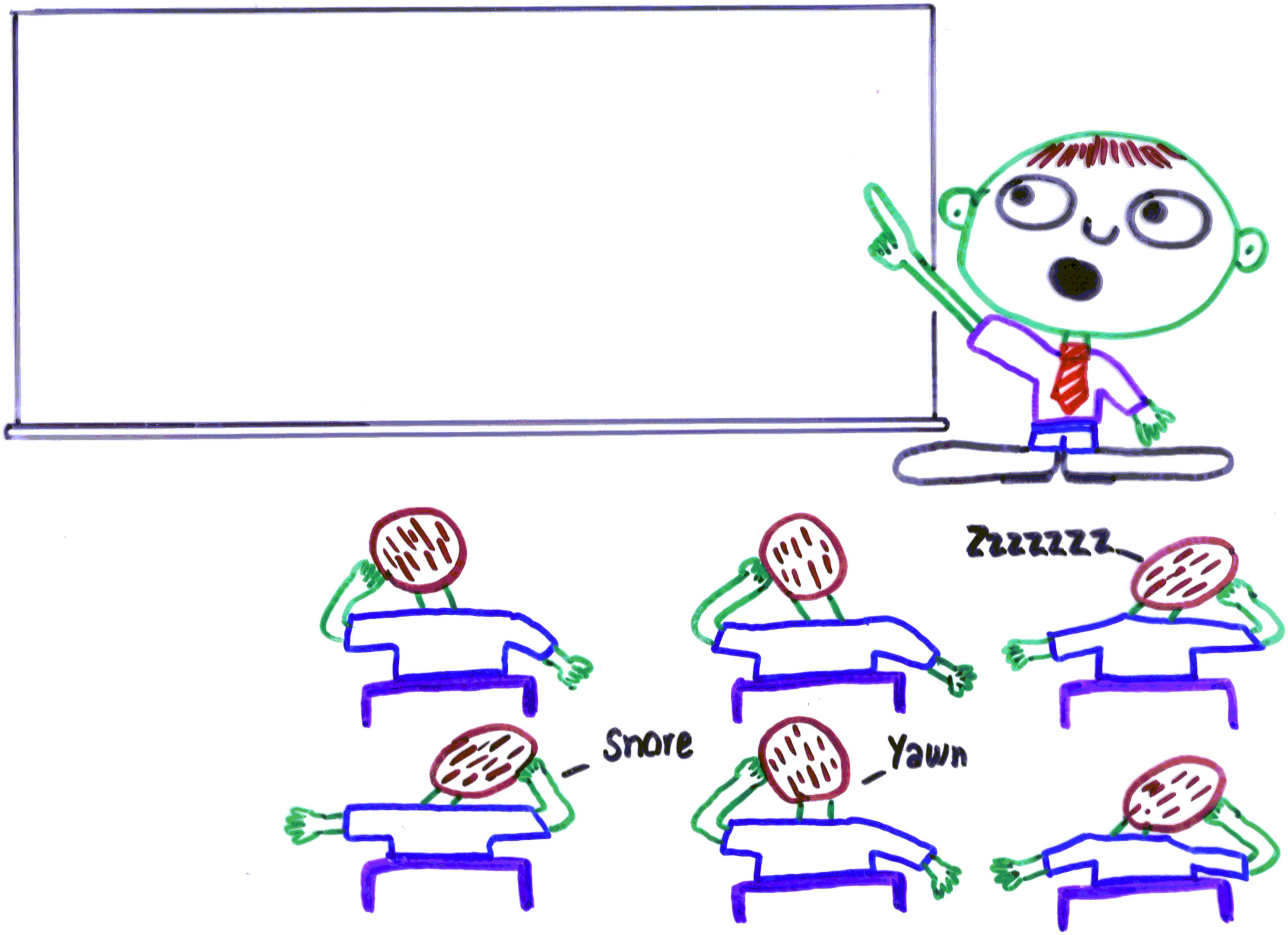
My Relationship with Insertion Sort

Fall 1990. I'm an undergraduate.



