registration method provides a reliable way to detect statistically significant areas of group differences for brain morphometry research; and (2) our proposed conformal invariance based feature vector can be used for template selection, dramatically improving the overall detection power.
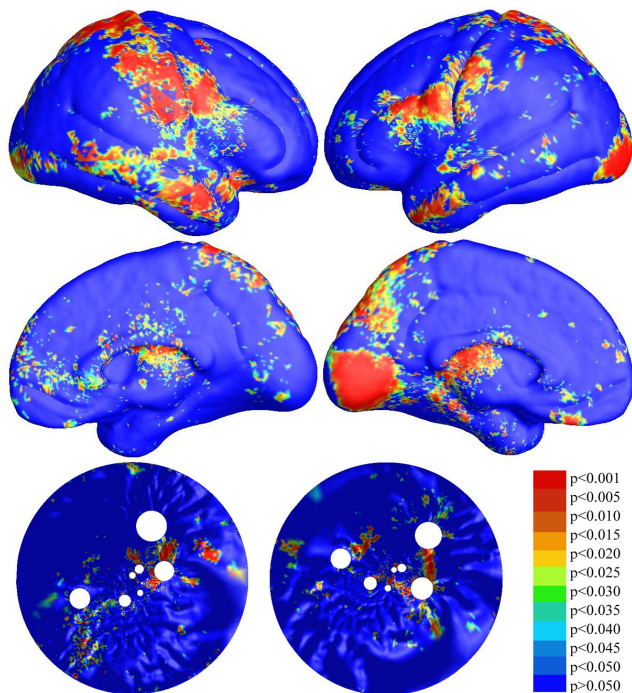

Figure 4. Map of statistically significant differents in cortical morphometry between 40 controls and 40 WS subjects. In the color-coded scale, non-blue colors denote the area where there is a significant statistical difference, at the $p = 0.05$ level. The first column shows the results for the right hemisphere and the second column shows the left hemisphere. The overall permutation significance p values are 0.001 and 0.0006, respectively, which correspond to highly significant group differences after controlling for the multiple comparisons involved in computing statistics at each point.

5. Conclusion and Future Work

In this paper, we propose a stable way to compute a conformal invariant (a vector-valued shape index) for a multiply connected domain. With the surface Ricci flow method, we can conformally map a $3D$ surface with open boundaries onto a punctured disk with multiple circular holes. The perimeters of inner circles under hyperbolic metric are the shape index. The computed shape index is invariant under

rigid-body transformation and conformal transformations of the original surface. It can also be used to measure the conformal deformation between two given surfaces with the same boundary conditions. We also applied the proposed method to detect systematic morphometric abnormalities in brain mapping research. We illustrated our method on hippocampal surfaces and cortical surfaces. The proposed method can reliably detect shape differences, at the group level, between control subjects and patients. For a group study of surface morphology, the proposed method can also determine which surface should be used as the target surface for registration, with provably better power for detecting abnormalities. As a result, we created a map of statistically significant differences in morphology in the cortical surface for patients with Williams syndrome, versus matched healthy controls.

Conformal invariants are robust to rigid-body transformations and conformal transformations of the surfaces from which they are derived. Local geometry is well preserved under conformal mapping so conformal invariants are good candidate features for brain research on cortical and subcortical surface morphology. In future, we will study more conformal invariants, such as the period matrix for high-genus surfaces [12], etc. Conformal maps clearly provide several reliable and intrinsic shape features. By applying some popular statistical classifiers, such as the decision tree and support vector machine, we plan to carefully study and validate numerous applications of these conformal invariants for shape analysis in neuroimaging research.

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