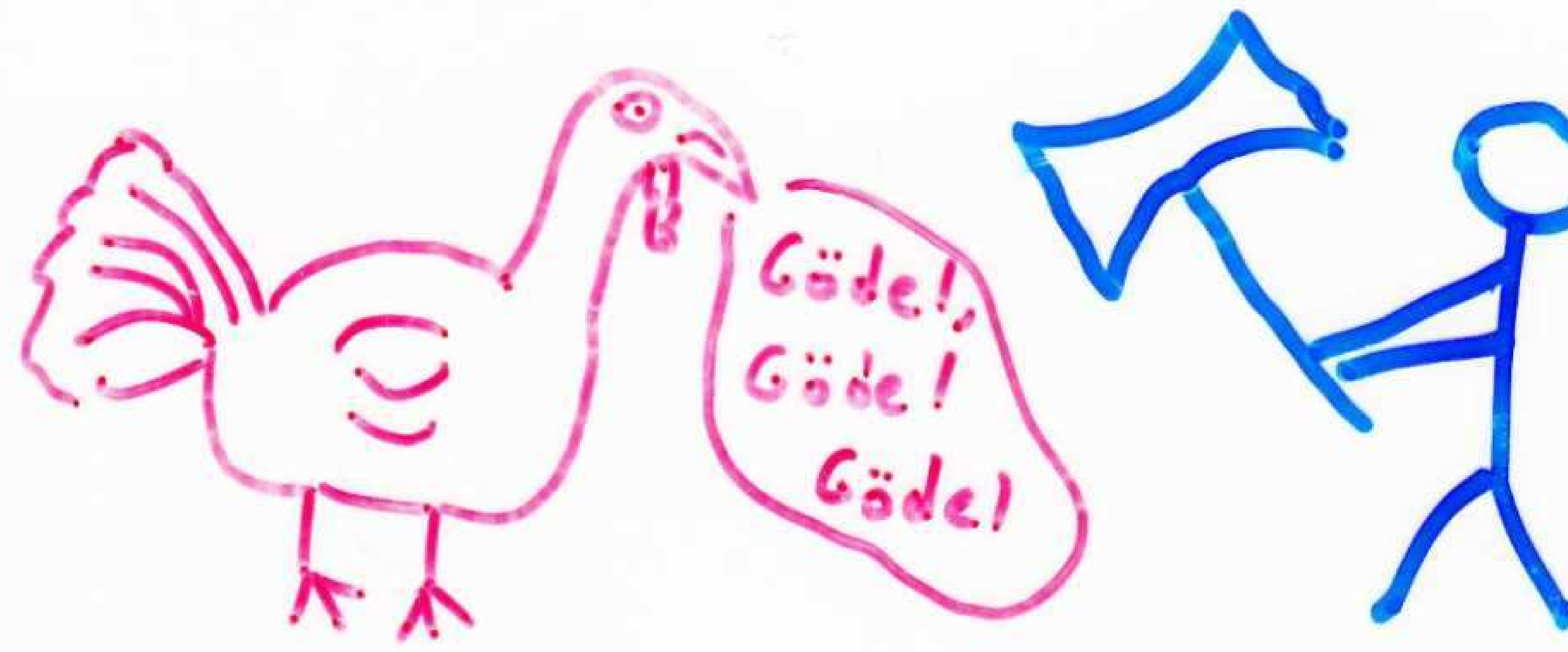
Connectivity

Connectivity in graphs may seem like a simple issue - either you are or you aren't:



Soon to be disconnected Turkey

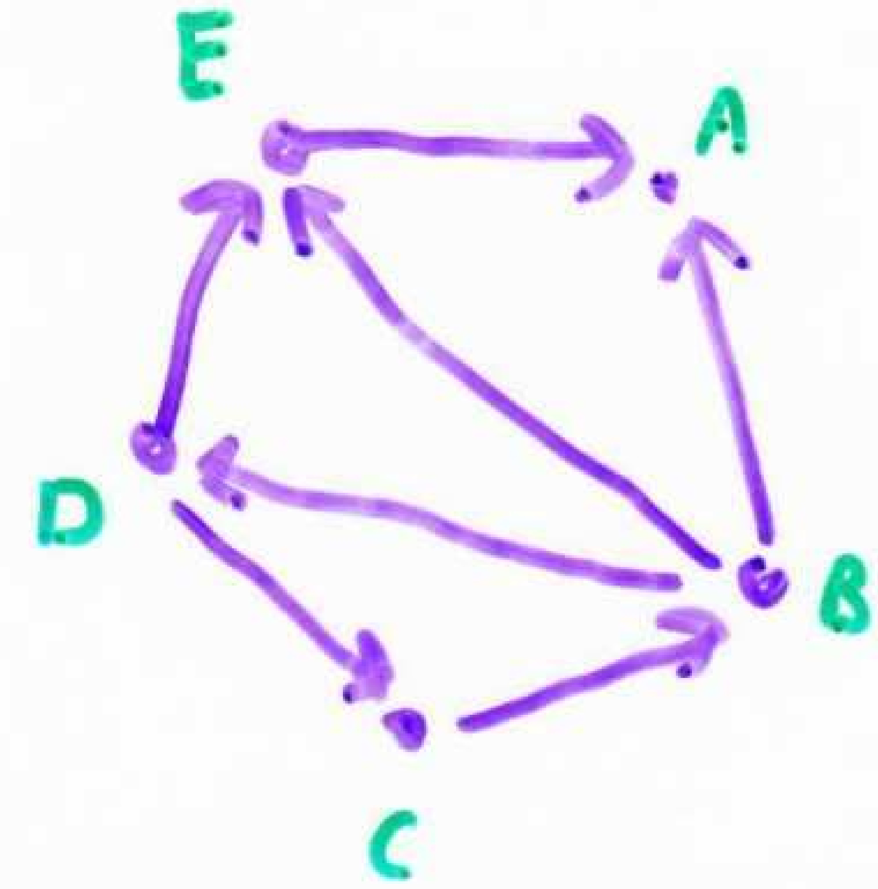
However, there are two questions which make things more interesting

