Performance Guarantees for B-trees with Different-Size Atomic Keys

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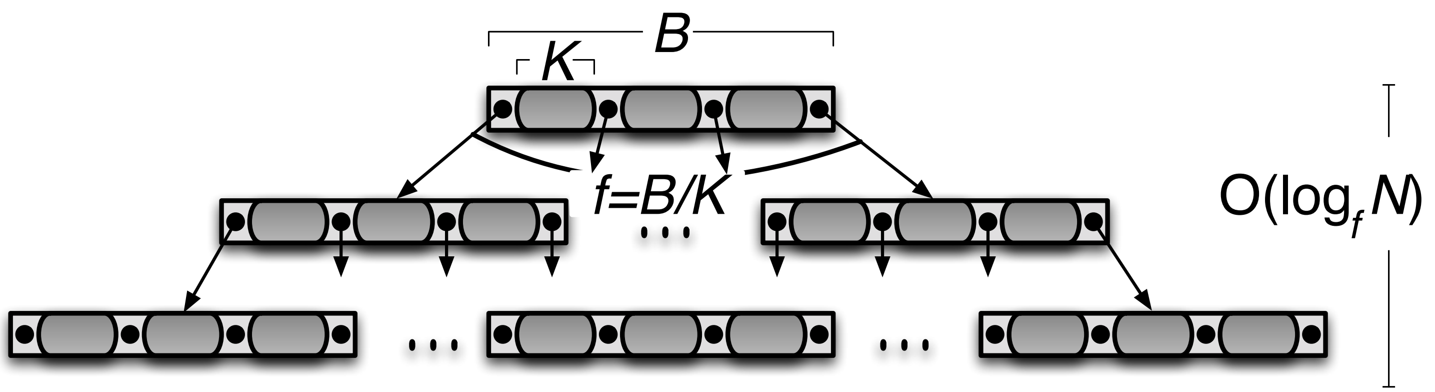
This talk

In B-trees in textbooks, all keys have the same size.



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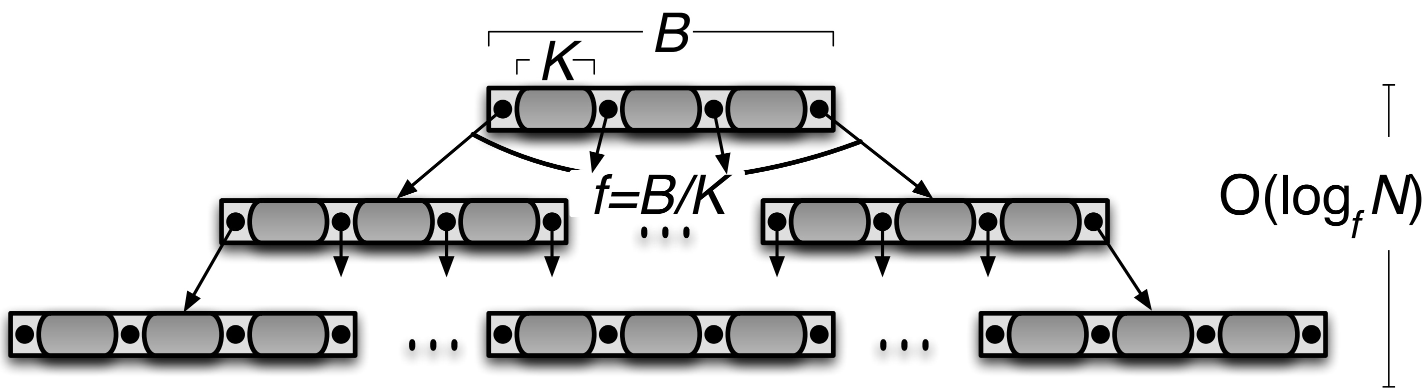

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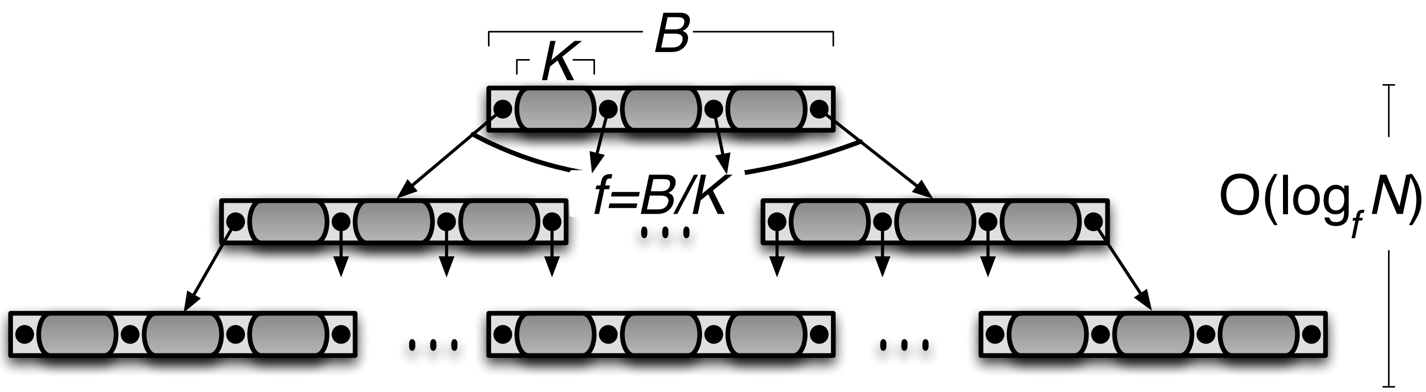

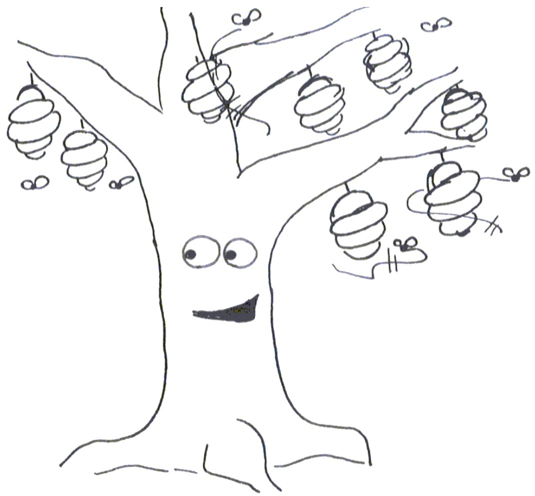

Production B-trees support different-size keys, but with no nontrivial performance guarantees.

This talk: give provably good guarantees in an only slightly modified B-tree.

