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Visualizing the Evolutions of Silhouettes

Abstract Silhouettes play a crucial role in visualization, graphics and vision. This work focuses on the global behaviors of silhouettes, especially their topological evolutions, such as their splitting, merging, appearing and disappearing. The dynamics of silhouettes are governed by the topology, the curvature of the surface, and the viewpoint.

Some theoretical results are established: the integration of signed geodesic curvature along a silhouette is equal to the view cone angle; critical events can only happen when the view point is on the aspect surfaces (ruled surface of the asymptotic lines of parabolic points).

We introduce a method to visualize the evolution of silhouettes, especially all the critical events where the topologies of the silhouettes change. The results have broad applications in computer vision for recognition, graphics for rendering and visualization.

Keywords silhouette · geodesic curvature · topological change · normal curvature · cusp · projection · aspect graph

1 Introduction

Silhouettes refer to the locus of points on the surface where the view rays tangentially touch the surface. *Projected silhouettes* refer to the projection images of silhouettes.

Silhouettes play a crucial role in computer vision. Projected silhouettes convey rich geometric information about the original surface. For example, the inflection point of projected silhouettes corresponds to the parabolic point on the original surface. The curvature signs of points on the projected silhouettes are consistent with the Gaussian curvature sign of their pre-images on the surface. Silhouettes have been applied for surface reconstruction, pattern recognition and aspect graphs in computer vision.

Silhouettes are one of the major research focuses in Non-Photorealistic Rendering in graphics. For human visual perception, silhouettes carry the most important shape information. Accurately and efficiently computing the silhouettes has attracted many researchers. In order to improve the efficiency of computing silhouettes, it is highly desirable to interpolate silhouettes from pre-computed ones when the view is moved between sampled views. If the topological structures of the pre-computed silhouettes are consistent, the interpolation is sensible and easy to perform. Therefore, it is critical to fully understand the topological evolutions of silhouettes when the view point is moved in space.

The major goal of this work is to study when, where, and how the topologies of the silhouettes will change, along with the global properties of the silhouettes. Silhouettes can shrink to a point and disappear, intersect each other either transversally or tangentially and reconnect. These critical events can only happen when the view point is on the ruled surface of asymptotic lines of parabolic points. Furthermore, we show that the integration of the geodesic curvature along a closed smooth silhouette is equal to the view cone angle.

Contributions In this work, we make the following contributions:

– Visualization of all possible topological changes of a silhouette.
– Development of a theorem of the relation of geodesic curvature of a silhouette and the view cone angle.
– Introduction of the concept of the aspect surface, all topological changes happen when the view is on the aspect surface. The silhouettes are homotopic to each other if two views can be connected by a curve which doesn't cross the aspect surface.

1.1 Previous Works

Non-Photorealistic Rendering Finding silhouettes from a given object is a key ingredient in Non-Photorealistic Rendering.

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A comprehensive review of silhouettes for NPR is not scope of this paper, and we refer the reader to the excellent books and survey papers [13, 37, 16] for details.

Usually, for non-photorealistic rendering silhouettes are rendered with a hybrid approach; object-space detection and image-space detection [28]. The object-space silhouette detection is performed with simple $\mathbf{n}(p)\cdot\mathbf{v}(p)$ computation, where $\mathbf{n}(p)$ and $\mathbf{v}(p)$ are the normal direction and the view direction defined with a point p on the object, respectively. The silhouettes are detected with the contours of zero-points on the object [15]. The image-space algorithm detects the visible portions of silhouette in the image-space, and generates meaningful strokes for stylization.

In contrast to previous work in computer graphics, we study the silhouette revolution on the object space with respect to viewpoint changes. Our methods could be utilized to the animation with non-photorealistic rendering scene, since the key problem in here is to smoothly interpolate the silhouettes between coherent frames [17].

Silhouette for Shape Recognition and Aspect Graph For 3D shape analysis and recognition from images [4, 11, 32], silhouettes are well studied in computer vision literatures [12, 25].

Since image-space silhouettes cannot convey sufficient information for non-trivial objects, several studies have been done on the connections between the evolution of images-space silhouettes and the 3D shape of objects. Pae and Ponce [29] studies the structural changes of silhouette of algebraic surfaces.

The aspect graph presents the topological changes of silhouettes in the image-spaces with respect to viewpoint changes. A node of the aspect graph corresponds to a topological configuration of image silhouettes of the object. The status changing from a node to another happens when a view point is at the critical position [19].