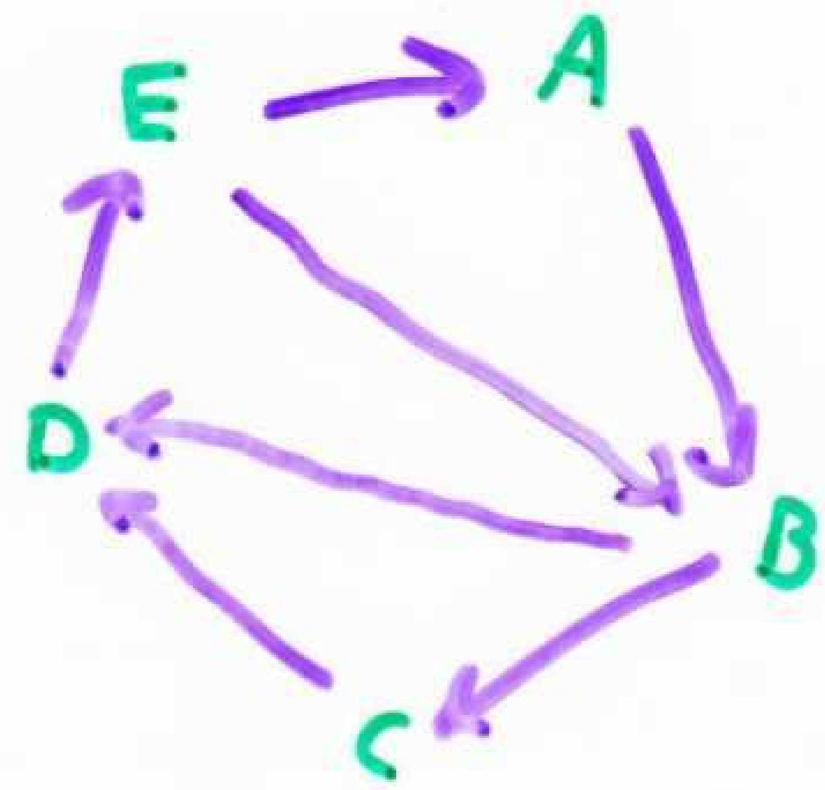
- What about directed graphs?
- Can one graph be "more connected" than another?

An undirected graph is connected if there exists a path between each pair of vertices.

Directed graphs have two notions of connectivity. A graph is strongly connected if there exists a directed path between each pair of vertices. A graph is weakly connected if it would be connected ignoring the direction of edges.

Examples:



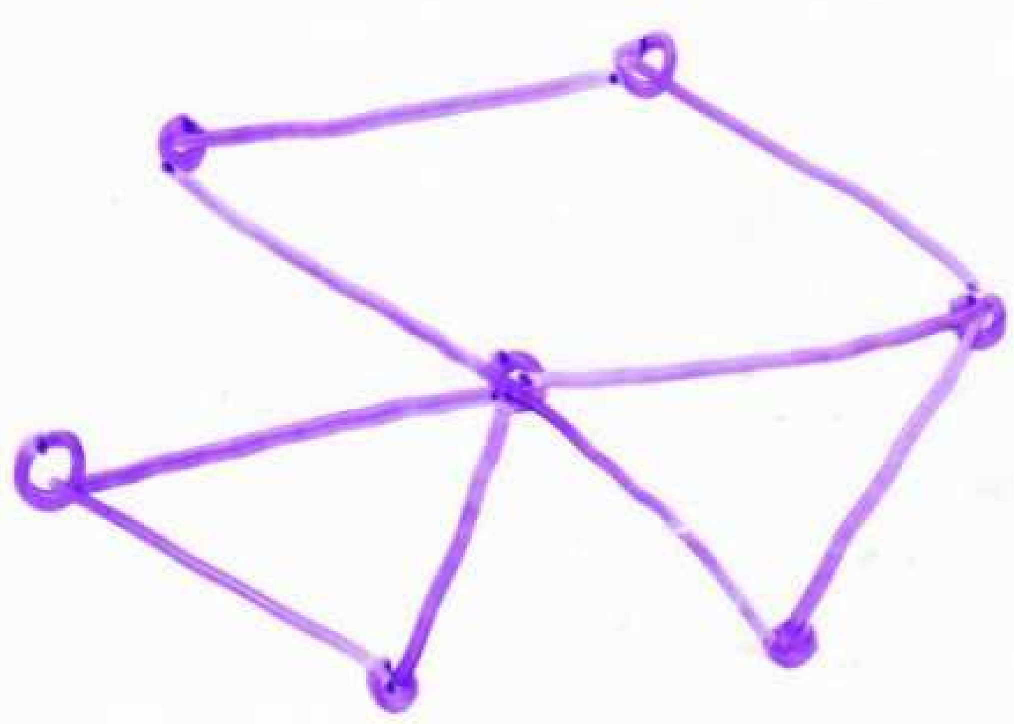


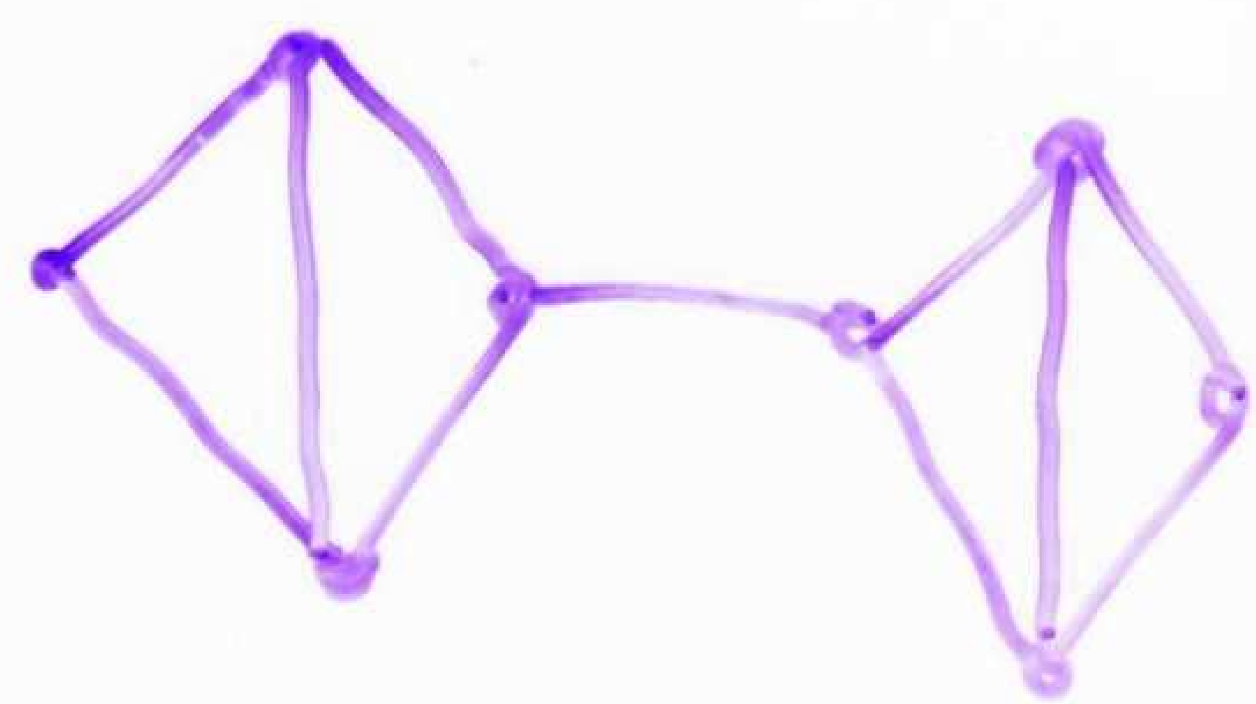
weakly connected - no path from A to anywhere