Example Showing Problem

a bbbbbbbb c dddddddd e ffffffff g hhhhhhhh

i jjjjjjjj k llllllll m nnnnnnnn o

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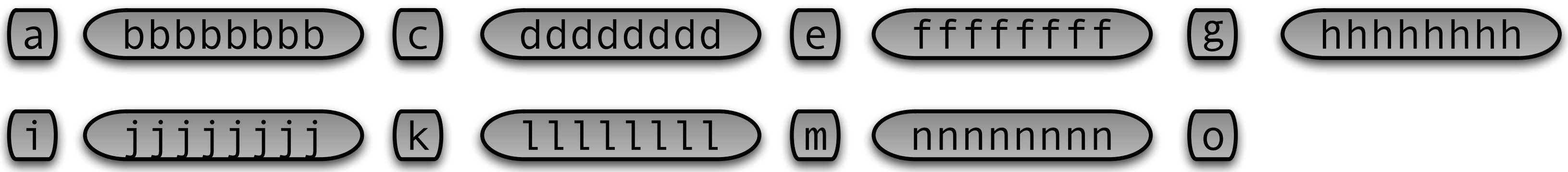
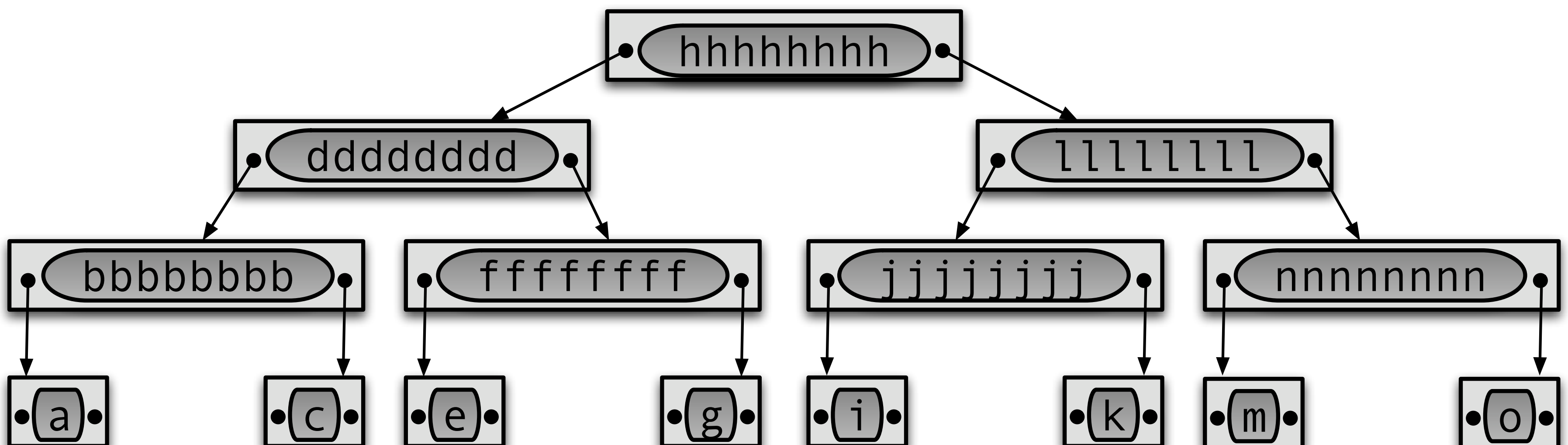


When the length-8 keys are pivots (and the block size is 8), the tree height is 4:



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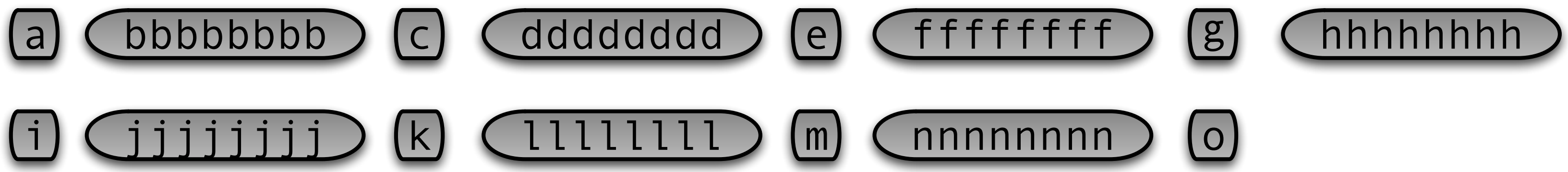
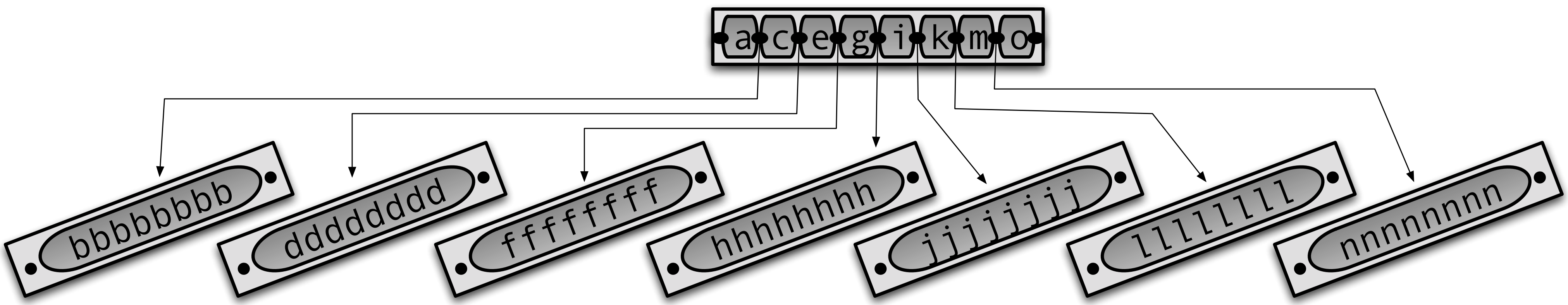


When the length-1 keys are pivots (and the block size is 8), the tree height is 2:



Choice of pivot affects the B-tree performance.

Keys are Atomic, Not Strings

Cannot compare first byte of bbbbbbbb **with** c

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Only the comparison function understands the keys.

Keys are opaque. Need to send entire key to comparison function and store entire key in node.

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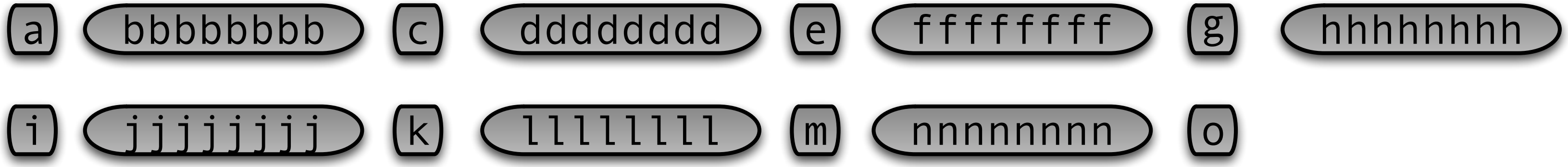
String B-tree techniques don't work [Ferragina, Grossi 98].

Front compression (prefix compression) doesn't work [Bayer, Unterauer 77] [Wagner 73].