In contrast to the approaches in computer vision, we study the evolution of silhouettes on the object-space since we have exact 3D geometry in graphics applications. The topological changes of silhouettes on a surface are much rare than those of projected silhouettes in images, because projection introduces many singularities (e.g., the image of a smooth silhouette may contain several cusps). Moreover, we propose the novel concept of aspect surfaces, which is the locus of all the critical view points. By identifying the aspect surfaces, we can easily detect the topological changes of silhouettes for computer graphics and visualization applications.

Silhouette in singularity and catastrophe theory Also, in mathematics, silhouette is analyzed as the projections of surfaces to planes from a visual perspective[20]. All the critical events of projected silhouettes have been thoroughly classified in the works of Arnold [1], McCrory [26], Platonova [31] and Landis [21] in the setting of singularity and catastrophe theory.

Arnold et al[1][26] give us a classification of apparent contours of surfaces. Related work has been done by [31] and [21].

2 Local Properties of Silhouettes

This section aims at visualizing all possible critical events for silhouettes. The local properties of silhouettes have been thoroughly studied in computer vision, singularity theory and catastrophe theory. The curvature of projected silhouettes and the normal curvature are strongly related to the Gaussian curvature. All the possible critical events of projected silhouettes have been completely categorized. We follow the categories as explained in [1].

2.1 Preliminaries

The *Gauss map* $G:\Sigma\to\mathbb{S}^2$ maps a point $\mathbf{r}(u,v)\in\Sigma$ to its normal $\mathbf{n}(u,v)\in\mathbb{S}^2$.

Definition 1 The derivative map of the Gauss map $DG: T\Sigma\to T\mathbb{S}^2$ is called the *Weingarten map*,

$$d\mathbf{n} = -W(d\mathbf{r}) \quad (1)$$

$$d\mathbf{r} = \mathbf{r}_u du + \mathbf{r}_v dv \quad (2)$$

$$d\mathbf{n} = \mathbf{n}_u du + \mathbf{n}_v dv \quad (3)$$

A Weingarten map is a self conjugate linear map, where the eigenvalues k_1, k_2 of W are called *principal curvatures*, and the eigen vectors $\mathbf{e}_1, \mathbf{e}_2$ are called *principal directions*. The product of k_1, k_2 are called the *Gaussian curvature*.

The local behavior of a silhouette is determined by the view direction and the Gaussian curvature at the touching point. We classify the points on a surface according to their Gaussian curvature and study the local behavior of silhouettes on each class respectively.

Definition 2 On a smooth surface with at least C^2 continuity, k_1, k_2 are principal curvatures. All points are classified,

1. *elliptic*, $0 < k_1 \le k_2$,
2. *hyperbolic*, $k_1 < 0 < k_2$,
3. *parabolic*, $k_1 k_2 = 0$.

A special class of parabolic points are called *flat*, if both k_1 and k_2 are zeros.

For general surfaces, the locus of parabolic points is a finite number of curves on the surface. Silhouettes on them are also curves. For special surfaces with flat regions, silhouettes may contain some flat regions, and they are difficult to analyze with classical differential geometry. In the following discussion, we assume the flat regions are with zero measures.

Definition 3 An *asymptotic direction* for a parabolic point is the principal direction with zero principal curvature. An asymptotic direction $d\mathbf{r}$ on a hyperbolic point satisfies $< d\mathbf{r}, W(d\mathbf{r}) >= 0$, where $<,>$ is the inner product in $\mathbb{E}^3$. Two tangent vectors $d\mathbf{r}_1, d\mathbf{r}_2$ are *conjugate*, if $< d\mathbf{r}_1, W(d\mathbf{r}_2) >= 0$.

Asymptotic directions play important roles in analyzing the local behavior of silhouettes.

2.2 Relation between curvatures

The curvature of the projected silhouettes and the normal curvature along the view direction are strongly related to the Gaussian curvature. Assume the orthographic projection is π, a point p is on the silhouette γ, then $\pi(p) \in \pi(\gamma)$. The curvature of $\pi(p)$ is denoted as k_c. The view direction and the normal at p to the surface determines a plane, the plane intersect the surface at the sectional curve τ. The curvature of τ at p is denoted as k_r. According to Koenderink [19], the following equation holds

$$k_r(p)k_c(\pi(p)) = K(p),$$

where $K(p)$ is the Gaussian curvature of p, as shown in figure 1.

