Lecture 5:
Elementary Data Structures (1997)

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Hashing, Hashing, and Hashing

Udi Manber says that the three most important algorithms at Yahoo are hashing, hashing, and hashing.
Hashing has a variety of clever applications beyond just speeding up search, by giving you a short but distinctive representation of a larger document.

- *Is this new document different from the rest in a large corpus?* – Hash the new document, and compare it to the hash codes of corpus.
- *Is part of this document plagerized from part of a document in a large corpus?* – Hash all overlapping windows of length w in the document and the corpus. If

there is a match of hash codes, there is possibly a text match.

- *How can I convince you that a file isn't changed?* – Check if the cryptographic hash code of the file you give me today is the same as that of the original. Note that any changes to the file will result in changing the hash code.

(a) Is $2^{n+1} = O(2^n)$?
(b) Is $2^{2n} = O(2^n)$?

(a) Is $2^{n+1} = O(2^n)$?

Is $2^{n+1} \leq c * 2^n$?

Yes, if $c \geq 2$ for all n

(b) Is $2^{2n} = O(2^n)?$

Is $2^{2n} \leq c * 2^n$?

note $2^{2n} = 2^n * 2^n$

Is $2^n * 2^n \leq c * 2^n$?

Is $2^n \leq c$?

No! Certainly for any constant c we can find an n such that this is not true.

Elementary Data Structures

“Mankind’s progress is measured by the number of things we can do without thinking.”

Elementary data structures such as stacks, queues, lists, and heaps will be the “of-the-shelf” components we build our algorithm from. There are two aspects to any data structure:

- The abstract operations which it supports.
- The implementation of these operations.

The fact that we can describe the behavior of our data structures in terms of abstract operations explains why we can use them without thinking, while the fact that we have different implementation of the same abstract operations enables us to optimize performance.

Stacks and Queues

Sometimes, the order in which we retrieve data is independent of its content, being only a function of when it arrived.

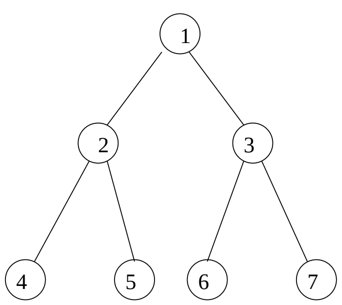
A *stack* supports last-in, first-out operations: push and pop.

A *queue* supports first-in, first-out operations: enqueue and dequeue.

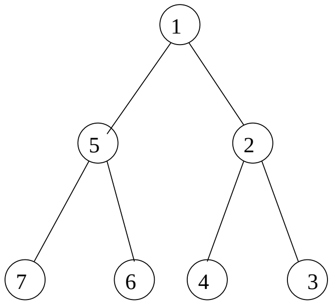
A *deque* is a double ended queue and supports all four operations: push, pop, enqueue, dequeue.

Lines in banks are based on queues, while food in my refrigerator is treated as a stack.

Both can be used to traverse a tree, but the order is completely different.



Queue



Stack

Which order is better for WWW crawler robots?

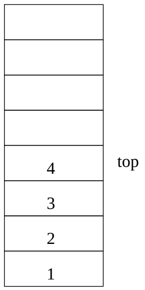
Stack Implementation

Although this implementation uses an array, a linked list would eliminate the need to declare the array size in advance.

```
STACK-EMPTY(S)
    if top[S] = 0
        then return TRUE
        else return FALSE


PUSH(S, x)
    $top[S] \leftarrow top[S] + 1$
    $S[top[S]] \leftarrow x$
```

```
POP(S)
    if STACK-EMPTY(S)
        then error “underflow”
        else $top[S] \leftarrow top[S] - 1$
            return $S[top[S] + 1]$
```

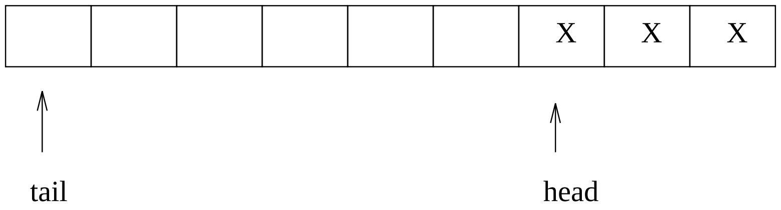


All are $O(1)$ time operations.

Queue Implementation

A circular queue implementation requires pointers to the head and tail elements, and wraps around to reuse array elements.

```
ENQUEUE(Q, x)
        Q[tail[Q]] $\leftarrow$ x
        if tail[Q] = length[Q]
                then tail[Q] $\leftarrow$ 1
                else tail[Q] $\leftarrow$ tail[Q] + 1
```



```
DEQUEUE(Q)
    x = Q[head[Q]]
    if head[Q] = length[Q]
        then head[Q] = 1
        else head[Q] = head[Q] + 1
    return x
```

A list-based implementation would eliminate the possibility of overflow.