strongly connected

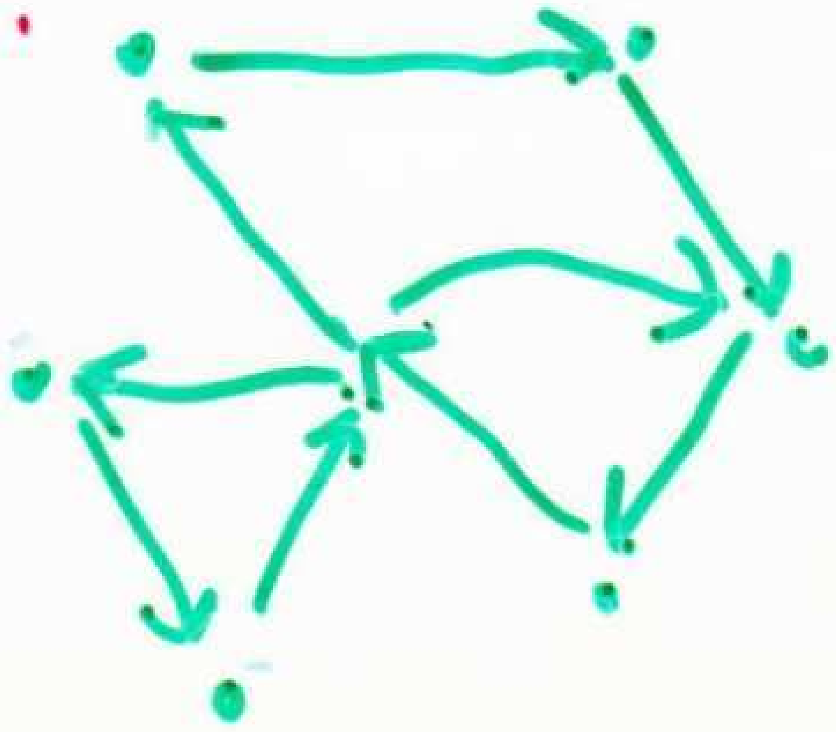
Clearly any strongly connected graph is weakly connected, but not necessarily visa-versa.

Suppose I give you an undirected graph. Under what conditions can you orient the graph - ie. find a labeling of the edges so it is strongly connected?





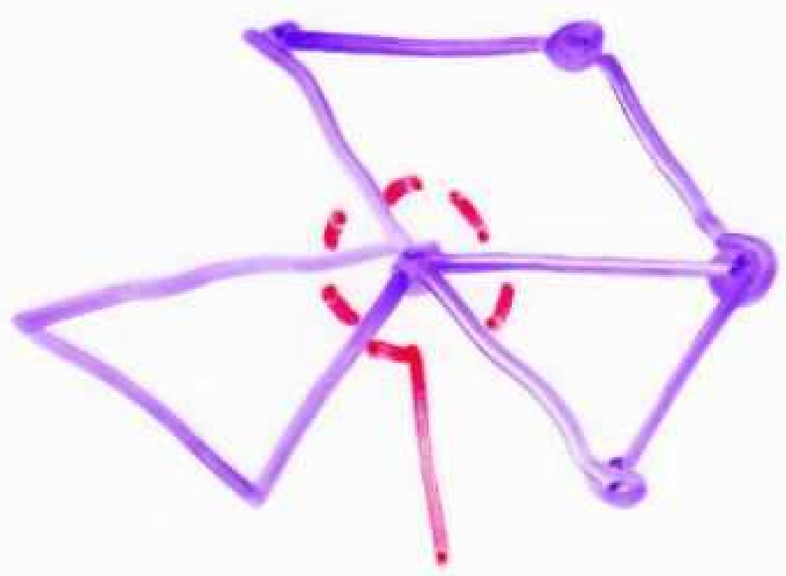
Yes!



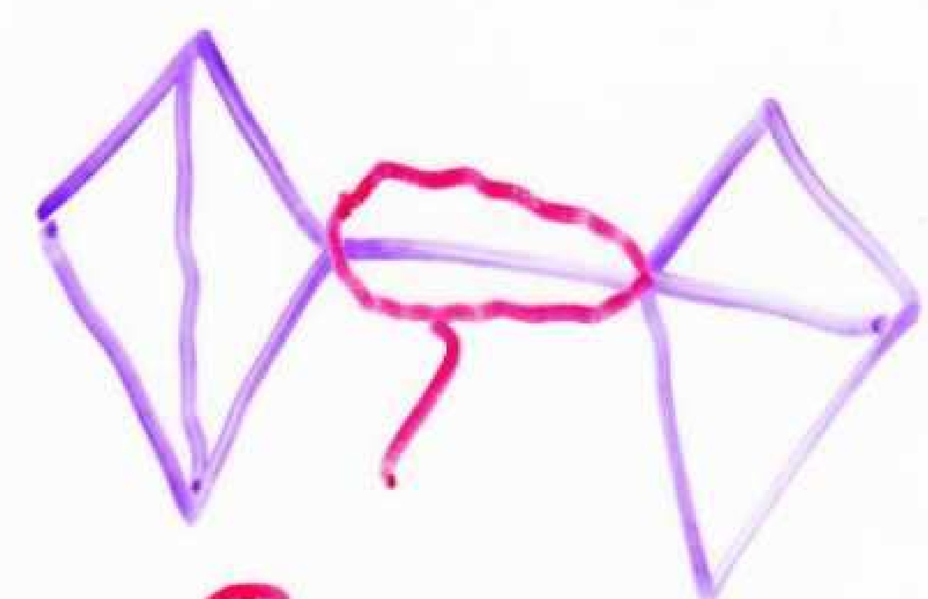
No! Once the center edge is oriented, there will be no path between components in the other direction

A bridge is an edge whose deletion disconnects the graph.

An articulation vertex is a vertex whose deletion disconnects the graph.