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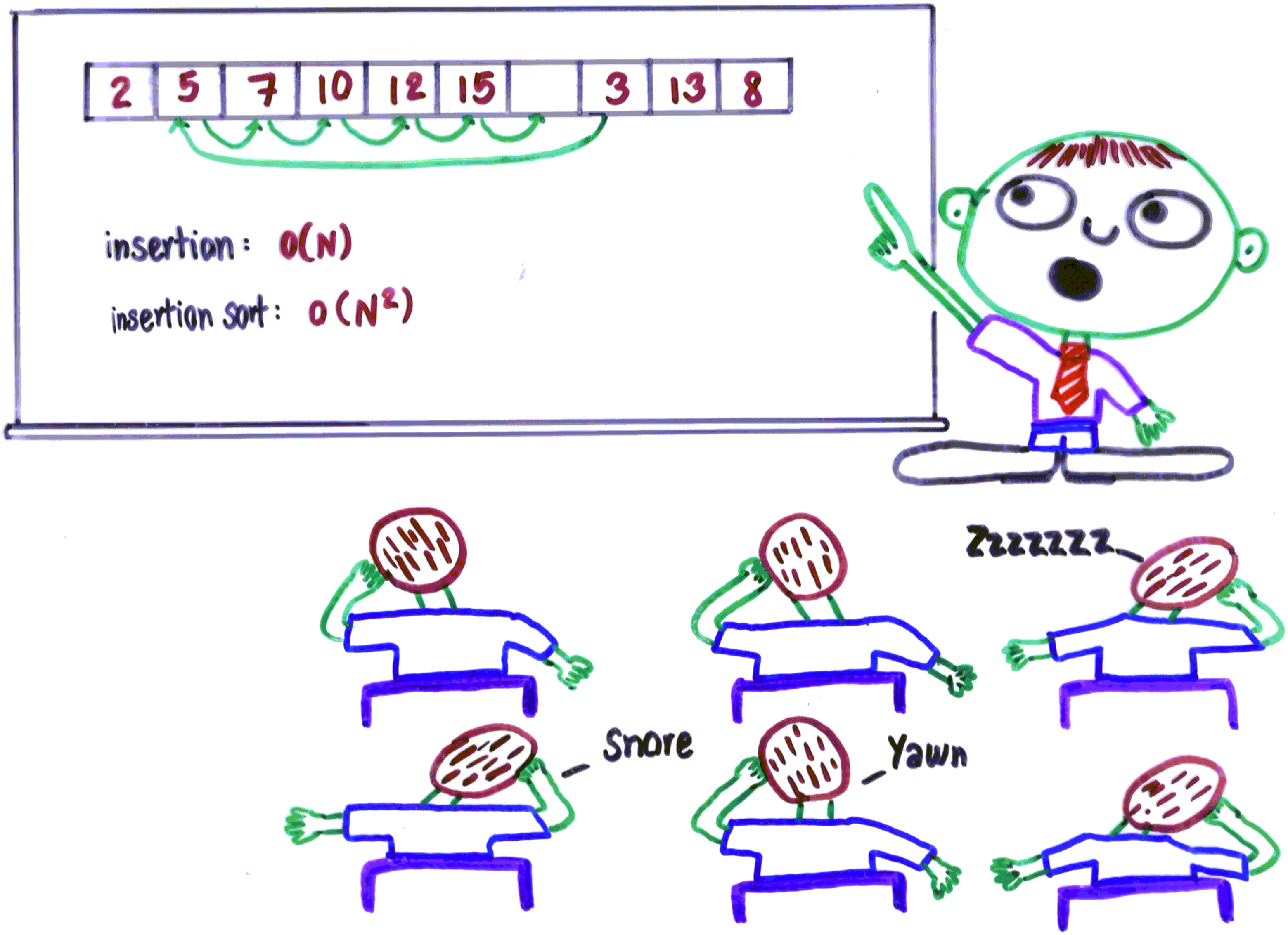