All are $O(1)$ time operations.

Dynamic Set Operations

Perhaps the most important class of data structures maintain a set of items, indexed by keys.
There are a variety of implementations of these *dictionary* operations, each of which yield different time bounds for various operations.

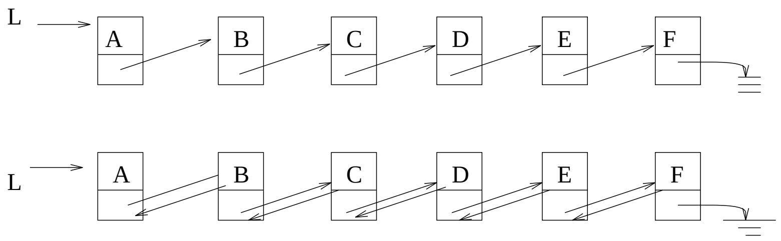
- *Search(S,k)* – A query that, given a set S and a key value k, returns a pointer x to an element in S such that $key[x]$ = k, or nil if no such element belongs to S.
- *Insert(S,x)* – A modifying operation that augments the set S with the element x.
- *Delete(S,x)* – Given a pointer x to an element in the set S,

remove x from S. Observe we are given a pointer to an element x, not a key value.

- *Min(S)*, *Max(S)* – Returns the element of the totally ordered set S which has the smallest (largest) key.
- *Next(S,x)*, *Previous(S,x)* – Given an element x whose key is from a totally ordered set S, returns the next largest (smallest) element in S, or NIL if x is the maximum (minimum) element.

Pointer Based Implementation

We can maintain a dictionary in either a singly or doubly linked list.



We gain extra flexibility on predecessor queries at a cost of doubling the number of pointers by using doubly-linked lists. Since the extra big-Oh costs of doubly-linkly lists is zero, we will usually assume they are, although it might not be necessary.

Singly linked to doubly-linked list is as a Conga line is to a Can-Can line.

Array Based Sets

Unsorted Arrays

- Search(S,k) - sequential search, $O(n)$
- Insert(S,x) - place in first empty spot, $O(1)$
- Delete(S,x) - copy nth item to the xth spot, $O(1)$
- Min(S,x), Max(S,x) - sequential search, $O(n)$
- Successor(S,x), Predecessor(S,x) - sequential search, $O(n)$

Sorted Arrays

- Search(S,k) - binary search, $O(\lg n)$

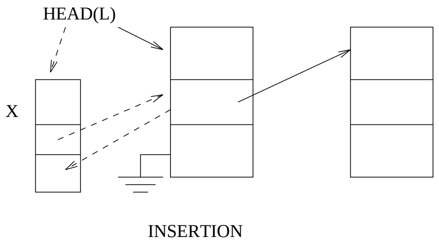
- Insert(S,x) - search, then move to make space, $O(n)$
- Delete(S,x) - move to fill up the hole, $O(n)$
- Min(S,x), Max(S,x) - first or last element, $O(1)$
- Successor(S,x), Predecessor(S,x) - Add or subtract 1 from pointer, $O(1)$

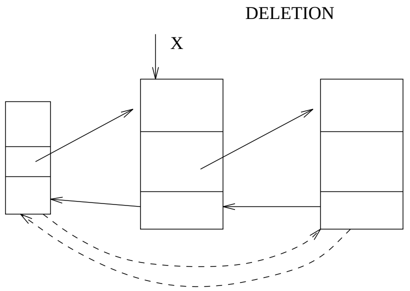
What are the costs for a heap?

Unsorted List Implementation

```
LIST-SEARCH(L, k)
    $x$ = head[L]
    while $x <> NIL$ and $key[x] <> k$
        do $x$ = next[x]
    return $x$
```

Note: the while loop might require two lines in some programming languages.