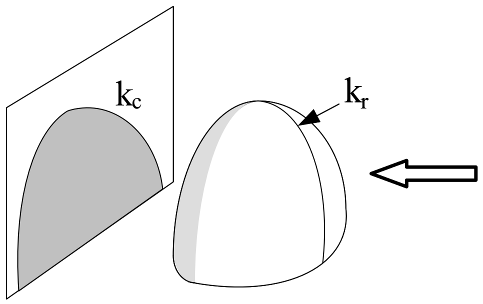


Fig. 1 Curvature relation $k_r k_c = K$. K is the Gaussican curvature at p, k_c is the curvature of $\pi(p)$, π is an orthographic projection, k_r is the normal curvature of the view ray direction.

At the cusps of the projected silhouette, the view direction is asymptotic to a hyperbolic point, $k_r = 0$, therefore k_c is infinite. We will show that at those cusps, the geodesic curvature is also zero. Therefore, those cusps correspond to an inflection point on the silhouette on the view cone surface.

An important corollary is that the sign of the curvature of a point on a visible projected silhouette is consistent with the sign of the Gaussian curvature of its pre-image on the surface. Therefore, the projected silhouette of a convex surface must be convex.

2.3 Critical events

Catastrophe theory [1] has indicated all possible critical events for projected silhouettes. The major approach to study the critical events is to classify points on the surface according to the maximum contact order of all tangency directions.

Definition 4 Suppose Σ is a generic smooth surface with position function $\mathbf{r}(u,v)$. A point $p \in \Sigma$, a tangent direction $\mathbf{t} \in T\Sigma(p)$ has *contact of order n*, if

$$\frac{\partial^k \mathbf{r}(u,v)}{\partial^k \mathbf{t}} = 0, k = 1, 2, \cdots, n-1,$$

and

$$\frac{\partial^n \mathbf{r}(u,v)}{\partial^n \mathbf{t}} \neq 0.$$

All points on the surface can be classified according to their order of tangency as shown in figure 2. The critical events corresponding to the special directions of each class are illustrated in figure 6.

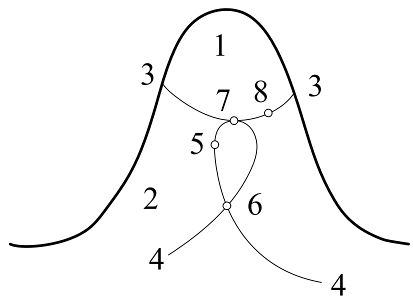


Fig. 2 Classification of points on a generic surface. The class types are labeled numerically (adopted from [1])

1. The elliptic domain, where all tangents are of order 2.
2. The hyperbolic domain, where each point has two asymptotic directions. The asymptotic directions have tangency order of 3. On the projected silhouettes, there will be a cusp as shown in the type 2 of the figure.
3. The curve of parabolic points, each point has one asymptotic direction. The lip event is shown as the type 3 and 10, the beak-to-beak event is shown as the type 4 and 11.
4. The envelops of asymptotic directions are called asymptotic curves. The inflections of the projection of the asymptotic curves onto the tangent plane form a flecnodal curve, where each point has a tangent of order 4 or greater. This is shown in type 6.
5. The self intersection point of a flecnodal curve, which is with two tangents of order 4.
6. The *biflecnodes*, which are the inflections of flecnodal curve, with a tangent of order 5. This is shown in types 8,9 and 13.
7. The *godrons*, which are points of tangency of the parabolic curve and the flecnodal curve, with a tangent of order 4. This is shown in type 7 and type 12.
8. The gutterpoints on parabolic curves, which are stationary points of the asymptotic tangents, as shown in type 5.

The topological changes happen when the view is along the asymptotic directions of some parabolic points: this case give birth to the lip (type 3,10)and beak-to-beak events (type 4, 11); the asymptotic directions along the parabolic curves and flecnodal curves give birth to the self intersection of the silhouettes on the surface (type 7, 12). The other critical events change the topologies of the projected silhouettes, but preserve the topologies of the silhouettes on the surface.

3 Global Properties of Silhouettes

This section aims at developing some theoretical results for the global properties of silhouettes without using any advanced machinery from singularity theory or catastrophe theory.