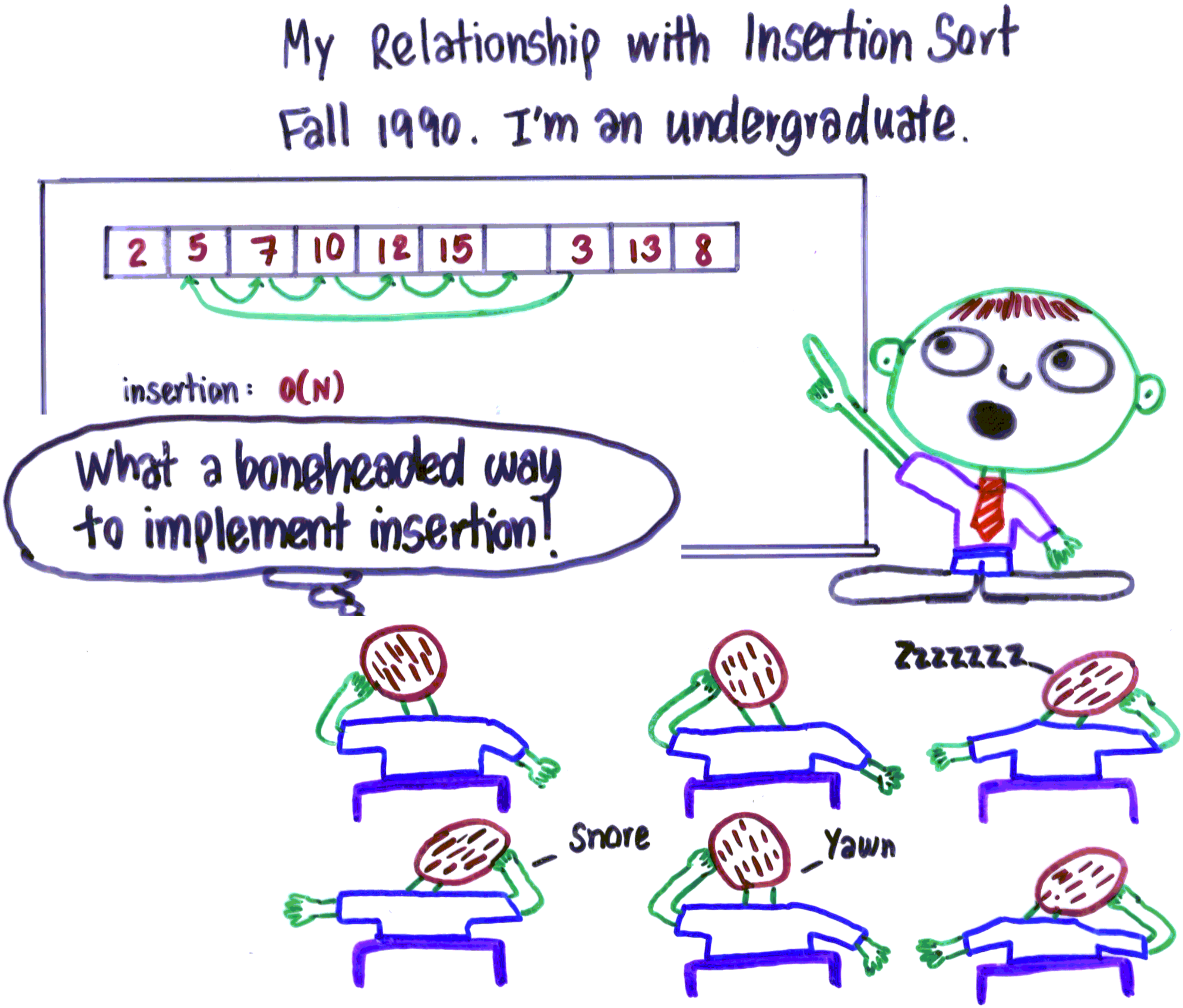
My Relationship with Insertion Sort
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2 5 7 10 12 15 3 13 8
insertion: O(N)
What a boneheaded way to implement insertion!
Zzzzzzz
Snore
Yawn

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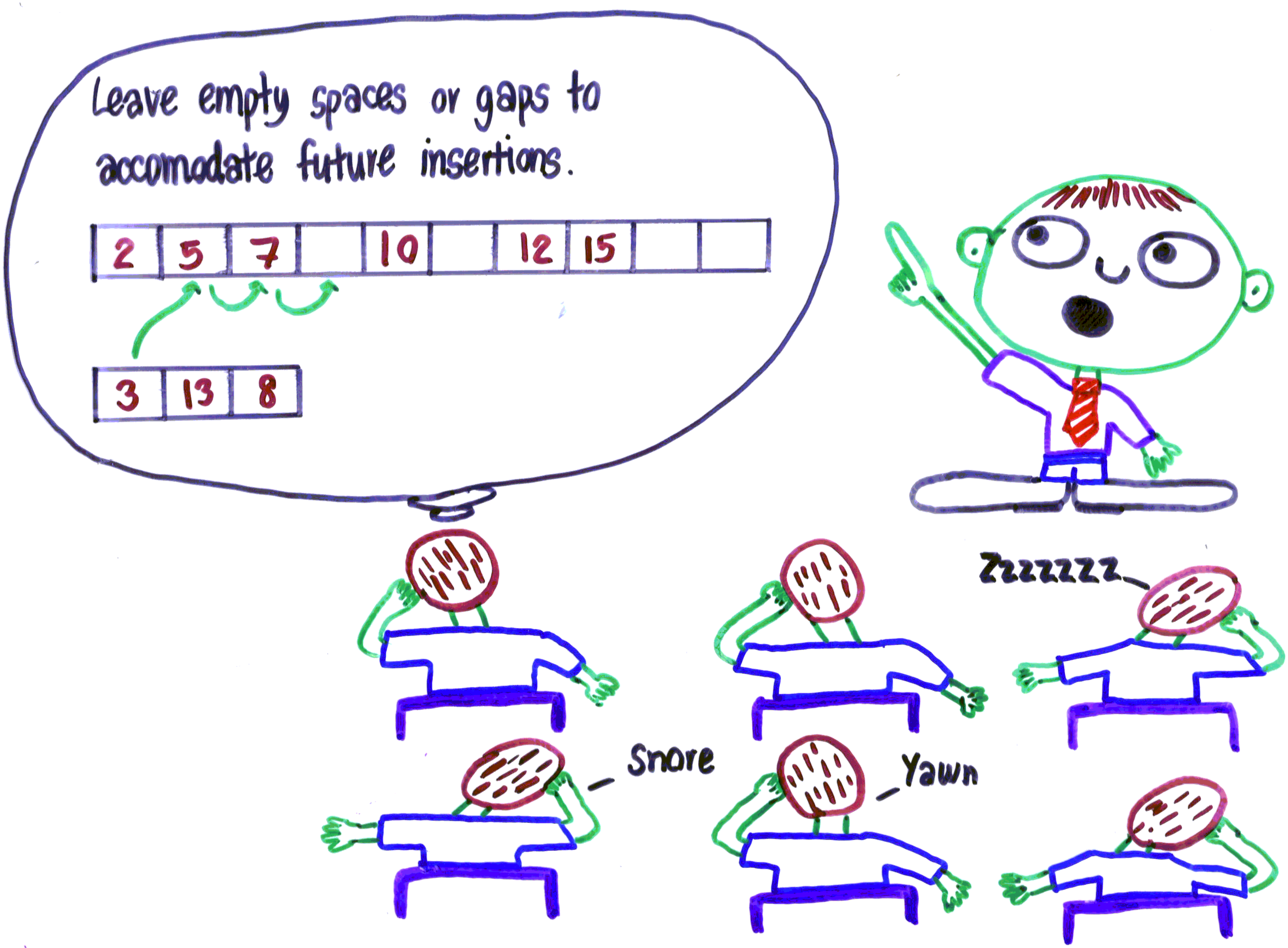