Terminology

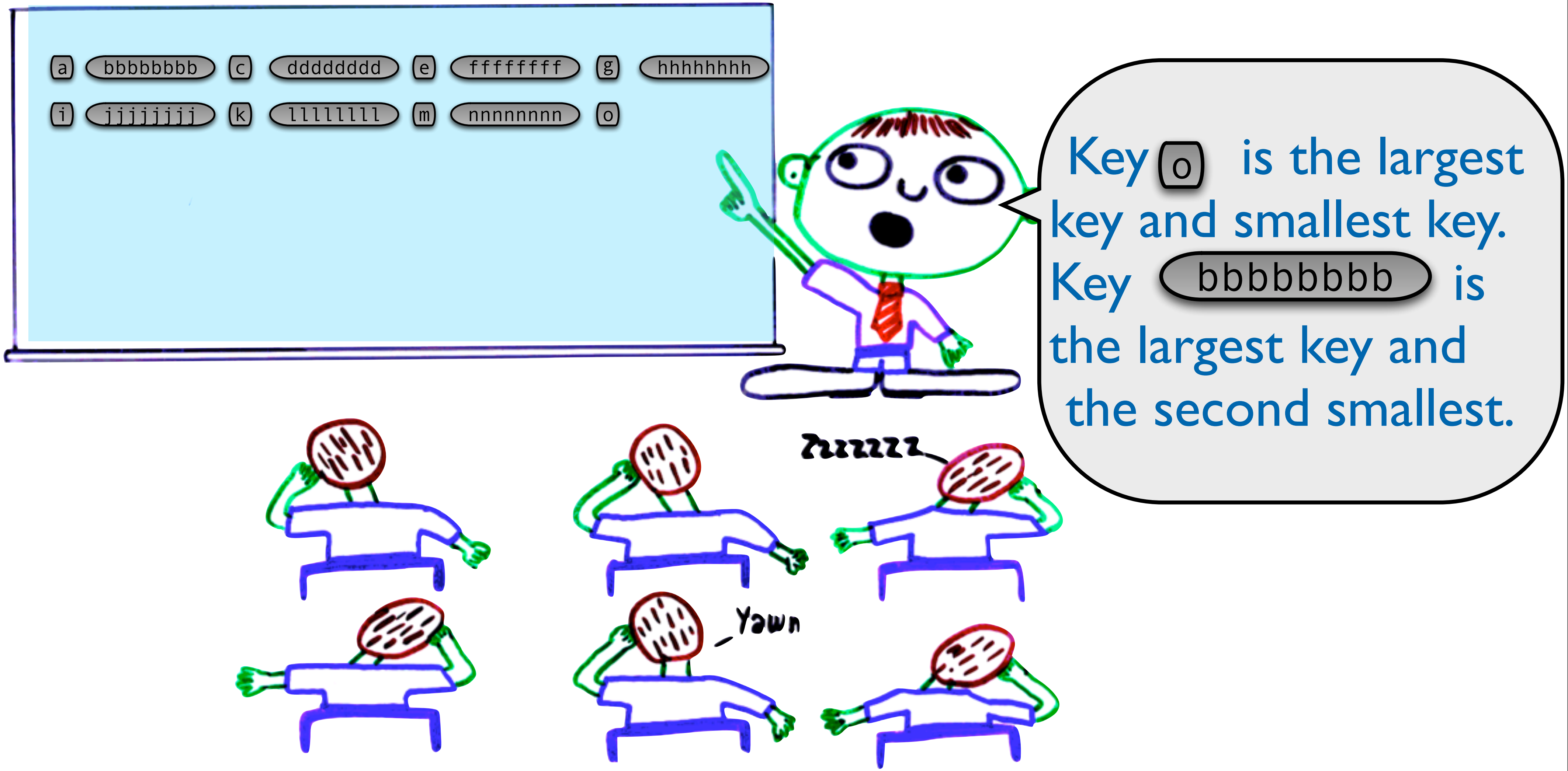


earliest vs. *latest*: order determined by comparison function.

shortest versus *longest*: how many bytes to store key

(Words to avoid: *small*, *large*, *minimum*.)

So if I say....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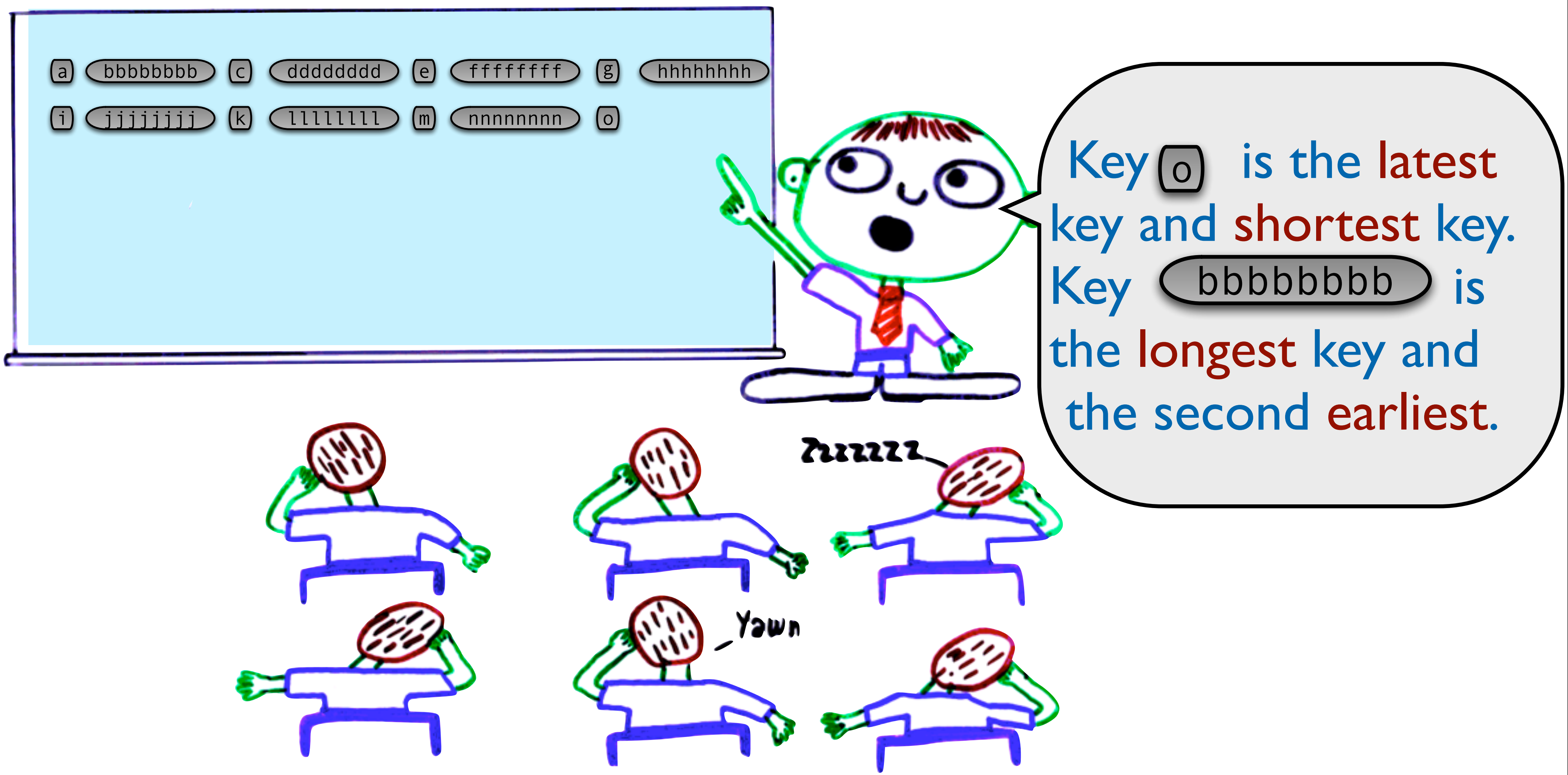
Key o is the largest key and smallest key. Key bbbbbbbb is the largest key and the second smallest.
Zzzzzzz
Yawn

... I mean...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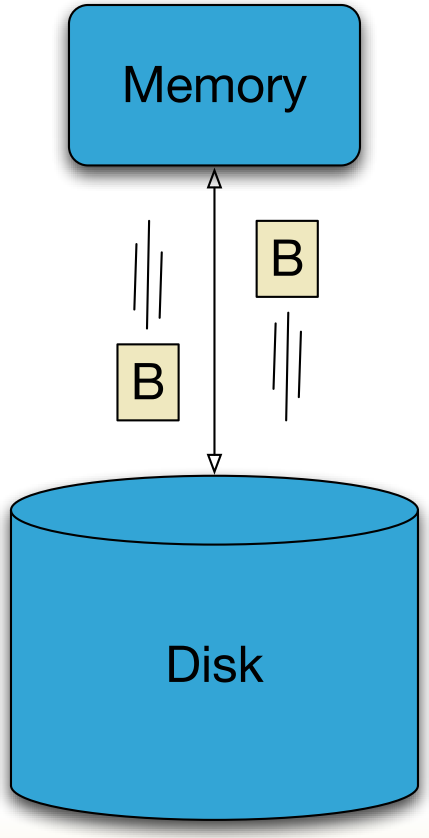
Key o is the latest key and shortest key.
Key bbbbbbbb is the longest key and the second earliest.
Zzzzzzz
Yawn

Algorithmic Performance Model

Disk-Access Machine (DAM) [Aggrawal, Vitter 88]

- Two-levels of memory.
- Two parameters: block-size ***B***, memory-size ***M***.



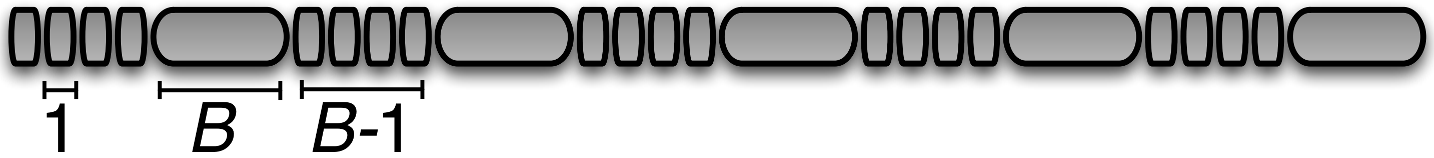
Performance metric:

- Minimize # of block transfers

Choice of Pivot Matters For Variable-Size Keys

Example: N **keys with average size** <2**.**

- N/B keys with size B and N-N/B keys with size 1.