3.1 Aspect Surface

In this section, we define the *aspect surface*, and prove that all critical events for silhouettes happen when the view is on the aspect surface. If two views can be connected by a curve without intersecting the aspect surface, then the silhouettes are homotopic. The proofs use only local differential geometry.

Lemma 1 *Suppose* $\mathbf{r}(u,v)$ *is a generic smooth surface, with local parameters* (u,v)*. The view point is* ν*,* $\mathbf{r}(s)$ *is a silhouette. Then the tangent direction* $\dot{\mathbf{r}}$ *is conjugate to the view ray direction* $\mathbf{r}-\nu$*.*

Proof According to the definition of silhouette, $<\mathbf{r}-\nu,\mathbf{n}>=0$, take the derivative on both sides,

$$<\dot{\mathbf{r}},\mathbf{n}>+<\mathbf{r}-\nu,\dot{\mathbf{n}}>=0,$$

therefore

$$<\mathbf{r}-\nu,\dot{\mathbf{n}}>=<\mathbf{r}-\nu,W\dot{\mathbf{r}}>=0,$$

where W is the Weingarten map. □

Lemma 2 *Suppose* Γ *is a silhouette on a generic smooth surface* Σ *with a view point* ν*, which is not on the surface* $\nu\notin\Sigma$*. A point* $p\in\Gamma$ *is in one of the three cases*

- p *is elliptic,* $K(p)>0$*;*
- p *is hyperbolic,* $K(p)<0$*;*
- p *is parabolic, but the view direction* $p-\nu$ *is not along the asymptotic direction of* p*.*

then in a neighborhood of p*, the silhouette* Γ *is a one dimensional manifold.*

Proof We take a special local parameterization, such that ,at point p, $\mathbf{r}_u=\mathbf{e}_1,\mathbf{r}_v=\mathbf{e}_2$, where $\mathbf{e}_1,\mathbf{e}_2$ are principal directions. Then $\mathbf{n}_u=-k_1\mathbf{e}_1$, $\mathbf{n}_v=-k_2\mathbf{e}_2$.

The silhouette is the zero level set of the function

$$f(u,v)=<\mathbf{r}(u,v)-\nu,\mathbf{n}(u,v)>.$$

At the point p,

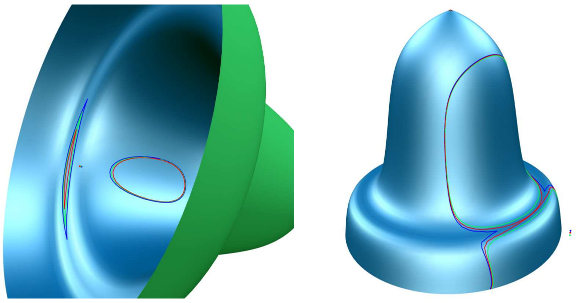
$$\frac{\partial f}{\partial u}=<\mathbf{r}-\nu,\mathbf{n}_u>$$
$$\frac{\partial f}{\partial v}=<\mathbf{r}-\nu,\mathbf{n}_v>$$

When $K(p)\neq 0$, since $\mathbf{r}-\nu\neq 0$, $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ cannot be both zeros.

When $K(p)=0$, assume $\mathbf{e}_1$ is the asymptotic direction and $\mathbf{r}(p)-\nu$ is not along $\mathbf{e}_1$, then $\mathbf{r}(p)-\nu$ is not orthogonal to $\mathbf{e}_2$, therefore $\frac{\partial f}{\partial v}$ is not zero.

From the implicit function theorem, we know that locally the solution of $f=0$ around p is a one dimensional submanifold. □

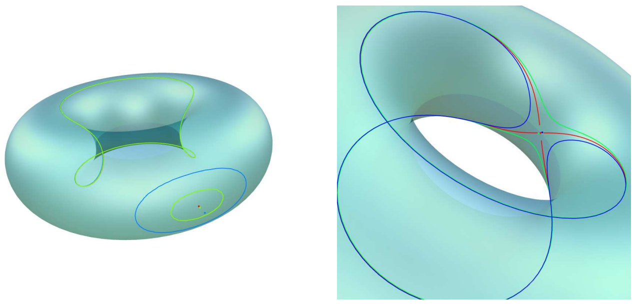
When the view is along the asymptotic direction of a parabolic point, the silhouettes may not be a one dimensional manifold, thus the lip event or the beak-to-beak event may happen as shown in figure 3, the three colored curves show the progress of topology changes of silhouettes.



a. lip event b. beak-to-beak event

Fig. 3 Critical events for a bell shape.