My Relationship with Insertion Sort

Anybody who has spent time in a library knows that insertions are cheaper than linear time.

Everybody (except computer scientists) know that gaps make inserts cheaper than $O(n)$ ("folk computer science").

Question: how much.

Small library

big library

14 years after that lecture on insertion sort

(1 BA, 1 DEA, 1 Magistère, 1 Ph.D., tenure-almost)...

Insertion Sort is $O(N \log N)$

Michael A. Bender

Martin Farach-Colton

Miguel Mosteiro

Results

- **LibrarySort**, a natural implementation of **InsertionSort** with gaps.

- **Theorem: LibrarySort** runs in $O(N \lg N)$ time w.h.p. and uses linear space. Each insertion is exp $O(1)$ and $O(\lg N)$ w.h.p..

Library Sort is $O(N \log N)$

for $k = 1$ to N do

find location to insert x_k (binary search)

insert x_k

if k is power of 2 then

rearrange elements evenly in $(2+\epsilon)k$-sized region.

Library Sort is $O(N \log N)$

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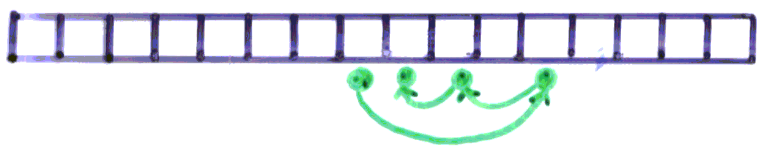

$O(\log N)$

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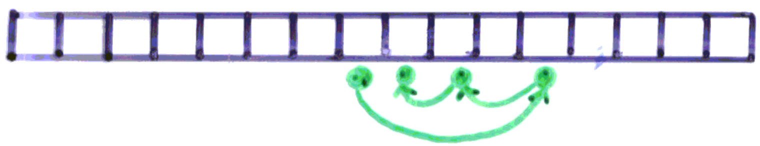

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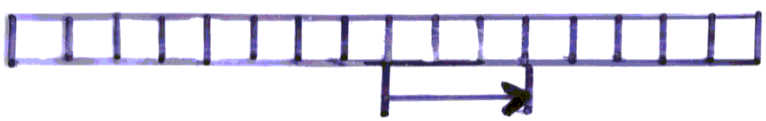

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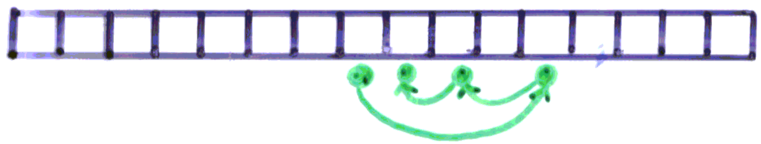

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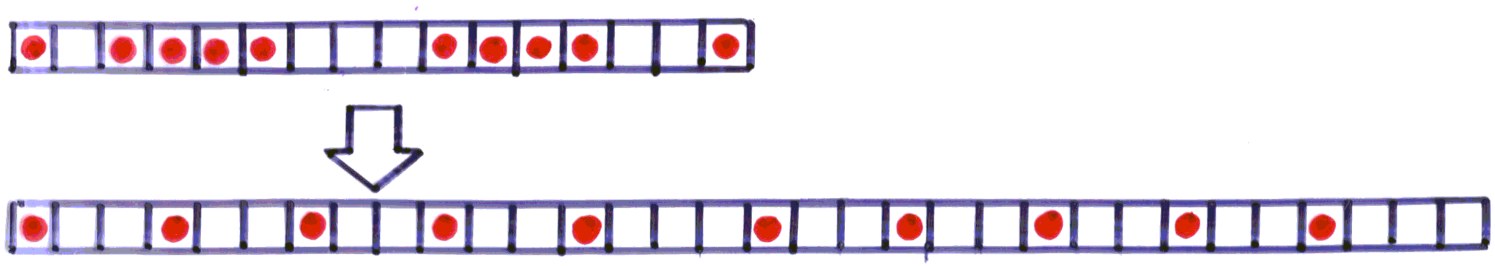