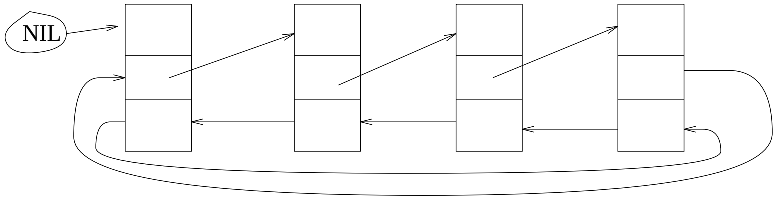


```
LIST-INSERT(L, x)
    next[x] = head[L]
    if head[L] <> NIL
        then prev[head[L]] = x
    head[L] = x
    prev[x] = NIL

LIST-DELETE(L, x)
    if $prev[x] <> NIL$
        then next[prev[x]] = next[x]
        else head[L] = next[x]
    if $next[x] <> NIL$
        then prev[next[x]] = prev[x]
```

Sentinels

Boundary conditions can be eliminated using a sentinel element which doesn't go away.



```
LIST-SEARCH'(L, k)
    $x$ = next[nil[L]]
    while $x <> NIL[L]$ and $key[x] <> k$
        do $x$ = next[x]
    return $x$
```

```
LIST-INSERT'(L, x)
        next[x] = next[nil[L]]
        prev[next[nil[L]]] = x
        next[nil[L]] = x
        prev[x] = NIL[L]


LIST-DELETE'(L, x)
        next[prev[x]] <> next[x]
        next[prev[x]] = prev[x]
```

Hash Tables

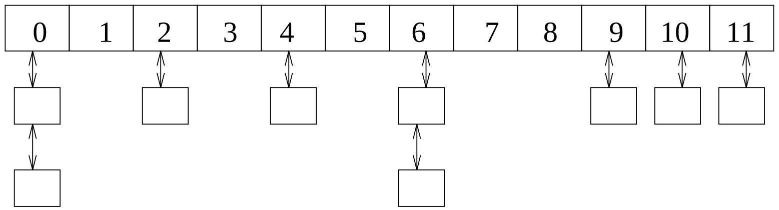
Hash tables are a *very practical* way to maintain a dictionary. As with bucket sort, it assumes we know that the distribution of keys is fairly well-behaved.

The idea is simply that looking an item up in an array is $\Theta(1)$ once you have its index. A hash function is a mathematical function which maps keys to integers.

In bucket sort, our hash function mapped the key to a bucket based on the first letters of the key. “Collisions” were the set of keys mapped to the same bucket.

If the keys were uniformly distributed, then each bucket contains very few keys!

The resulting short lists were easily sorted, and could just as easily be searched!



Hash Functions

It is the job of the hash function to map keys to integers. A good hash function:

1. Is cheap to evaluate
2. Tends to use all positions from $0 \ldots M$ with uniform frequency.
3. Tends to put similar keys in different parts of the tables (Remember the Shifletts!!)

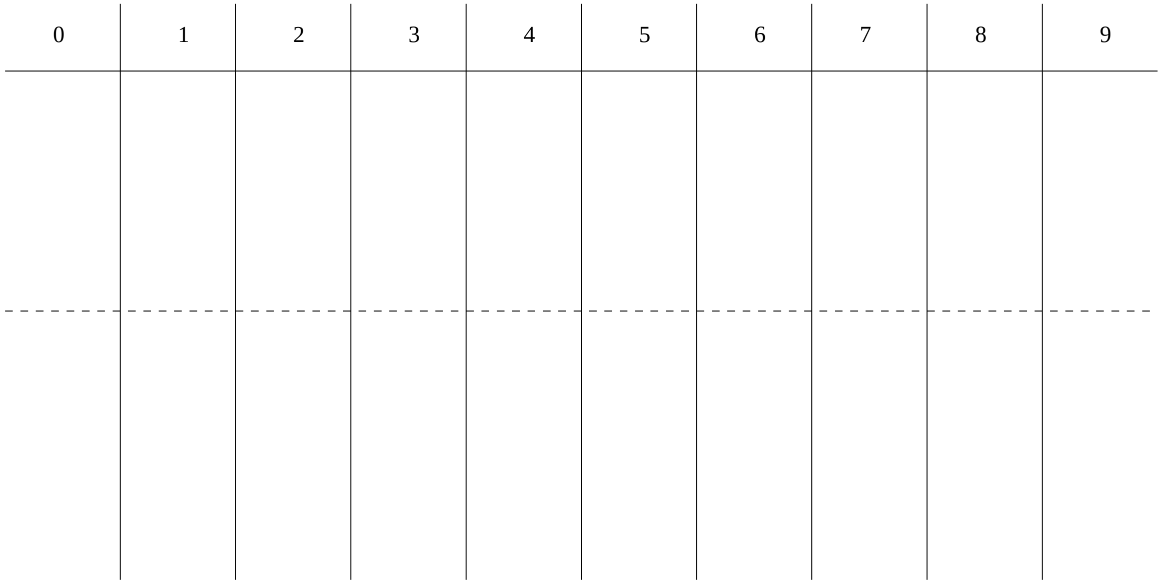
The first step is usually to map the key to a big integer, for example

$$h = \sum_{i=0}^{keylength} 128^i \times char(key[i])$$

This large number must be reduced to an integer whose size is between 1 and the size of our hash table.
One way is by $h(k) = k \bmod M$, where M is best a large prime not too close to $2^i - 1$, which would just mask off the high bits.
This works on the same principle as a roulette wheel!