If the view point is on the surface, the following lemma depicts the topological change of the silhouettes. Figure 4 illustrates the evolution of the silhouettes when the view crosses the surface.



a. View crosses an elliptic point b. View crosses a hyperbolic point.

Fig. 4 Topological changes when the view crosses the surface. The view point and its corresponding silhouette are marked with same color.

Lemma 3 *Suppose* Σ *is a generic smooth surface, view point* ν *crosses the surface along the normal direction through* p *from outside to inside, then*

– *if p is elliptic, then a closed silhouette will shrink to the point p and disappear.*
– *if p is hyperbolic, two silhouettes will intersect and reconnect, the silhouettes are along the asymptotic directions of p.*

Proof For elliptic and hyperbolic points, we can use quadratic models $(x,y,z(x,y))$ to locally approximate the surface, where

$$z = \frac{1}{2}(k_1x^2 + k_2y^2),$$

By examining the silhouette evolutions when v crosses the surface along z direction, we can straight forwardly obtain the conclusion. □

For parabolic points, one has to use higher order approximation model as described in the section 2. From lemma 3 and 2, it is obvious that the topological changes of silhouettes can only happen when the view points are either on the object surface itself or along asymptotic directions of parabolic points.

Definition 5 Suppose Σ is a generic smooth surface, $\gamma(s)$ is a parabolic curve, at each point $\mathbf{e}(s)$ is the asymptotic direction of $\gamma(s)$. The following surface

$$\Gamma(s,t) = \gamma(s) + t\mathbf{e}(s),$$

is called the *aspect surface* of $\gamma(s)$. The union of the aspect surfaces of all parabolic curves and the surface Σ itself is called *the aspect surface* of Σ, and denoted as $\Omega(\Sigma)$.

Theorem 1 *Suppose Σ is a generic smooth surface. The topological changes of the silhouettes only happens when the view point v is on the aspect surface of Σ, $v \in \Omega(\Sigma)$.*

Proof It is obvious from lemma 2 and 3. □

Suppose Σ is a generic smooth surface, two views v_0 and v_1 are connected by a curve $v(t)$. Suppose the view curve doesn't intersect the aspect surface $\Omega(\Sigma)$, Γ_k is the silhouette for v_k, $k = 0,1$, then

$$\Gamma_k = \{\gamma_0^k, \gamma_1^k, \cdots, \gamma_n^k\},$$

such that γ_i^k are curve segments or loops, γ_i^0 is homotopic to γ_i^1.

3.2 Geodesic Curvature Relation

In this section, we discover a relation between the integration of the geodesic curvature of a silhouette and the view cone angle. To our surprise, this simple relation has not been discussed before. Perhaps, most research focuses only on extrinsic geometry.

Definition 6 Suppose Σ is a generic smooth surface. A view v is at a *generic position*, if it is not on the aspect surface, $v \notin \Omega(\Sigma)$. Otherwise, it is at a *critical position*.

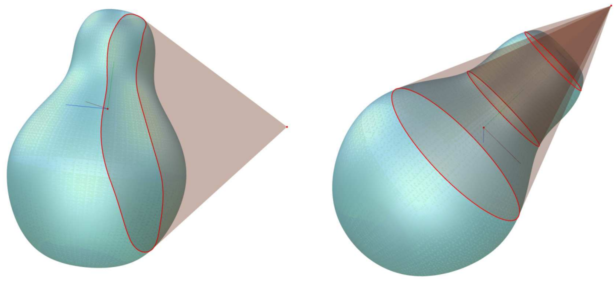


Fig. 5 View cones for a pear shape.

First we define the view cone surface as shown in figure 5,

Definition 7 Suppose the view v is in a generic position for a generic smooth surface. The distance between v and Σ is finite. A connected component of the silhouette is $\gamma(s)$. The *view cone surface* $\Lambda(s,t)$ is defined

$$\Lambda(s,t) = vt + \gamma(s)(1-t), t \in [0, +\infty)$$

The view cone surface Λ is tangent to the original surface Σ at the silhouette γ. But the orientation of the silhouette with respect to the visible region of Σ and the orientation with respect to the view cone may not be consistent. More precisely, we define an orientation of γ, such that if one travels along γ with the orientation, the visible region on Σ is always on the left hand side of γ. Then we treat γ as a curve on the view cone Λ, sometimes the view point v is on the left hand side of γ, sometimes it is on the right hand side. Then we define the following orientation function:

Definition 8 Suppose Σ is a generic smooth surface, v is the generic view point. γ is a silhouette with consistent orientation of the visible region on Σ. The view cone surface is Λ. Then orientation function ϕ is defined as

$$\phi : \gamma \to \{+1, 0, -1\}, \phi(p) = sign < (v - p) \times \dot{\gamma}, \mathbf{n}(p) >; \quad (4)$$

where $\mathbf{n}(p)$ is the normal to the surface Σ.