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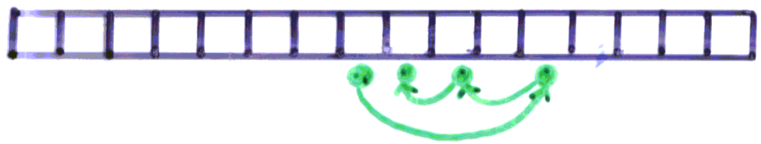

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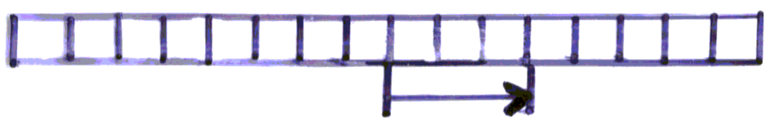

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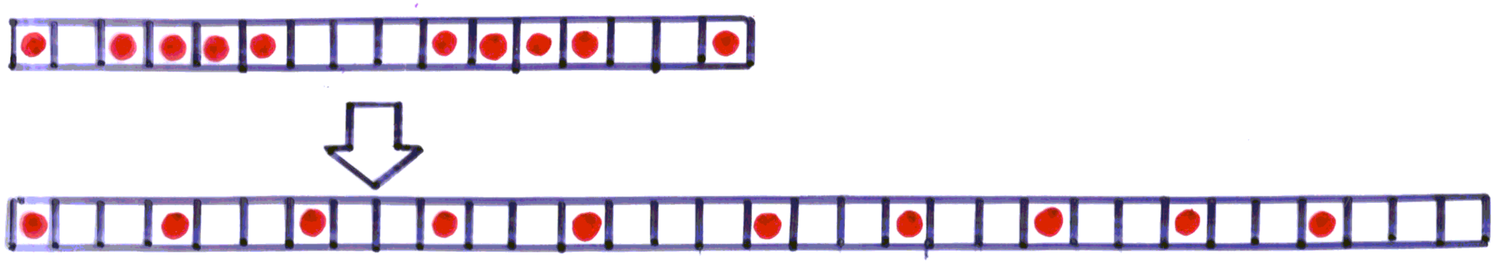

Thm: each insertion has cost $O(\lg N)$ w.h.p.

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Pf idea: Phase ℓ: elements $2^{\ell} \longrightarrow 2^{\ell+1}$ inserted.

beginning of phase: elements are rebalanced (evenly spread in array).

2^{ℓ} support elements

Thm: each insertion has cost $O(\lg N)$ w.h.p.

Pf idea: Phase ℓ: elements $2^{\ell} \longrightarrow 2^{\ell+1}$ inserted.

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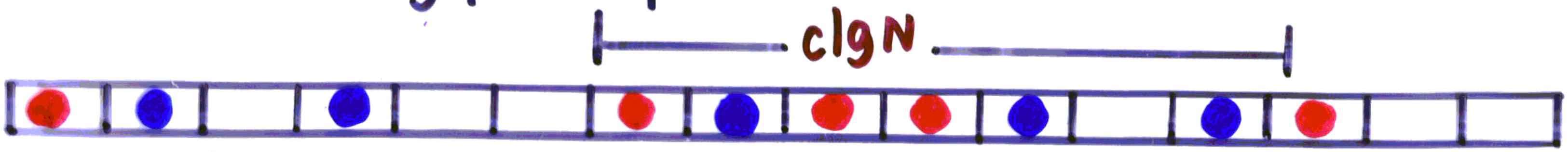
2^{ℓ} support elements

end of phase: for sufficiently large constant c, any region of size $c \lg N$ has gaps w.h.p.



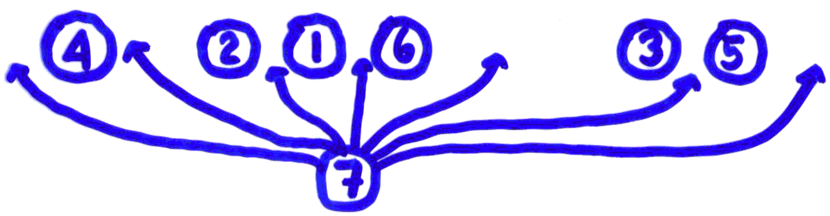
2^{ℓ} support and 2^{ℓ} intercalated elements

Invariant

The $(k+1)$st element is equally likely to be inserted between any two of the k elements already in the array.