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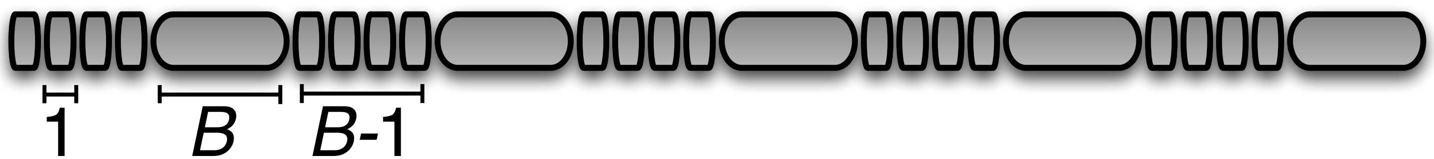
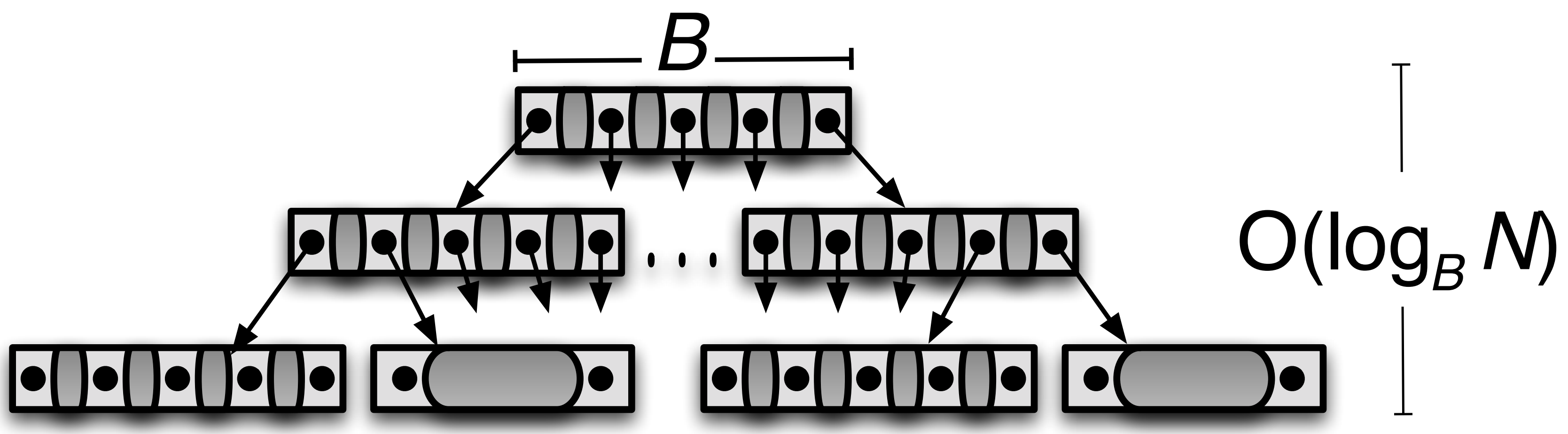


Size 1 keys as pivots: optimal.



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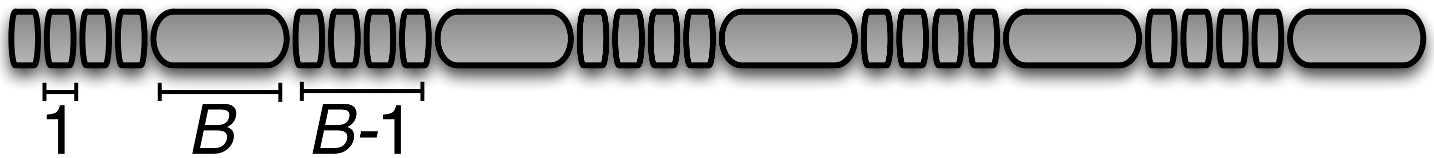
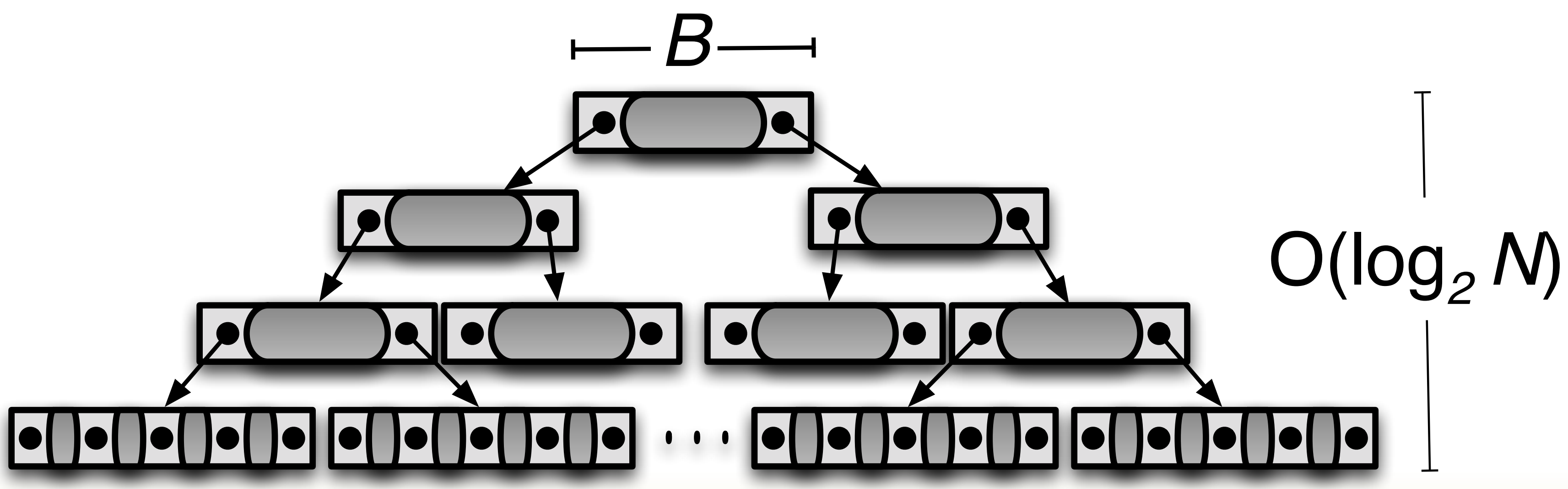
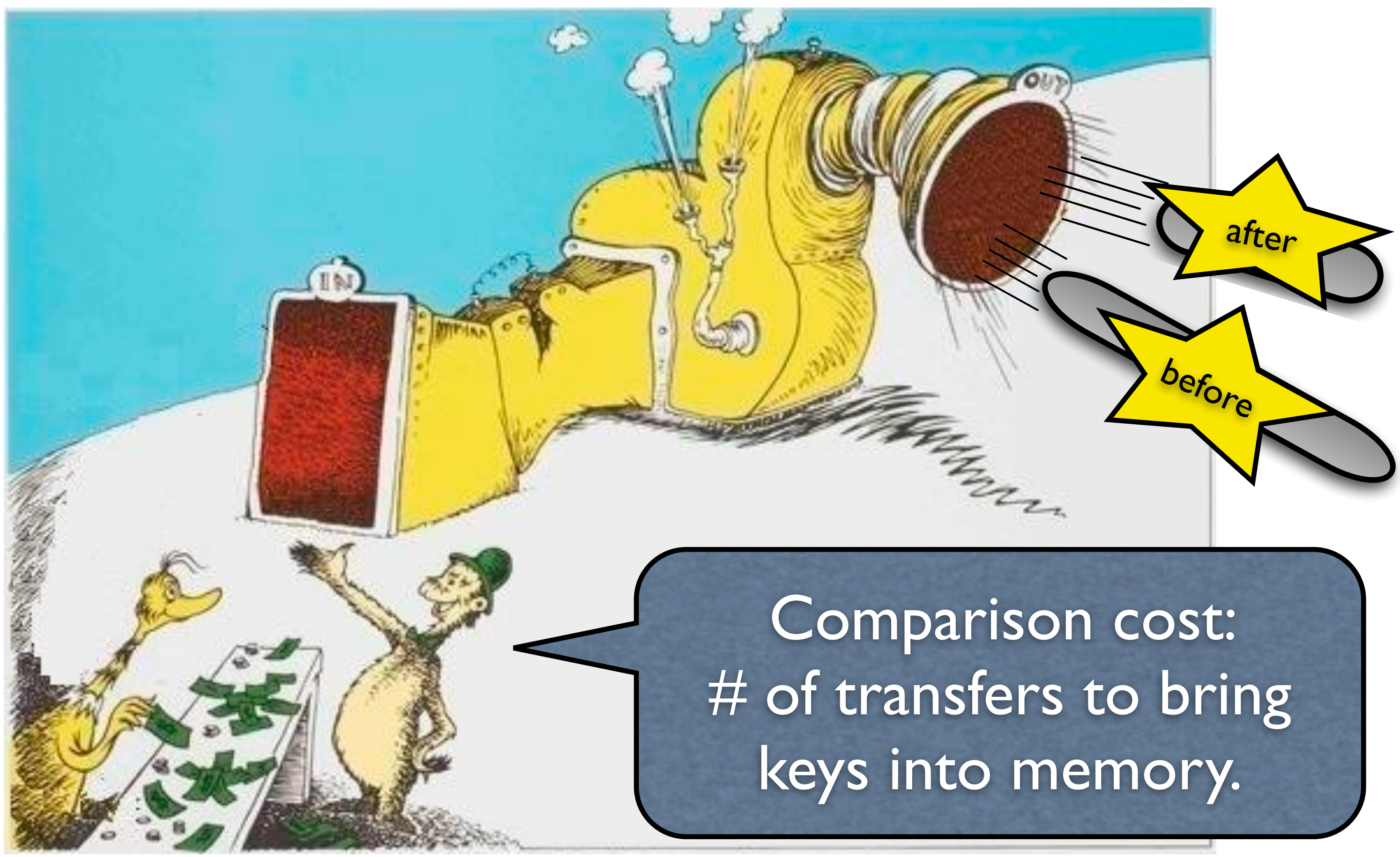