It is obvious that if $\phi(p) = 0$, then p must be a hyperbolic point, the view direction is along its asymptotic direction.

Lemma 4 *The geodesic curvature of a point p on the silhouette is zero when p is hyperbolic and the view direction is along its asymptotic direction.*

Proof At the asymptotic direction, the line of sight is tangent to the surface of order 3, therefore the curvature k of the silhouette at p is zero.

$$k^2 = k_n^2 + k_g^2,$$

therefore both k_n and k_g vanish. □

The view cone surface $\Lambda(s,t)$ is a developable surface. One can slice $\Lambda(s,t)$ open along a straight line $\Lambda(0,t)$, the angle at the view point is called the *cone angle at the view point*.

Theorem 2 *Suppose Σ is a generic smooth surface. The view v is not on the surface $v \notin \Sigma$. A closed silhouette $\gamma(s)$ is smooth (without cusps on Σ), where s is the arc length; the cone angle at the view point is Φ, then*

$$\int_{\gamma} k_g(s)ds = \Phi. \tag{5}$$

Proof Fix the orientation of γ such that it is consistent with both Σ and Λ. Denote its geodesic curvature as $k_g(s)$. Because the view cone surface Λ is tangent to the original surface Σ at the curve γ, the geodesic curvature of γ on Λ equals $k_g(s)$.

On the view cone surface Γ, $\gamma(s)$ bounds a topological disk. We use Gauss-Bonnet theorem,

$$\int_{\gamma} k_g(s)ds + K(v) = 2\pi,$$

where $K(v)$ is the discrete Gaussian curvature at the view point (v), $K(v) = 2\pi - \Phi$. □

When the view point moves to infinity, then the view cone angle Φ approaches 0, and the integration of the *signed geodesic curvature* goes to 0.

4 Experimental Results

We verified our theoretical results by tracing silhouettes as shown in figures 7, 8, and 9. The experimental results are consistent with our theorems and corollaries.

The silhouette tracing system is implemented in C++ on the windows platform with 3.6GHz CPU and 3.0G RAM. The silhouette tracing is efficient enough to allow the user to move views arbitrarily and visualize the evolution of the silhouettes in real time.

We computed the silhouettes, aspect surfaces for spline surfaces as shown in figure 9. When the view point crosses the aspect surfaces, the topologies of silhouettes are changed. From a to b, a beak-to-beak event happens, from b to e, the silhouette disappears, from e to h, a lip event happens. If the view is moved without touching the aspect surface, the silhouettes are homotopic.

5 Conclusion

This paper studies the global behavior of silhouettes. All the topological changes happen when the view is on the aspect surface. The typical topological changes are the lip event and the beak-to-beak event. Therefore, silhouettes of two views are homotopic if the views can be connected by a curve without crossing the aspect surface. The integration of a signed geodesic curvature along a silhouettes is equal to the view cone angle.

We also illustrate all possible critical events for projected silhouettes. Finally, we built a real time silhouette visualization.

In the future, we will design practical algorithms to interpolate silhouettes of different views for Non-Photo-Realistic Rendering purposes. Furthermore, we will explore other global properties of silhouettes.