Follows because elements are inserted in random order.

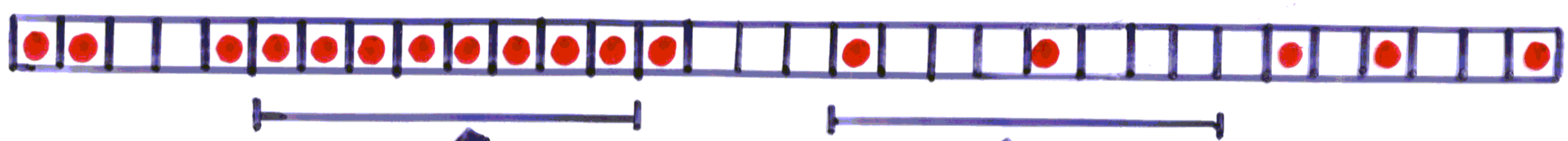
key order ⟶



(numbers = order of insertion)

Difficulty

Dense regions of array act as attractors.

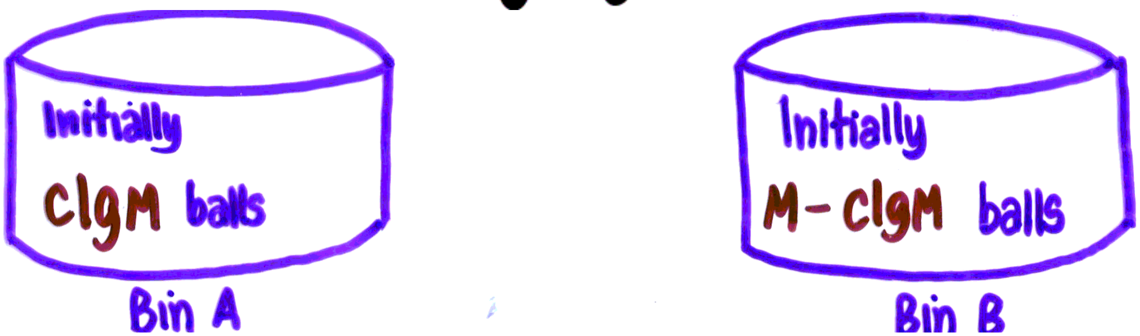


4x more likely to insert than .

Need to show that despite attraction dense regions do not get too big.

Balls and Bins Game

Idea: model attracting regions when M elmts in array.



M additional balls thrown in bins.

k^{th} ball thrown into Bin A or B with probability proportional to # balls in bins

$$X_k = \begin{cases} 1 & \text{if ball } k \text{ thrown into Bin } A \\ 0 & \text{otherwise.} \end{cases}$$

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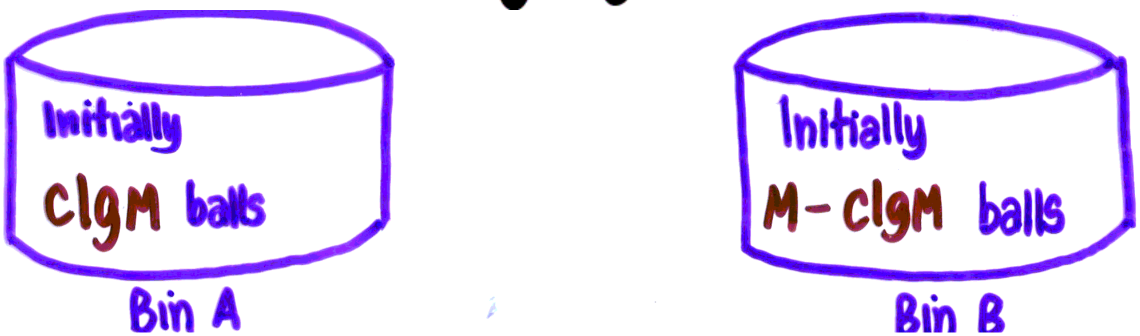

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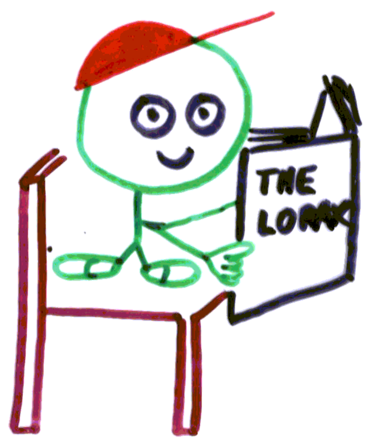
$$X_k = \begin{cases} 1 & \text{if ball } k \text{ thrown into Bin A} \\ 0 & \text{otherwise.} \end{cases}$$