Articulation vertex

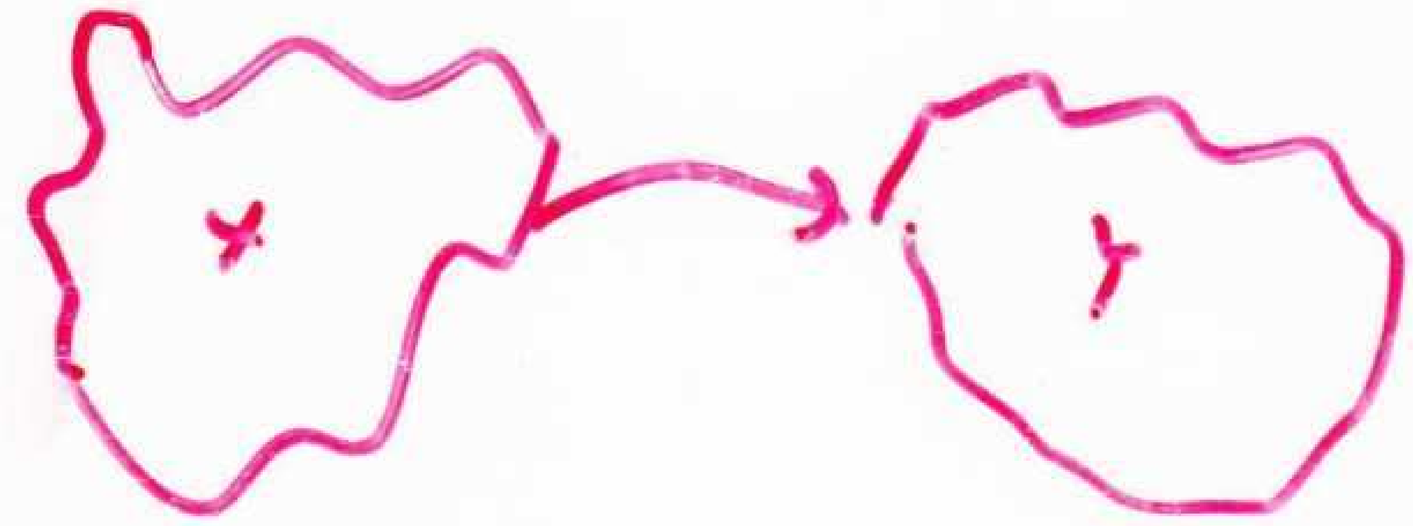


Bridge

(any edge in a tree is a bridge)

Theorem: A connected graph is Orientable iff and only if it does not contain a bridge.

Proof: Clearly, if it contains a bridge, it isn't orientable, as the bridge can only have one direction:



No path from y to x.

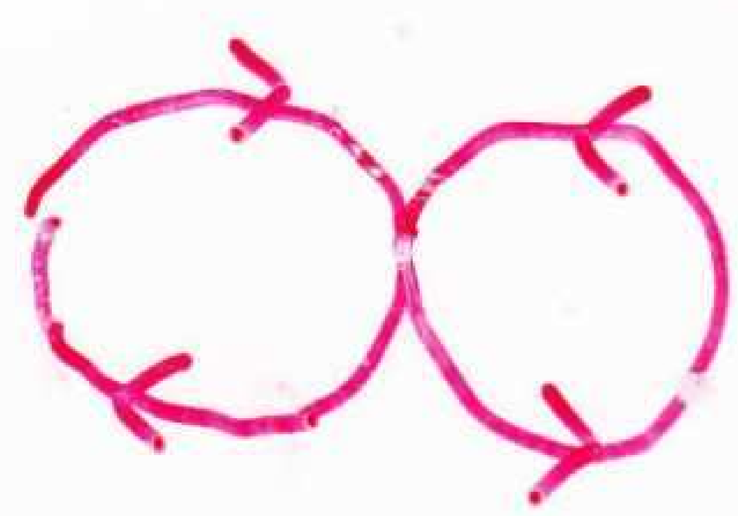
To prove the converse, we give an algorithm to orient a bridgeless graph.

while (G is not acyclic)
- find a cycle in G
- orient it as a directed cycle
- Delete the cycle from G

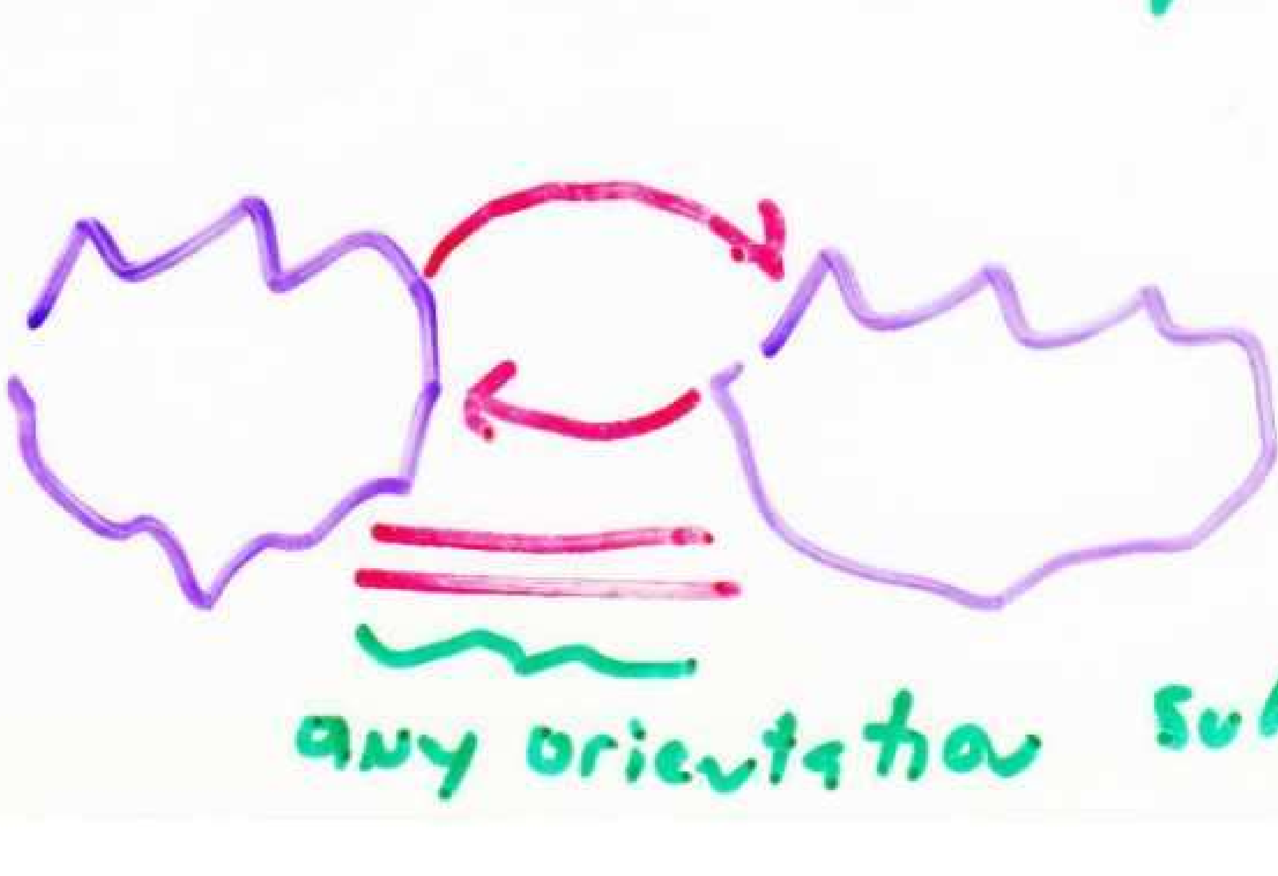
Between disconnected components, use two edges oriented in different directions.

Clearly, in a directed cycle, every pair of points is connected by a directed path.