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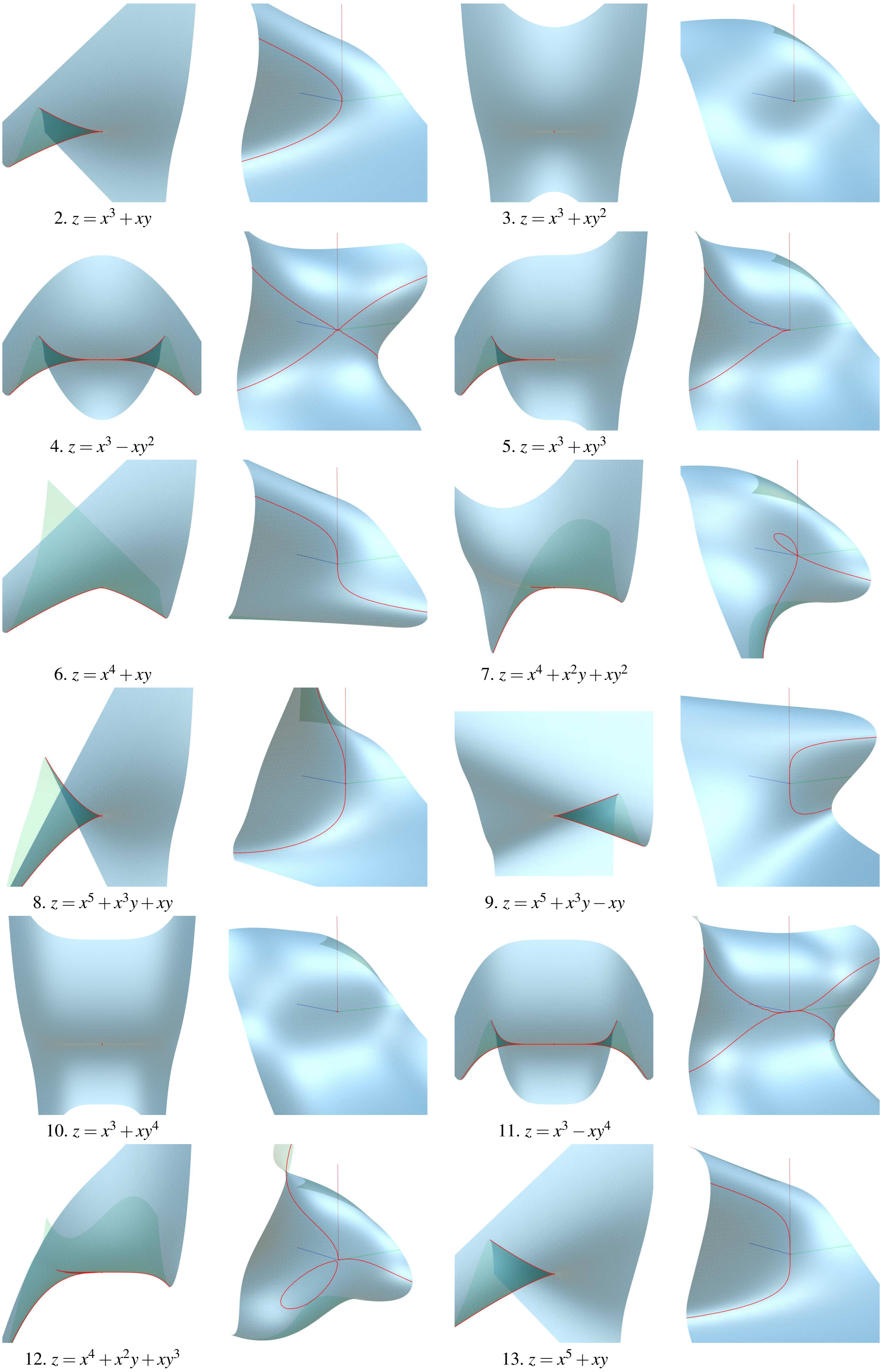

Fig. 6 All possible critical events for generic surfaces with arbitrary views.

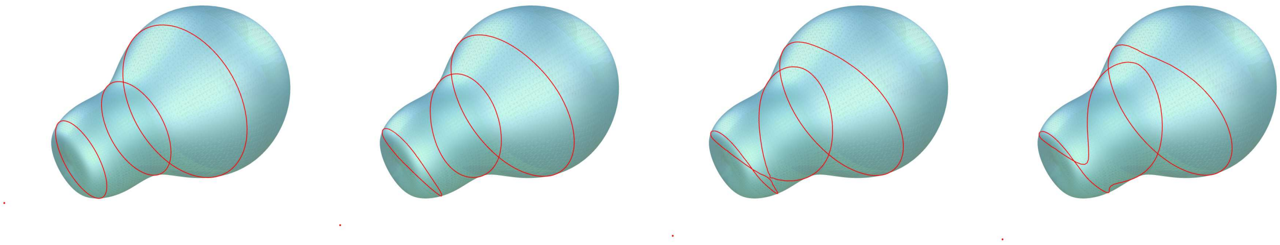

Fig. 7 Beak-to-beak critical event on a pear surface.