Thm: Number of balls thrown in Bin A is

$$X = X_{M+1} + X_{M+2} + \dots + X_{2M} = O(\lg N).$$

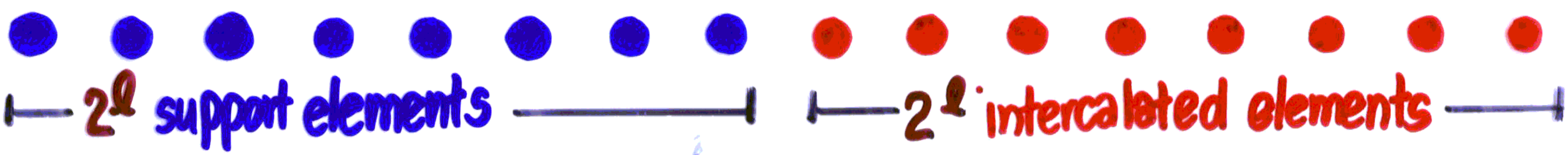
Issue: Random variables $X_{M+1} \dots X_{2M}$ are positively correlated.

Need an alternative approach



Alternative Analysis

Elements ordered by insertion order: random permutation on keys



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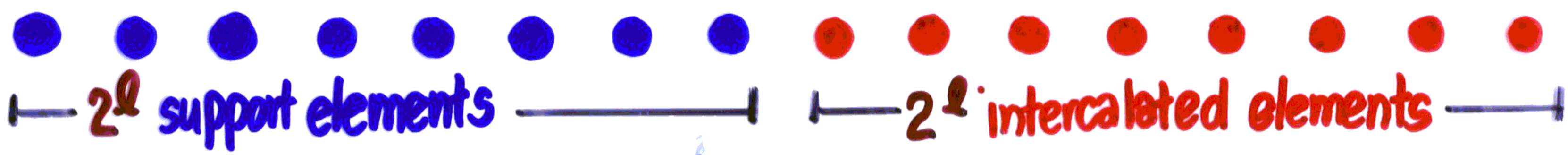

Elements ordered by keys: random permutation on insert order

Alternative Analysis

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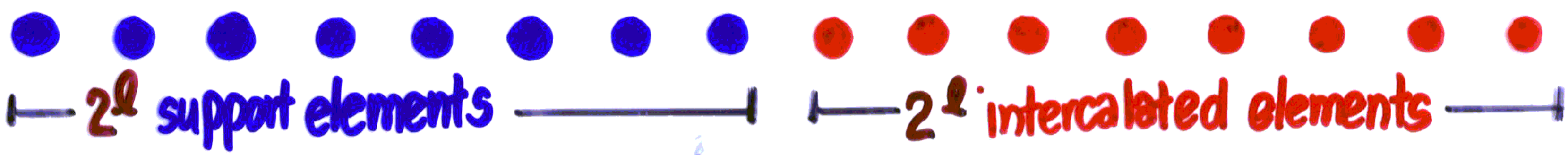

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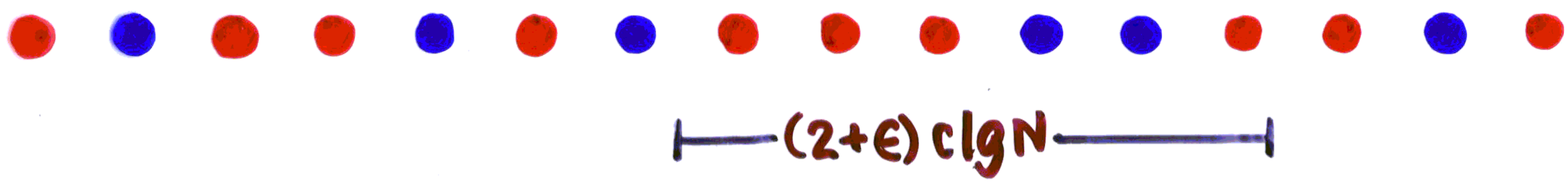
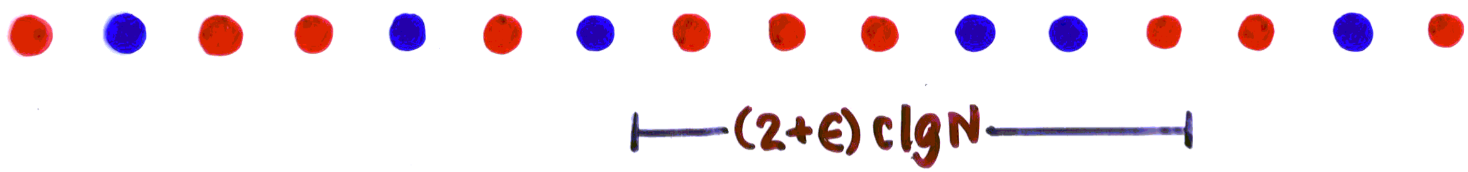


Claim: In any window of size $\Theta(\lg N)$ there are $\Theta(\lg N)$ support elements and $\Theta(\lg N)$ intercalated elements w.h.p.

$\Rightarrow$ evenly distributed.