If two directed cycles share a vertex, every pair of vertices is connected by a directed path:



If after all cycles have been deleted, we have disconnected oriented components, since the graph is bridgeless, there exist at least two edges between them



any orientation suffices.

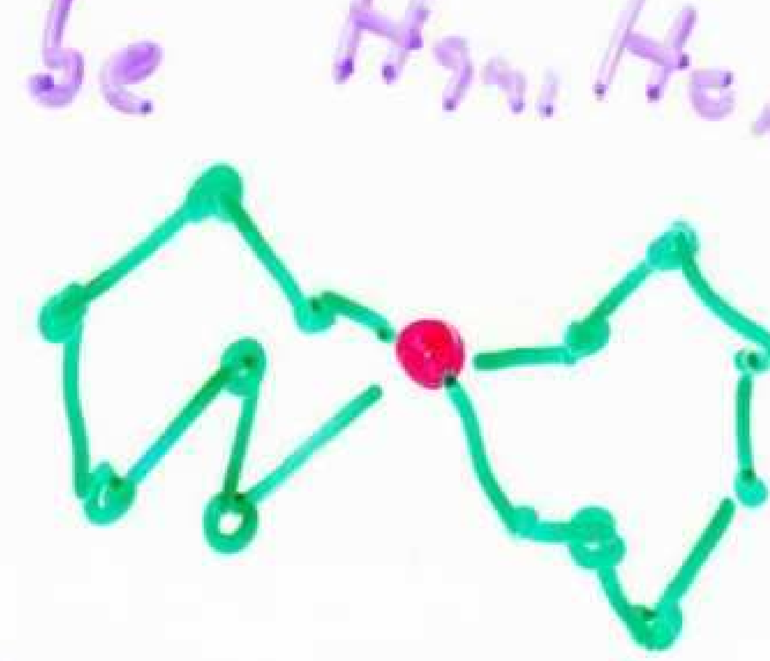
Biconnectivity

A graph G is biconnected if it contains no articulation vertex (ie. no vertex can be deleted to disconnect the graph). A graph G is edge-biconnected if it contains no bridge (ie. no edge can be deleted to disconnect G).

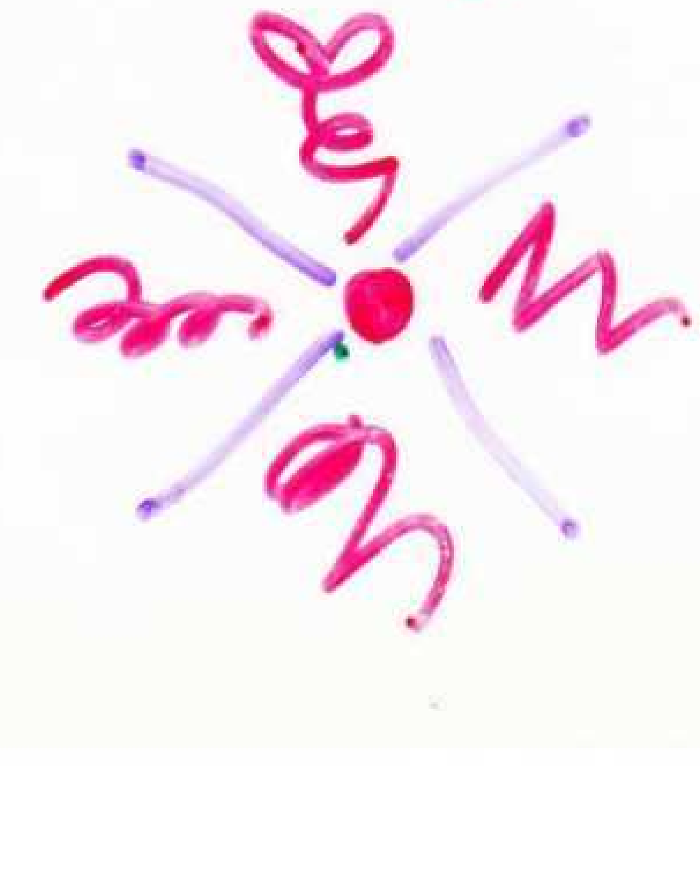
Except for the special case , any biconnected graph is edge biconnected but not necessarily visa-versa.

Biconnectivity is important from the standpoint of reliability - which telephone line should a NYNEX vandal cut or which node if it went down in a network would disrupt communications?

Only Biconnected graphs can be Hamiltonian

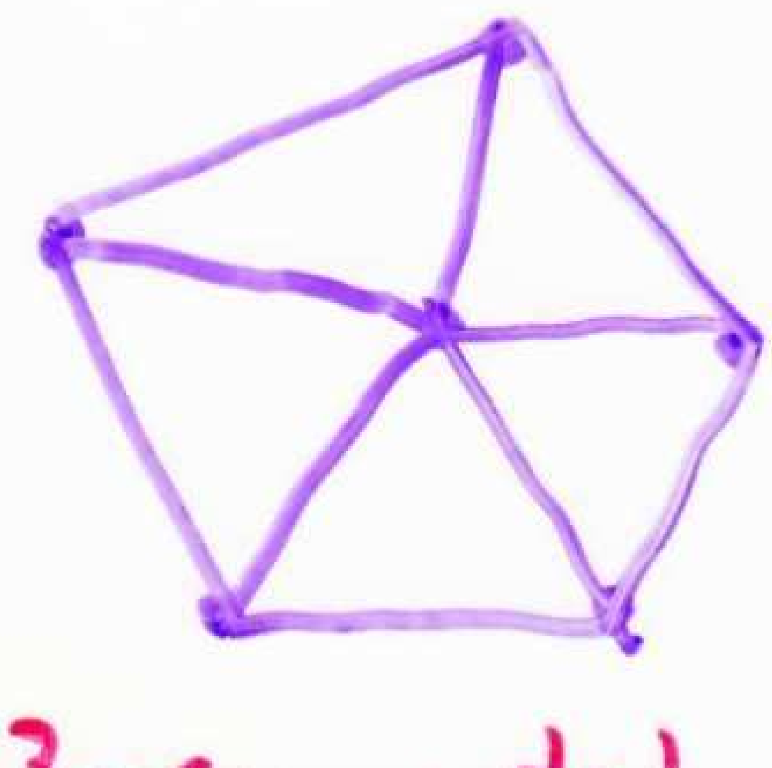


Testing Planarity reduces to testing for biconnected components: since any planar graph can have any vertex on the outside.