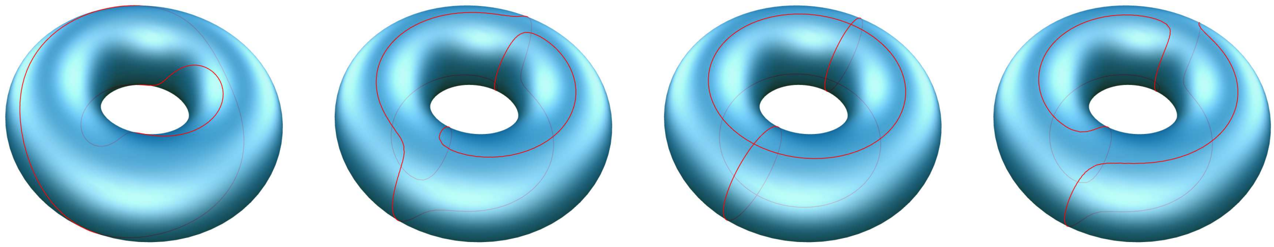

Fig. 8 Beak-to-beak critical event on a torus surface.

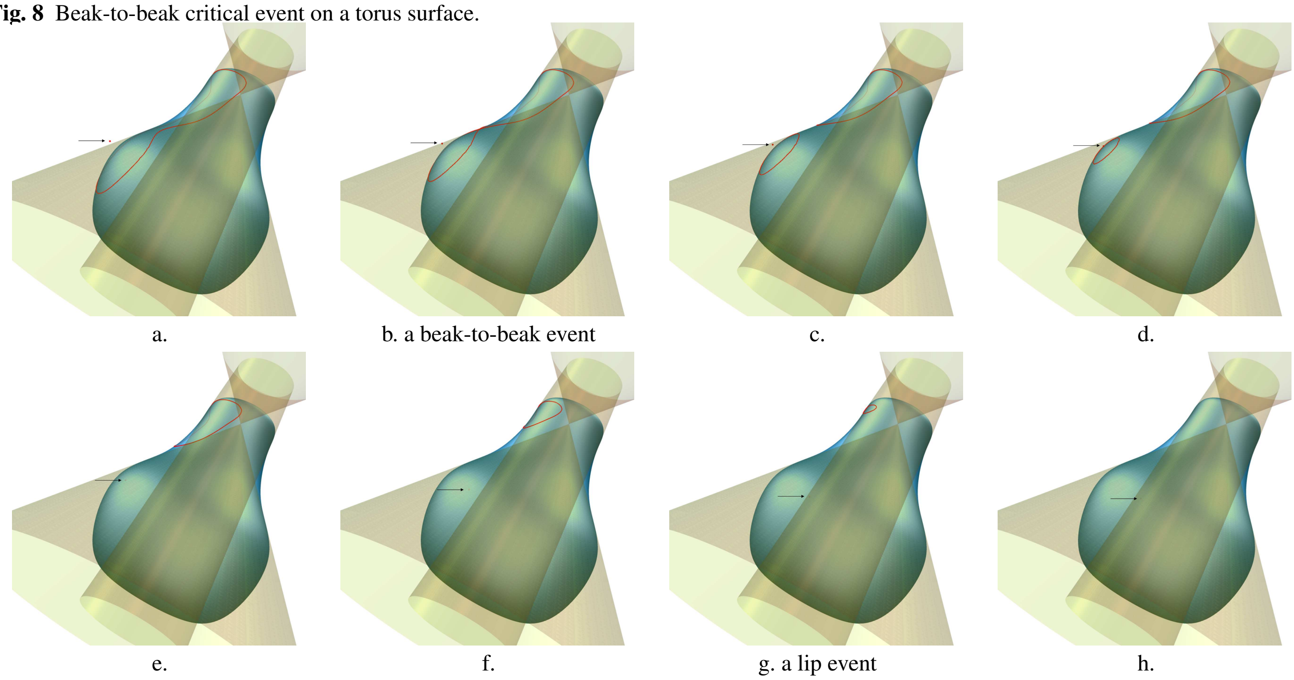

Fig. 9 Evolution of silhouettes when view crosses the aspect surface of a pear shape. The arrow points to the view point.