Size *B* keys as pivots: $O(\log B)$ factor worse.



Comparison Cost



Desired Guarantee

Let *K* be the *average* key size.

Goal: $O(\log_{B/K} N)$ memory transfers per operation.

- Generalizes what happens if keys all have the same size K.

Desired Guarantee

Let *K* be the *average* key size.

Goal: $O(\log_{B/K} N)$ memory transfers per operation.

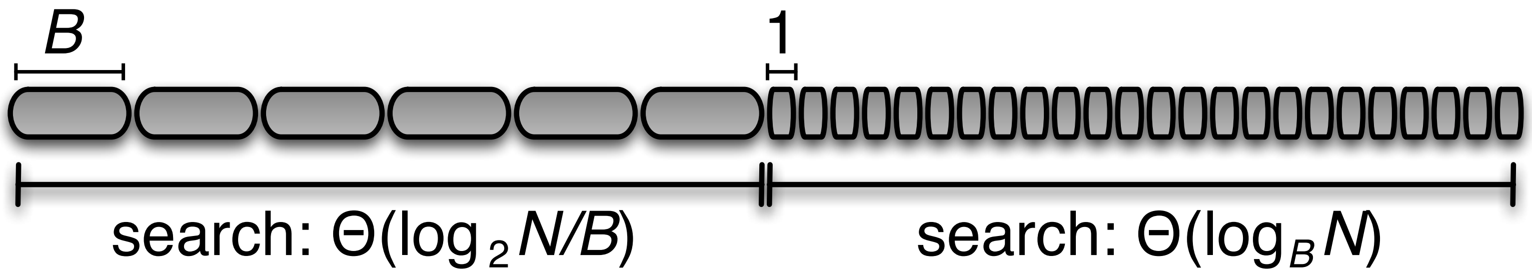
- Generalizes what happens if keys all have the same size *K*.

Unfortunately, we cannot get this for worst-case searches, but we'll get it in expectation.

Why We Cannot Attain Good Worst-Case Bounds

Example: *N* keys with average size *K*<2.

- *N*/*B* keys with size *B* and *N*-*N*/*B* keys with size 1.



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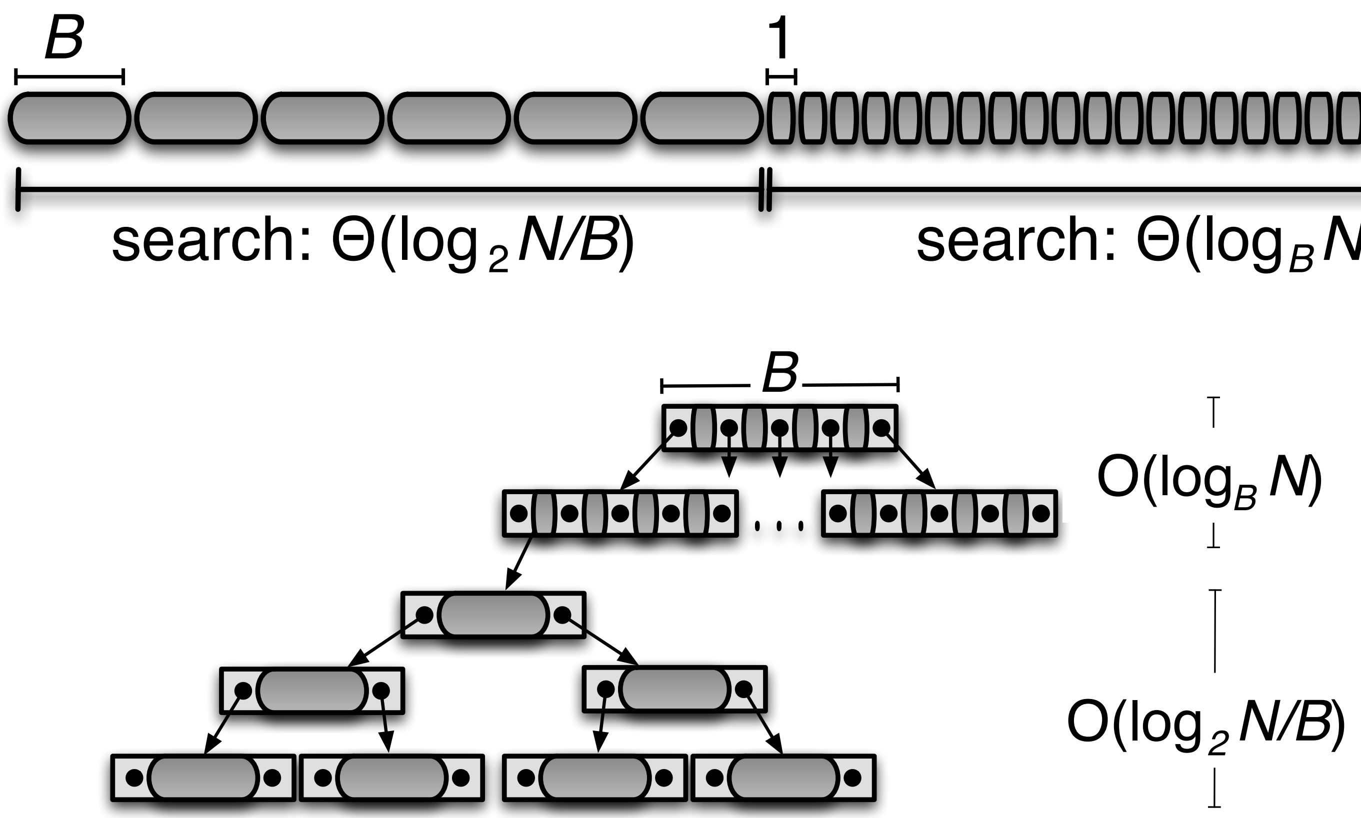