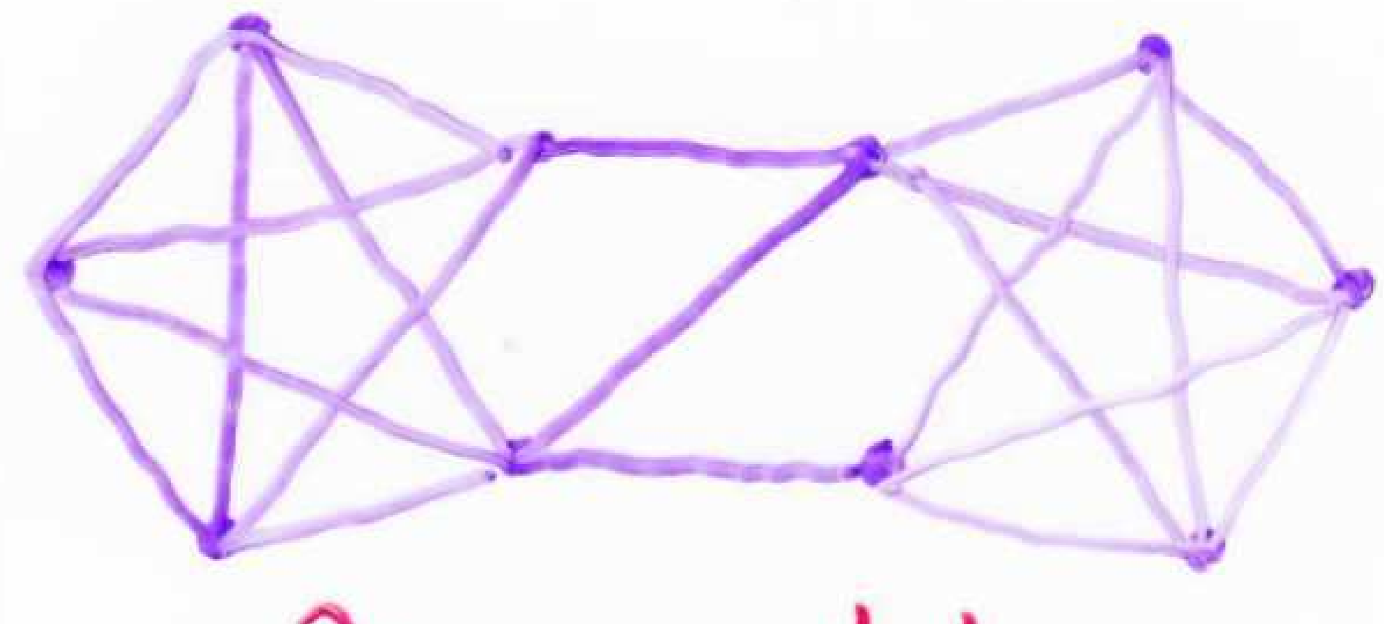


k-connectivity

We can generalize biconnectivity (or 2-connectivity) to k-connectivity, where k is the minimum number of vertices which must get blown away to disconnect G.



3-connected
3-edge connected



2-connected
3-edge-connected
minimum degree 4

There is simple relationship between vertex connectivity $\kappa(G)$, edge connectivity $\lambda(G)$, & minimum degree $\delta(G)$:

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

deleting the $\delta(G)$ incident edges or vertices disconnects the graph.

Let E be a set of edges which separate s & t. For each edge in E, eliminate a vertex that isn't s or t. At most |E| vertices are eliminated.