Good and Bad Hash functions

The first three digits of the Social Security Number



The last three digits of the Social Security Number

0	1	2	3	4	5	6	7	8	9

The Birthday Paradox

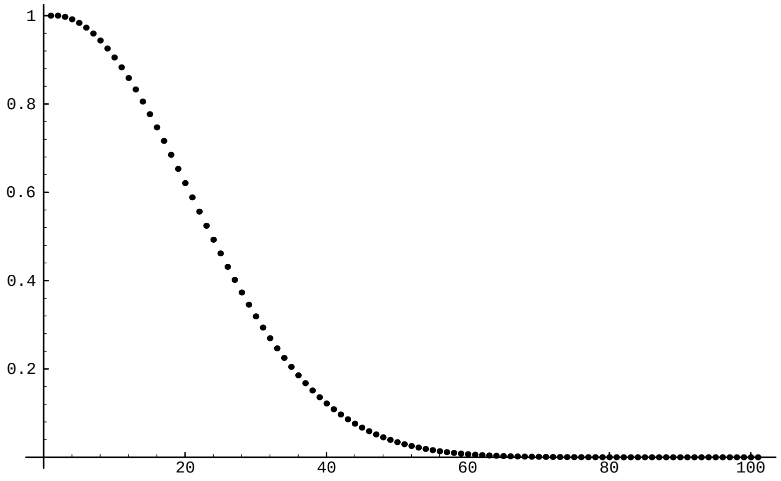
No matter how good our hash function is, we had better be prepared for collisions, because of the birthday paradox.

J	F	M	A	M	J	J1	A	S	O	N	D

The probability of there being *no* collisions after n insertions into an m-element table is

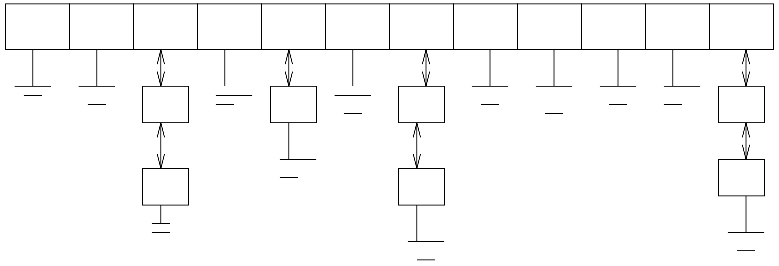
$$(m/m)\times((m-1)/m)\times\ldots\times((m-n+1)/m) = \Pi_{i=0}^{n-1}(m-i)/m$$

When $m = 366$, this probability sinks below 1/2 when $N = 23$ and to almost 0 when $N \geq 50$.


1
0.8
0.6
0.4
0.2
20
40
60
80
100

Collision Resolution by Chaining

The easiest approach is to let each element in the hash table be a pointer to a list of keys.

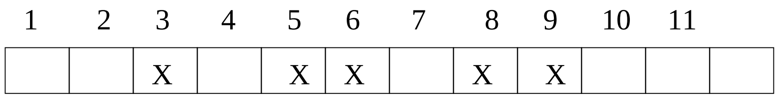


Insertion, deletion, and query reduce to the problem in linked lists. If the n keys are distributed uniformly in a table of size m/n, each operation takes $O(m/n)$ time.
Chaining is easy, but devotes a considerable amount of memory to pointers, which could be used to make the table

larger. Still, it is my preferred method.

Open Addressing

We can dispense with all these pointers by using an implicit reference derived from a simple function:



If the space we want to use is filled, we can examine the remaining locations:

1. Sequentially $h, h+1, h+2, \ldots$
2. Quadratically $h, h+1^2, h+2^2, h+3^2 \ldots$
3. Linearly $h, h+k, h+2k, h+3k, \ldots$

The reason for using a more complicated science is to avoid long runs from similarly hashed keys.

Deletion in an open addressing scheme is ugly, since removing one element can break a chain of insertions, making some elements inaccessible.

Performance on Set Operations

With either chaining or open addressing:

- Search - $O(1)$ expected, $O(n)$ worst case
- Insert - $O(1)$ expected, $O(n)$ worst case
- Delete - $O(1)$ expected, $O(n)$ worst case
- Min, Max and Predecessor, Successor $\Theta(n+m)$ expected and worst case

Pragmatically, a hash table is often the best data structure to maintain a dictionary. However, we will not use it much in proving the efficiency of our algorithms, since the worst-case time is unpredictable.

The best worst-case bounds come from balanced binary trees, such as red-black trees.