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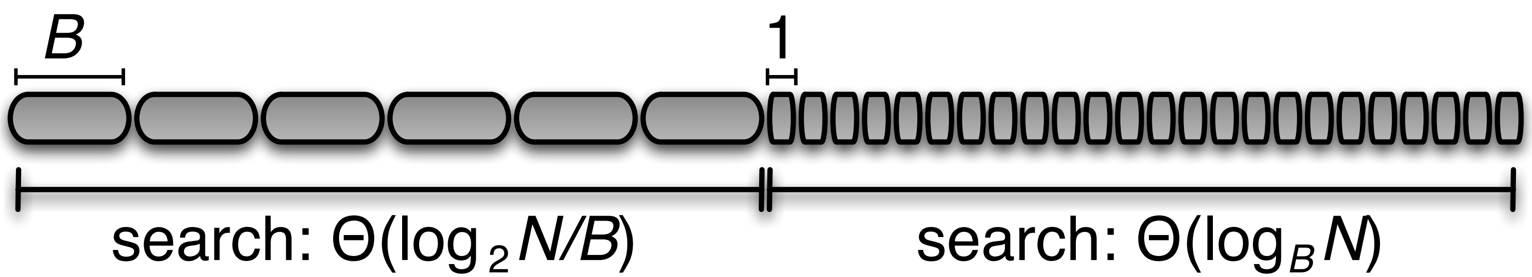

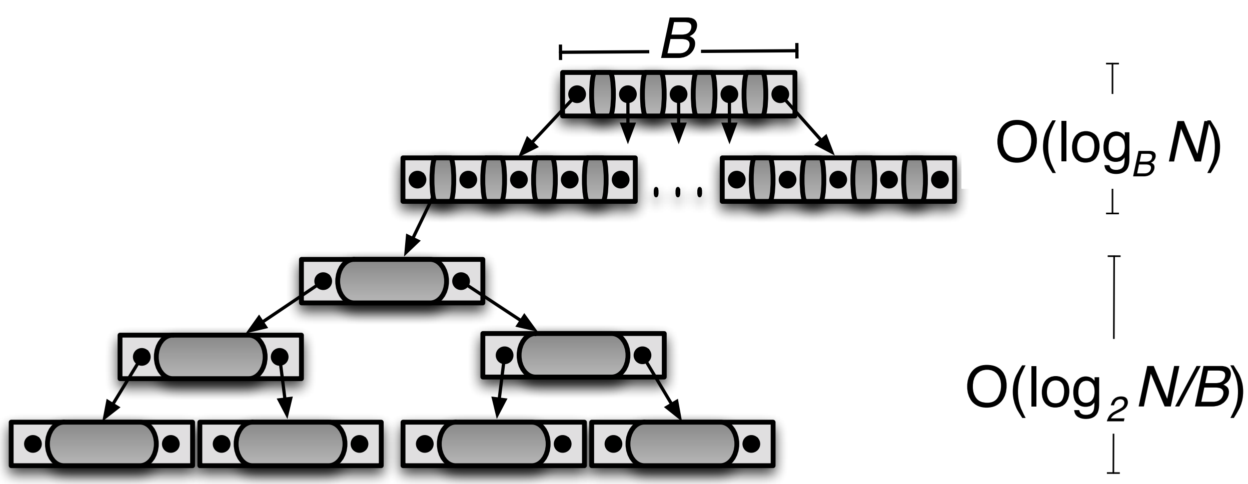

But we are ok in expectation: $(1-1/B)\log_B N + (1/B)\log_2 N = O(\log_B N)$.

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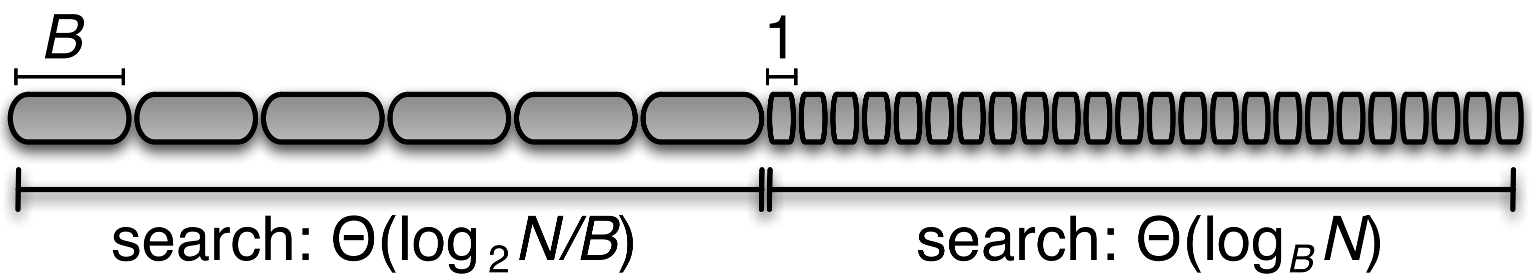


Related work: how to optimize B-tree height

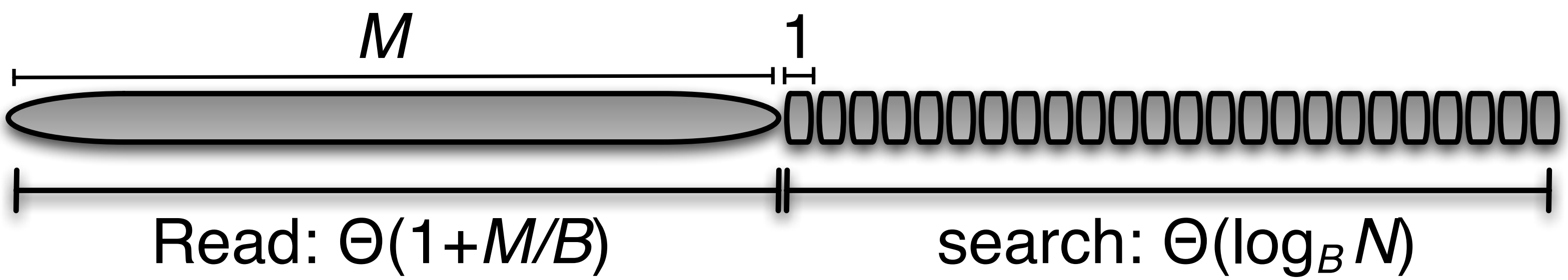
[Vaishnavi, Kriegel, Wood 80] [Gotleib 81] [Huang, Vishwanathan 90] [Becker 94]

- static (no inserts/deletes)
- DP-based
- (far from our target guarantee)

Why We Cannot Attain Good Worst-Case Bounds

Extreme example: N keys with average size K<2.

- 1 key with size M and N-1 keys with size 1.



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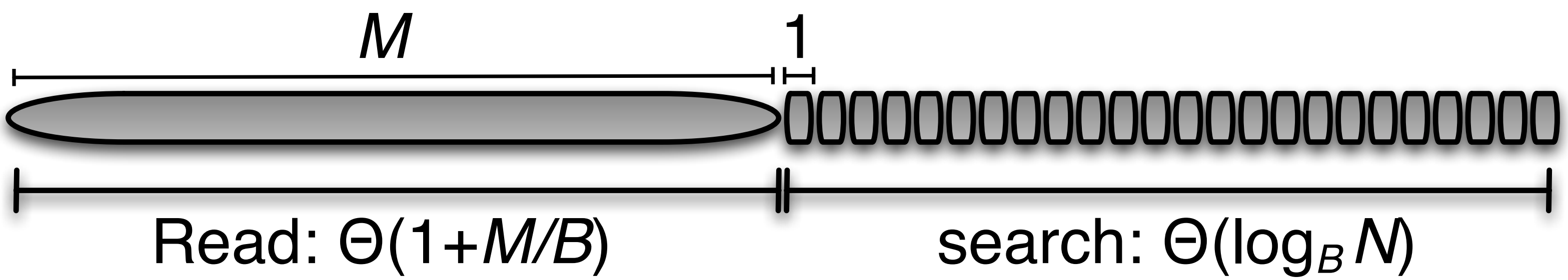

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These two examples have different flavors.

- LB for first example is based on tree structure.
- LB for for second example is based on reading the key.
- Both motivate why we consider expectation.