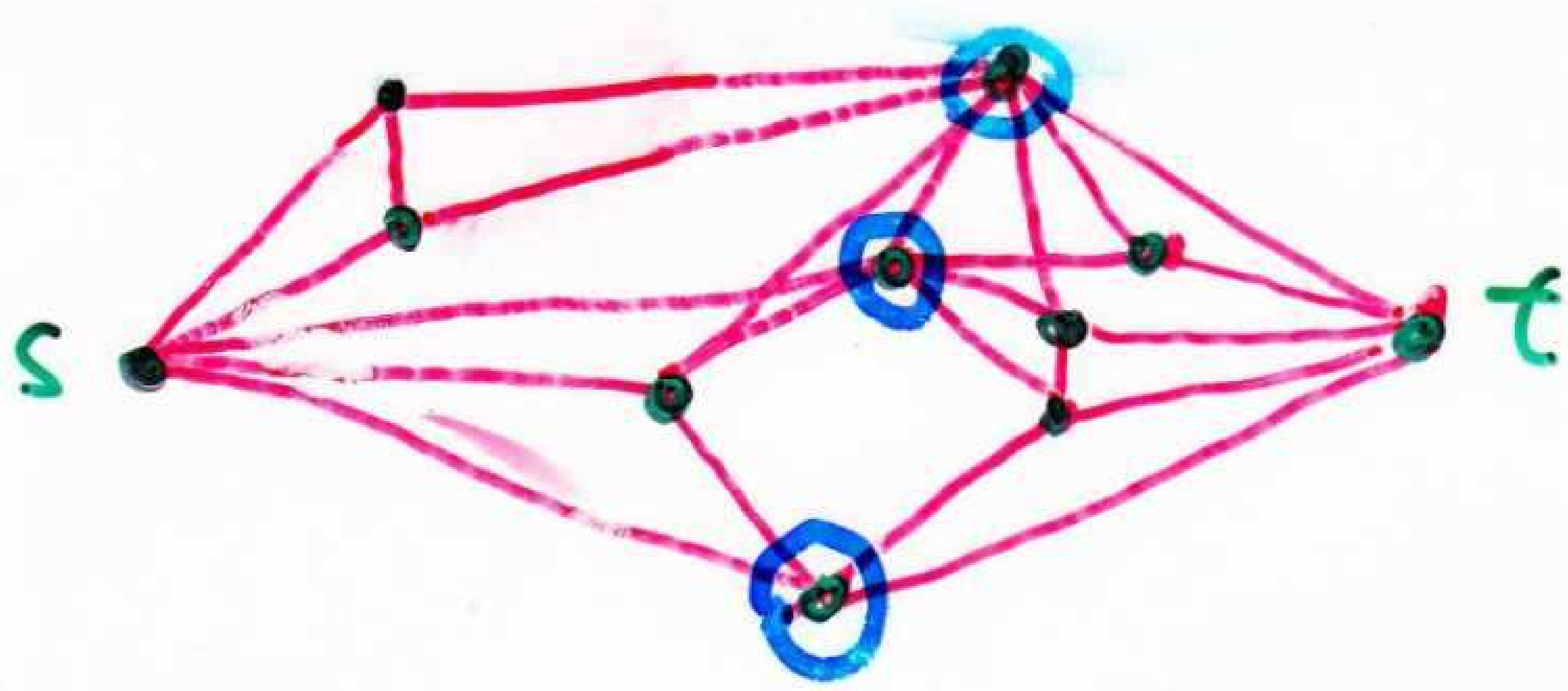
beware of the special case K_2 =

6

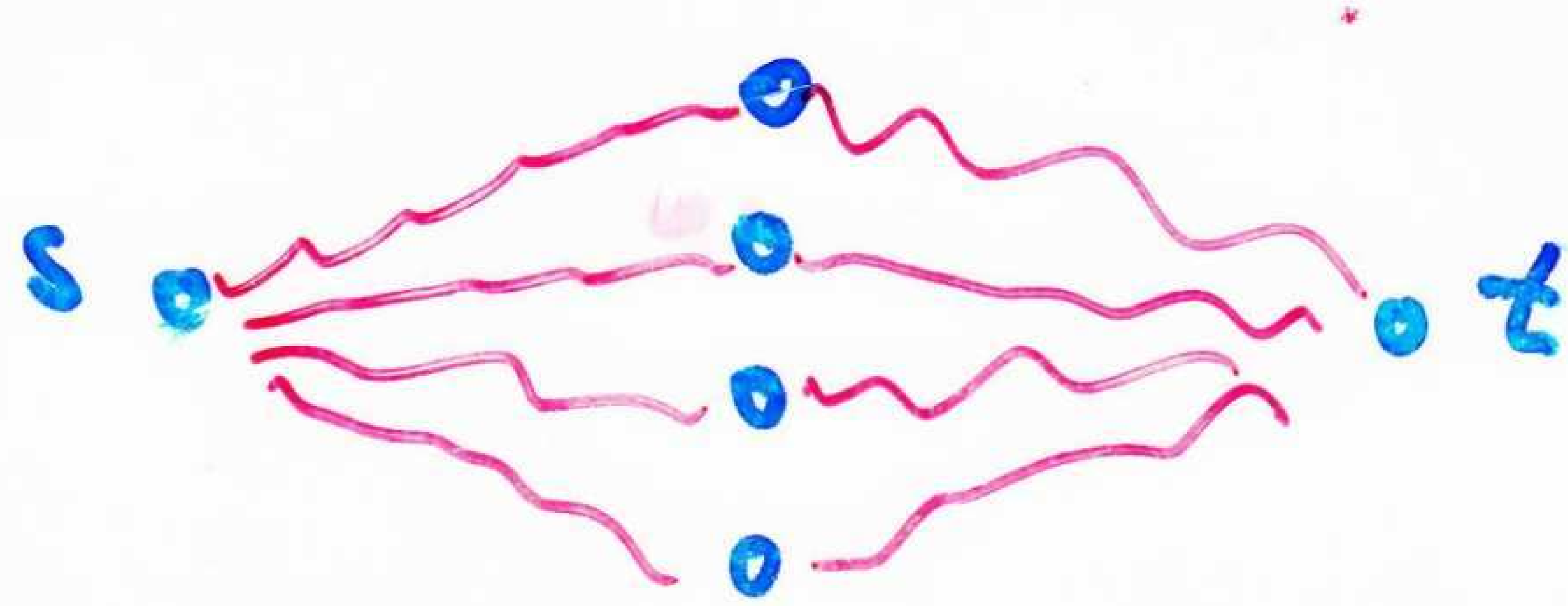
Menger's Theorem

The most important characterization of k-connectivity, which also leads to an algorithm to find the connectivity of a graph, is Menger's Theorem.

Theorem: The minimum number of vertices separating two non-adjacent vertices s + t is the maximum number of vertex disjoint s-t paths.



Proof: If k vertices separate s + t, there cannot be more than k vertex-disjoint paths from s to t.



We must now show that if k but not $k-1$ vertices separate s + t, there exists k vertex disjoint paths.

$k = 1$ True, since G is connected, there exists at least one path between s + t.

$k > 1$ We will proceed by contradiction.

meaning $\nexists$ h vertex disjoint paths

- Let h be the smallest k which fails.
- Let F be the graph with the fewest vertices such that it fails for h.
- Let G be the subgraph of F such that h vertices are necessary to separate G, but deleting any edge of G gives a graph which can be separated by $h-1$ vertices
  - fewer edges makes it easier to separate
  - deleting edges not incident to a set of h separating vertices will not change the fact that they separate, but can create other sets of separating vertices.

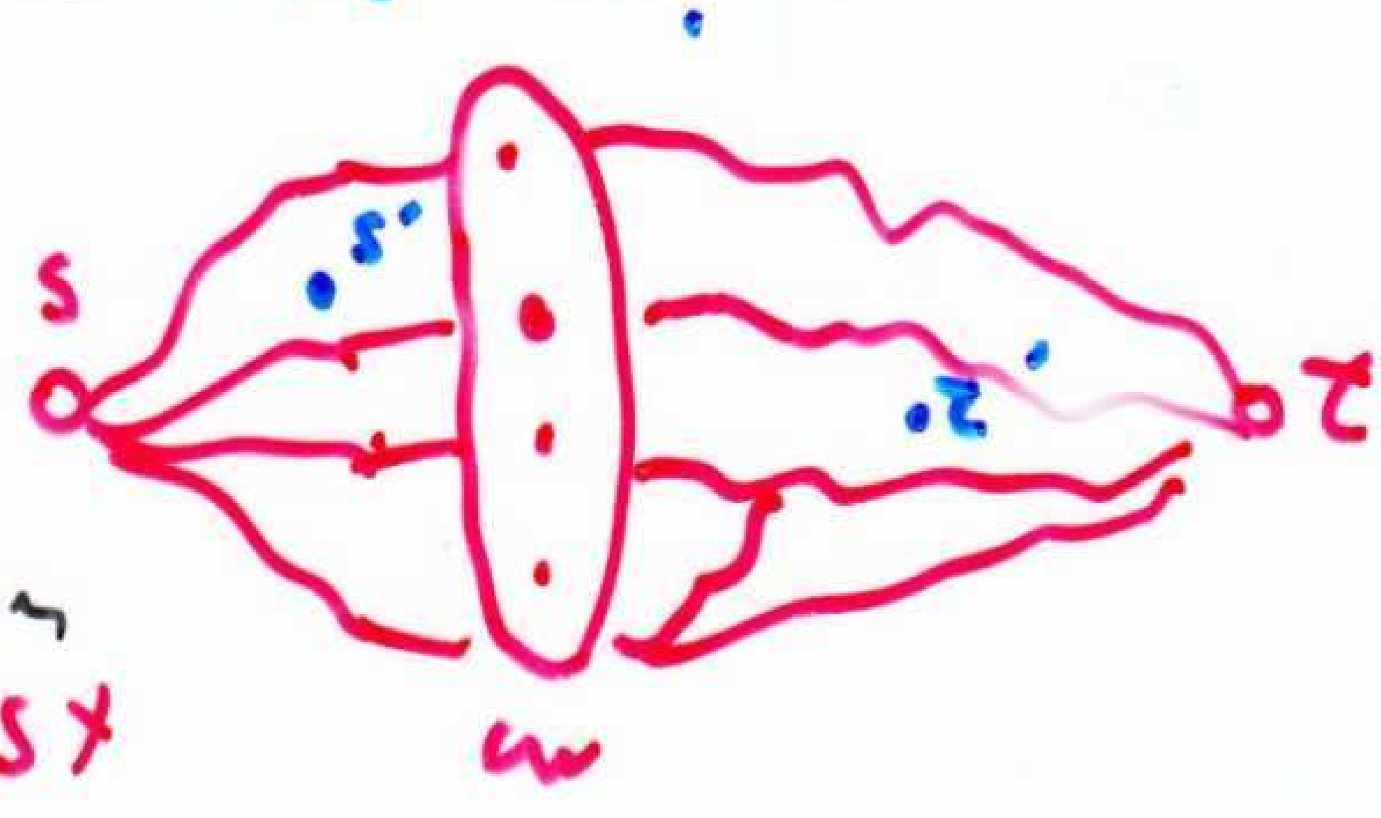
This graph G has two properties:

(I) No vertex is adjacent to both s & t.