Alternative Analysis

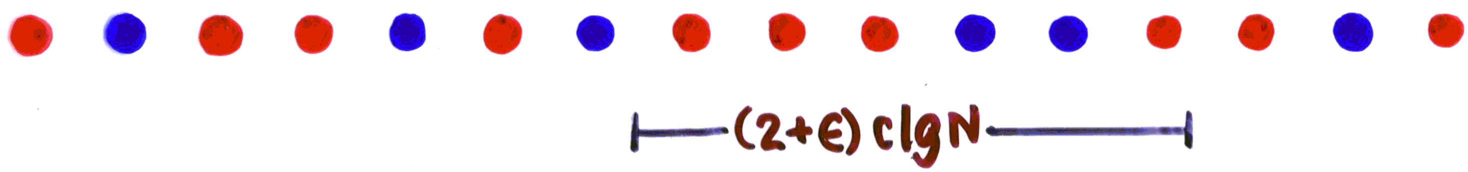
Elements ordered by keys: random permutation on insert order



Claim: For sufficiently large c, in any window of size $(2+\epsilon)\, c \lg N$, there are $> c \lg N$ support elements and $< (1+\epsilon)\, c \lg N$ intercalated elements.

Alternative Analysis

Elements ordered by keys: random permutation on insert order



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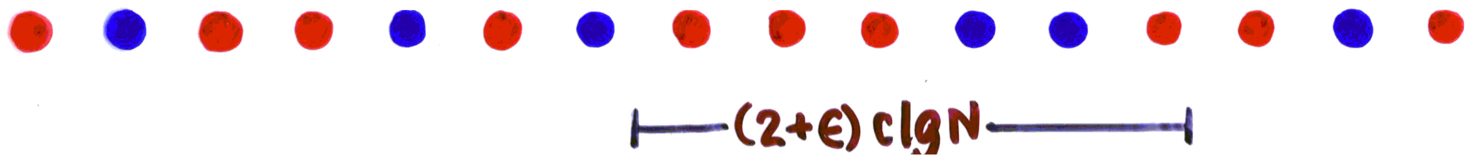
Recall: k support elements take space $(2+\epsilon)k$.

$\Rightarrow$ room for intercalated elements

Alternative Analysis



Claim: In any window of size $\Theta(\lg N)$ there are $\Theta(\lg N)$ support elements and $\Theta(\lg N)$ intercalated elements w.h.p.

Similar to # coin flips until we get a head.
$O(1)$ in expectation & $O(\lg N)$ w.h.p.

Coins are not independent, but negatively correlated.
Easy to solve using Chernoff bounds.
Can also solve directly using basic probability.

Concluding analysis

- Pr[given set C, $|C|=(2+\varepsilon)c \lg m$ has too few support elements]

$$\leq \sum_{j=0}^{c\log m} \binom{|C|}{j} \left(\frac{m}{2m-|C|+1}\right)^{j} \left(\frac{m}{2m-|C|+1}\right)^{|C|-j}$$

$$\leq \left(\frac{m}{2m-|C|+1}\right)^{|C|} \sum_{j=0}^{c\log m} \binom{|C|}{j}.$$

…which is polynomially small.

Minor Detail

Holds only when # elmts is large, but...

<u>claim:</u> while the number of elements $k \le \sqrt{n}$, the total cost for library sort is $O(n)$.

$\Longrightarrow$ only need to consider case $k \ge \Omega(\sqrt{n})$.

Thm: each insertion has cost $O(\lg N)$ w.h.p.

Gaps in My Knowledge

<u>This talk</u>: average-case analysis of naïve folk insertion.

- Related work: Ave-case priority queues [Itai,Konheim,Rodeh81]
  Contains most ideas of LibrarySort.
  LibrarySort simplifies.

Gaps in My Knowledge

Other work: rebalance schemes for worst case.

	Upper bound	Lower bound
$O(N)$ gaps Sequential File Maintenance	$O(\lg^2 N)$ insert [Itai,Konheim,Rodeh] [Willard]	$\Omega(\lg N)$ insert [Dietz][Seiferas]
poly(N) gaps order maintenance, list labeling	$O(\lg N)$ insert [Deitz][DietzSleator] [Tsakalidis] [Bender,Cole,Demaine,Farach Colton,Zito]	$\Omega(\lg N)$ insert [Dietz][Seiferas]
$O(N)$ gaps in external memory packed-memory structure in cache-oblivious algorithms	$O(1+\lg^2 N/B)$ insert	

Why do computer scientists say that insertionsort is $O(N^2)$?

Personal Experience?

Bookshelves of Distinguished Computer Scientists

Results

- **LibrarySort**, a natural implementation of **InsertionSort** with gaps.
- **Theorem: LibrarySort** runs in $O(N \lg N)$ time w.h.p. and uses linear space. Each insertion is exp $O(1)$ and $O(\lg N)$ w.h.p..