Static atomic-key B-tree (only searches)

- Expected leaf search cost: $O(\lceil K/B \rceil \log_{1+\lceil B/K \rceil} N)$
- Linear construction cost for sorted data: $O(NK/B)$

Results

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Optimal static atomic-key B-tree:

- $O(BN^3)$ operations
- Applies even for nonuniform search probabilities

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Optimal static atomic-key B-tree:

- $O(BN^3)$ operations ← RAM not external memory. (Won't discuss in talk.)
- Applies even for nonuniform search probabilities

Results

Static atomic-key B-tree (only searches) ← To discuss next

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Static Atomic-Key B-tree

Greedy construction algorithm

- Greedily select pivot elements for the root node
- Proceed recursively on all subtrees of the root.

Intuition

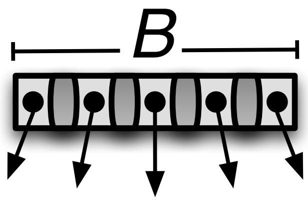
- Pick small keys in root to maximize fanout.
- Pick evenly distributed keys to reduce the search space.

To prove

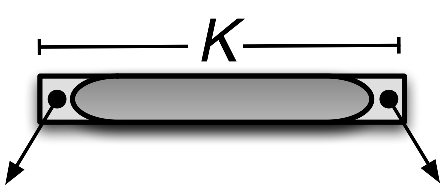
- Root has a good structure.
- Recursive substructures achieve good performance, even though subtrees may have different average key sizes.

Root Structure of Static Atomic Key B-tree

Case 1: $K=O(B)$. Root has size $O(B)$ and fanout $\Theta(B/K)$.

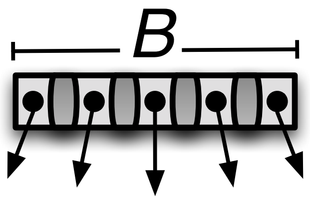


Case 2: $K=\Omega(B)$. Root has size $O(K)$ and fanout 2.



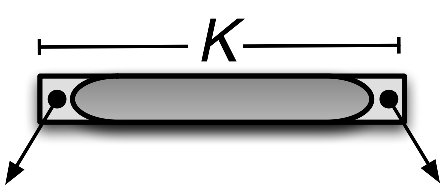
Root Structure of Static Atomic Key B-tree

Case 1: K=O(B). Root has size O(B) and fanout Θ(B/K).



Overall search cost: $O(\lceil K/B \rceil \log_{1+\lceil B/K \rceil} N)$

Case 2: K= Ω(B). Root has size O(K) and fanout 2.



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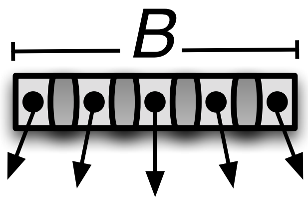
Case 1: *K*=O(*B*). Root has size O(*B*) and fanout Θ(*B*/*K*).

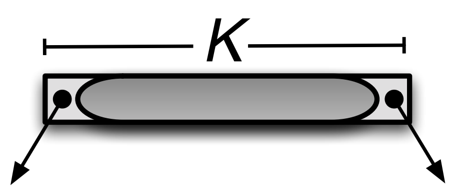


cost of reading root

ave fanout

Overall search cost: $O(\underbrace{\cancel{\lceil K/B \rceil}}_{=1} \log_{\underbrace{\cancel{1+\lceil B/K \rceil}}_{\approx\Theta(B/K)}} N)$

Case 2: *K*= Ω(*B*). Root has size O(*K*) and fanout 2.



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Root Structure of Static Atomic Key B-tree

***Useful Lemma*: Consider N #s whose average is K. Divide into f groups of equal cardinality (each has N/f). Take the min in each group (say K_i is min of group i). Then average of these minima is at most the overall average K (i.e., $(K_1+K_2+\ldots+K_f)/f \le K$).**

Ex. 4 4 3 4 | 1 2 2 2 | 3 2 5 4 | 3 1 4 4

Ave K=3.

Root Structure of Static Atomic Key B-tree

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Ave *K*=3. Ave of mins =1.75.

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Note: only true because groups have the same size.

Enough structure to bound the size and fanout of the root.

Root Construction

1. Divide keys into $f = \max\left\{3, \left\lfloor \frac{B}{K} \right\rfloor\right\}$ equal-size groups.
2. Pick shortest key in each group.
3. Store these keys in root (except 1st & last groups).

aaaa bbbb ccc dddd e ff gg hh iii jj kkkkk llll mmm n oooo pppp

Ex: *B* = 12, *K* = 3. **So** *f*=4.

STONY BROOK

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Root Construction

1. **Divide keys into** $\mathrm{f} = \max\left\{3, \left\lfloor \frac{\mathrm{B}}{\mathrm{K}} \right\rfloor\right\}$ **equal-size groups.**
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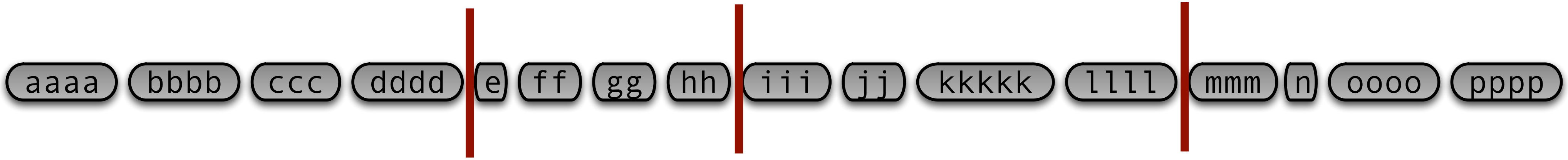

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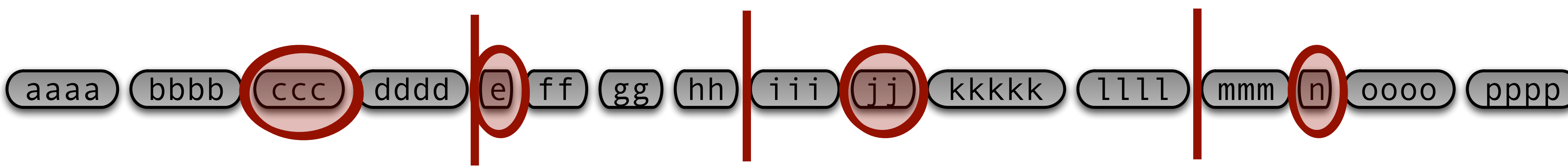