- If vertex w is adjacent to s and t, deleting w leaves a separating set of size $h-1$, which since G is the smallest counter-example, means $G/\{w\}$ contains $h-1$ vertex-disjoint paths. But then the addition of $s-w-t$ gives h vertex disjoint paths and G is not a counter-example!

(II) Any separating set W of G where $|W| = h$ is either all adjacent to s or are all adjacent to t.

- observe that all paths from s to t go from s to w_i & from w_i to t. The shortest such paths meet only in one vertex w_i



- Suppose this wasn't true, ie. $\exists$ a vertex t' "between" W + t. Deleting it and connecting t to W directly creates a smaller graph, meaning there exist h vertex-disjoint paths from s to W. Repeating the argument with s' gives h vertex disjoint paths from W to t, which together contradict that G is a counter-example.

As a consequence of two properties G must have, we show that no such graph exists.

(I) No vertex is adjacent to both S & t

(II) Any set W of 4 separating points are either all adjacent to S or all to t

Assuming these are true, let P = the shortest s-t path $\{s, v_1, v_2 \dots t\}$.

- $v_2 \neq t$ by (I)
- Let $x = \{v_1, v_2\}$. If we delete x from G, the resulting graph has a separation of 4-1 vertices, $S(x)$.
- There is no edge $\{v_1, t\}$, by (I)
- Set $S(x) \cup \{v_1\}$ is 4 vertices which separate G, so, by (II) all of these vertices are adjacent to S.
- But $S(x) \cup \{v_2\}$ are also a separating set of 4 vertices, so by (II) there is an edge (s, v_2).

But then $\{s, v_1, v_2 \dots t\}$ could not be the shortest path!

#9

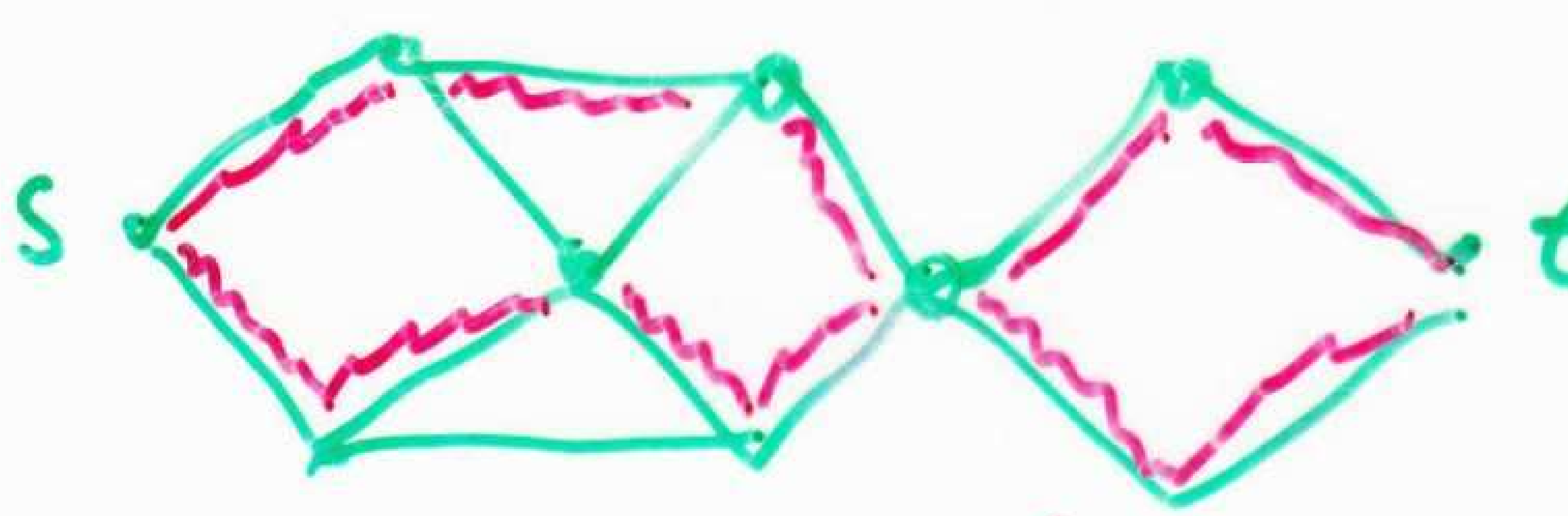
The MAX-flow, Min-cut Theorem

An alternate form of Menger's theorem is useful in finding the max flow through a network.

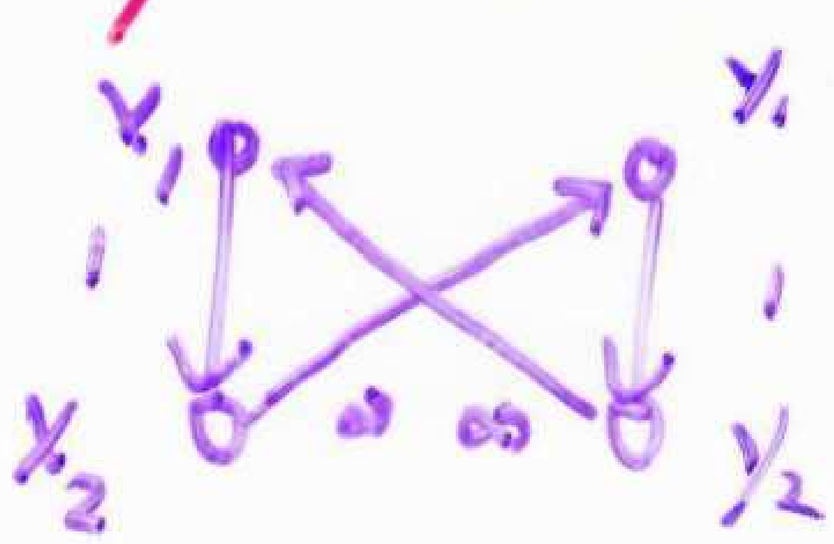
★ [The maximum flow from s to t = the capacity of the minimum cut.

Clearly the max-flow cannot exceed the min-cut, but realizing it is in some sense like proving Menger's theorem.

Observe that in a flow through a network, where each edge has capacity one, the flow counts the number of "edge" disjoint paths:



By replacing each vertex* by two vertices x_1 & x_2 and each edge x ∘——∘ y by



construction, we create a flow problem where only one path can go through a vertex, and so the max flow = min cut, the vertex connectivity of the graph.