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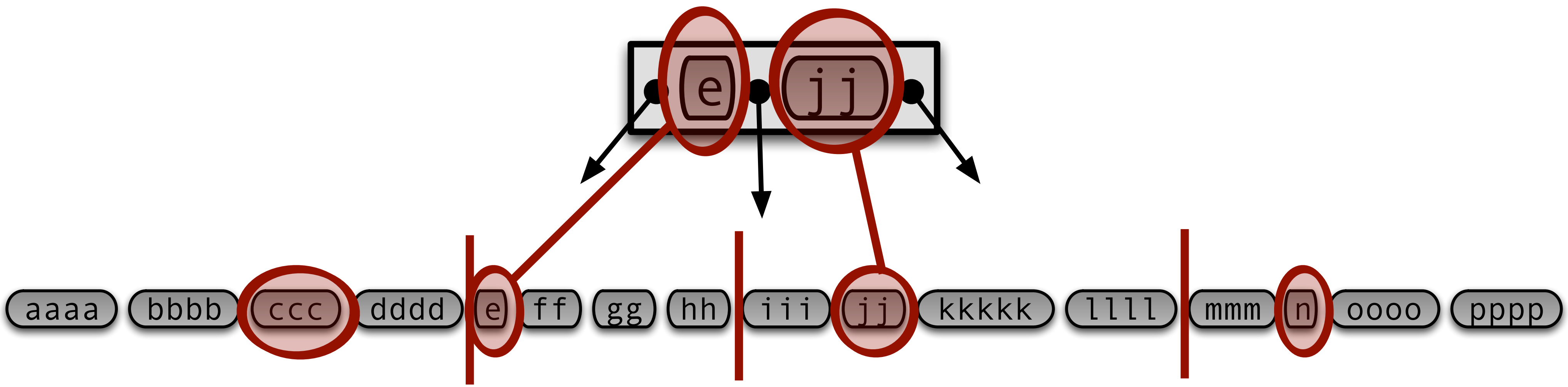
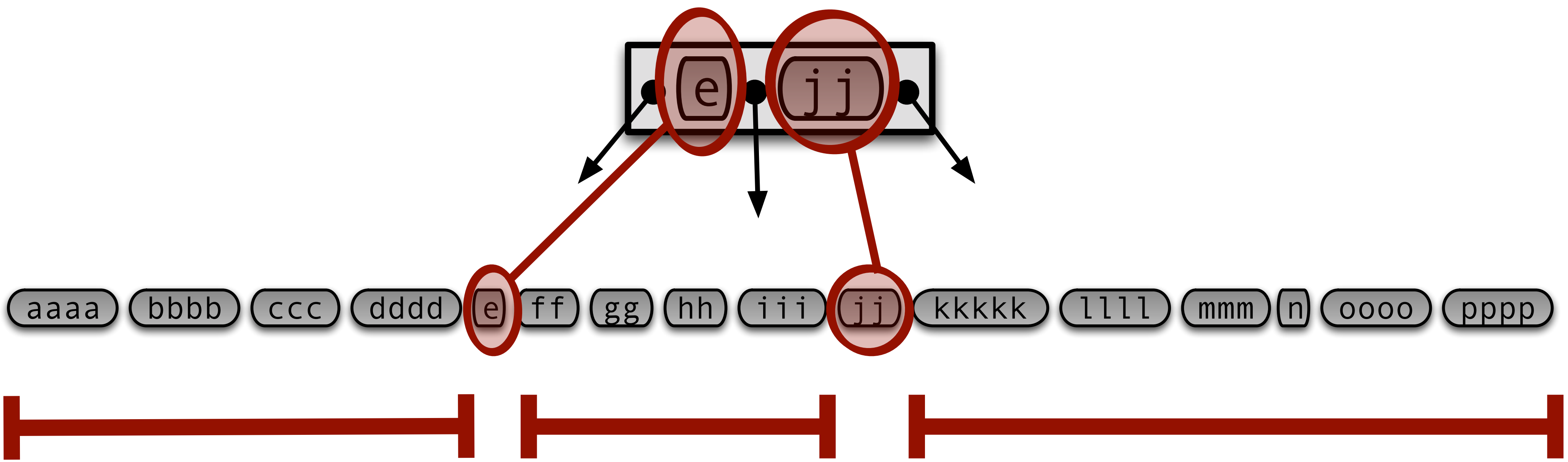
1. **Divide keys into** $f = \max\left\{3, \left\lfloor \frac{B}{K} \right\rfloor\right\}$ **equal-size groups.**
2. **Pick shortest key in each group.**
3. **Store these keys in root (except 1st & last groups).**



Ex: *B* = 12, *K* = 3. **So** *f*=4.

Construction of Rest of Tree

Proceed recursively in each subtree.
(Different value of *K* (and thus $f = \max\left\{3, \left\lfloor \frac{B}{K} \right\rfloor\right\}$)
in each subtree.



Results

Static atomic-key B-tree (only searches)

- Expected leaf search cost: $O(\lceil K/B \rceil \log_{1+\lceil B/K \rceil} N)$
- Linear construction cost for sorted data: $O(NK/B)$

Dynamic atomic-key B-tree

To discuss next

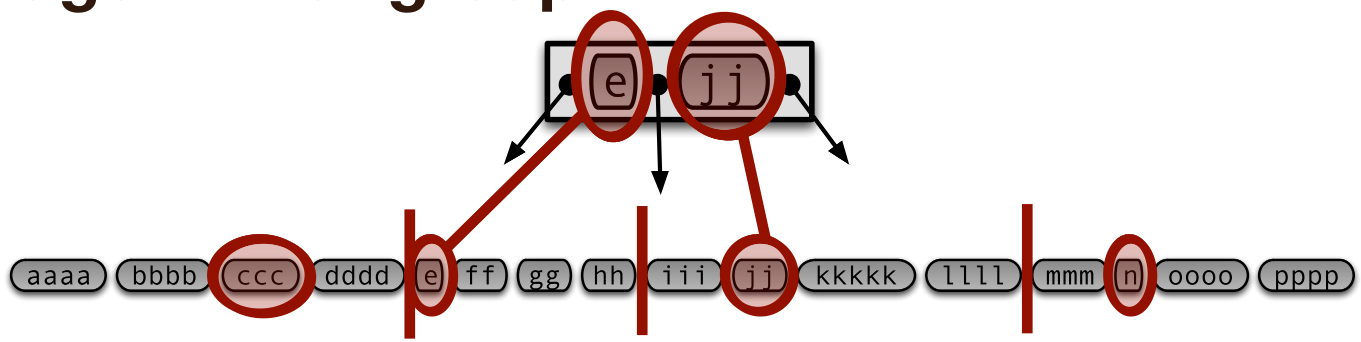
- Expected leaf search cost : $O(\lceil K/B \rceil \log_{1+\lceil B/K \rceil} N)$
- Cost to insert/delete/search for key *L* of *random* rank (amort): $O(\lceil K/B \rceil \log_{1+\lceil B/K \rceil} N + |L|/B)$
- Cost to insert/delete/search for key of *arbitrary* rank: ***modification cost is dominated by search cost.***

Optimal static atomic-key B-tree:

- $O(BN^3)$ operations
- Applies even for nonuniform search probabilities

Dynamic Atomic-Key B-tree

One idea: **Groups need not be of equal cardinality.**
Within constant factors is good enough.
We don't need the shortest key as a pivot.
We can choose a key whose length is at most twice the average in that group.



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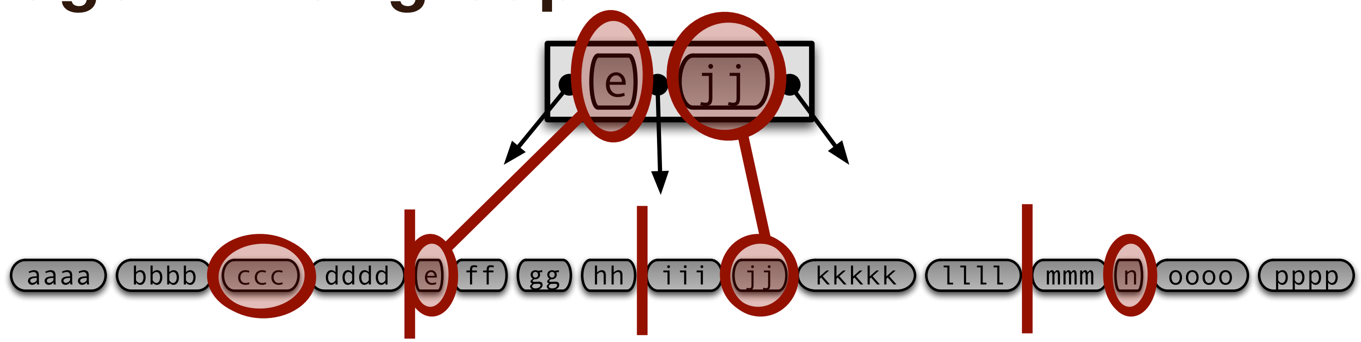

Thus, > half the keys in a group could be a pivot.

The shortest key can remain as pivot even if the group grows or shrinks by a constant factor.

Structure isn't brittle.

Dynamic Atomic-Key B-tree (Cont)

Second idea: **Insert elements directly into leaves. Rebuild entire subtrees whether there have been "too many" inserts/deletes.**
(Don't bother splitting and merging.)

$$\text{amortized update cost} = \frac{\text{rebuild cost}}{\text{\# updates between rebuilds}}$$

Problem: standard technique chokes because value of *K* changes over time. (Value of *K* during rebuild is different from value of *K* at actual insert/delete.)
Can be fixed. (Ask after talk.)

Waxing Philosophical

People want performance guarantees

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- As our results show, this folk theorem is incorrect.
- But the claim helps motivate the guarantees we achieve.

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