Basic Maneuvers:

You should see <u>WHY</u> each of these works!

Table ~~320~~ 334 Generating function manipulations.

addition: $$\alpha F(z) + \beta G(z) = \sum_n (\alpha f_n + \beta g_n) z^n$$

right shift: $$z^m G(z) = \sum_n g_{n-m} z^n, \qquad \text{integer } m \geqslant 0$$

Left shift: delete low order terms $$\frac{G(z) - g_0 - g_1 z - \cdots - g_{m-1} z^{m-1}}{z^m} = \sum_{n \geqslant 0} g_{n+m} z^n, \qquad \text{integer } m \geqslant 0$$

variable substitution: $$G(cz) = \sum_n c^n g_n z^n$$

Differentiation: Lowers degree by 1 $$G'(z) = \sum_n (n+1) g_{n+1} z^n$$

$$zG'(z) = \sum_n n g_n z^n$$

gives counting variable

Integration: raises degree by 1 $$\int_0^z G(t)\,dt = \sum_{n \geqslant 1} \frac{1}{n} g_{n-1} z^n$$

Convolution: $$F(z) G(z) = \sum_n \Big(\sum_k f_k g_{n-k} \Big) z^n$$

special case $f_i = 1$ gives prefix summations $$\frac{1}{1-z} G(z) = \sum_n \Big(\sum_{k \leqslant n} g_k \Big) z^n$$

You should understand how to move between each sequence, generating function, and closed form!

Table ~~321~~ 335 Simple sequences and their generating functions.

	sequence	generating function	closed form
Single terms	$\langle 1,0,0,0,0,0,\ldots \rangle$	$\sum_{n\geq 0}[n=0]\,z^n$	1
	$\langle 0,\ldots,0,1,0,0,\ldots \rangle$	$\sum_{n\geq 0}[n=m]\,z^n$	z^m
geometric series	$\langle 1,1,1,1,1,1,\ldots \rangle$	$\sum_{n\geq 0} z^n$	$\dfrac{1}{1-z}$
$z \to -z$	$\langle 1,-1,1,-1,1,-1,\ldots \rangle$	$\sum_{n\geq 0}(-1)^n z^n$	$\dfrac{1}{1+z}$
every n^{th} term of a geometric series	$\langle 1,0,1,0,1,0,\ldots \rangle$	$\sum_{n\geq 0}[2\backslash n]\,z^n$	$\dfrac{1}{1-z^2}$
	$\langle 1,0,\ldots,0,1,0,\ldots,0,1,0,\ldots \rangle$	$\sum_{n\geq 0}[m\backslash n]\,z^n$	$\dfrac{1}{1-z^m}$
derivative of geometric series	$\langle 1,2,3,4,5,6,\ldots \rangle$	$\sum_{n\geq 0}(n+1)\,z^n$	$\dfrac{1}{(1-z)^2}$
variable substitution	$\langle 1,2,4,8,16,32,\ldots \rangle$	$\sum_{n\geq 0} 2^n z^n$	$\dfrac{1}{1-2z}$
Binomial Theorem	$\langle 1,4,6,4,1,0,0,\ldots \rangle$	$\sum_{n\geq 0}\binom{4}{n} z^n$	$(1+z)^4$
	$\langle 1,c,\binom{c}{2},\binom{c}{3},\ldots \rangle$	$\sum_{n\geq 0}\binom{c}{n} z^n$	$(1+z)^c$
Binomial THEOREM: negative c	$\left\langle 1,c,\binom{c+1}{2},\binom{c+2}{3},\ldots \right\rangle$	$\sum_{n\geq 0}\binom{c+n-1}{n} z^n$	$\dfrac{1}{(1-z)^c}$
variable subst.	$\langle 1,c,c^2,c^3,\ldots \rangle$	$\sum_{n\geq 0} c^n z^n$	$\dfrac{1}{1-cz}$
Binomial Theorem: negative c	$\left\langle 1,\binom{m+1}{m},\binom{m+2}{m},\binom{m+3}{m},\ldots \right\rangle$	$\sum_{n\geq 0}\binom{m+n}{m} z^n$	$\dfrac{1}{(1-z)^{m+1}}$
Special functions	$\langle 0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots \rangle$	$\sum_{n\geq 1}\frac{1}{n} z^n$	$\ln\dfrac{1}{1-z}$
	$\langle 0,1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots \rangle$	$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n} z^n$	$\ln(1+z)$
	$\langle 1,1,\frac{1}{2},\frac{1}{6},\frac{1}{24},\frac{1}{120},\ldots \rangle$	$\sum_{n\geq 0}\frac{1}{n!} z^n$	e^z

MOST IMPORTANT

What is the Expansion of $\dfrac{1}{(1-z)^{m+1}}$?

The expansion will require binomial coefficents with negative arguments, which is ok since we define $\binom{N}{k} = \dfrac{N^{\underline{k}}}{k!}$

Since

$$N^{\underline{k}} = N(N-1)\cdots(N-k+1)$$
$$= (-1)^k(-N)(1-N)\cdots(k-N-1)$$
$$= (-1)^k(k-N-1)^{\underline{k}}$$

We have the identity

$$\boxed{\binom{N}{k} = (-1)^k\binom{k-N-1}{k}}$$

By the binomial theorem:

$$\frac{1}{(1-z)^{m+1}} = (1-z)^{-(m+1)} = \sum_{n \geq 0} \binom{-(m+1)}{n} (-z)^n \, 1^{-(m+1)-n}$$

$$= \sum_{n \geq 0} \binom{-m-1}{n} (-1)^n z^n$$

$$= \sum_{n \geq 0} \binom{n-m-n-1}{n} (-1)^n z^n$$

Since $\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$ with the substitution $k = n$ + $r = m+n$

$$\boxed{= \sum_{n \geq 0} \binom{n+m}{n} z^n}$$

Note that $\binom{n+m}{n}$ is an $m^{\underline{th}}$ degree polynomial in n.

Solving Recurrences

The main advantage of the generating function machinery is providing a systematic way to solve recurrences:

1. Express the recurrence as a single equation
2. Multiply both sides by z^N and sum over all N, giving generating function $G(z)$.
3. Find a closed form for $G(z)$
4. Expand $G(z)$ into a power series & read the coefficients.

Ex: Fibonacci Numbers

1. $F_N = f_{N-1} + f_{N-2} + [N=1]$ $\Big\{$ Consider $f_k = 0$ for all $k < 0$
2. $G(z) = \sum f_{N-1} z^N + \sum f_{N-2} z^N + z$

$$= z\,G(z) + z^2 G(z) + z$$

3. $G(z) = z/(1-z-z^2) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right)$ $\Big\}$ UGLY STEP
4. $F_N = \frac{1}{\sqrt{5}}(\phi^N - \hat{\phi}^N)$

Expansion Theorem for Rational Generating Functions

The heart of Step 4, expanding the power series, takes a closed form $G(z) = P(z)/Q(z)$ and expresses it as a partial fraction.

This means we take the denominator $Q(z)$ and factor it, and find appropriate numerators so

$$G(z) = \frac{A}{(1-\rho_1 z)} + \frac{B}{(1-\rho_2 z)} \cdots\cdots$$

where $1/\rho_i$ is a root of $Q(z)$

From this representation, we can expand each term into a geometric series. Thus if constants can be found for $A, B \cdot -$, it makes sense the solution of any recurrence will largely be powers of the reciprocals of the roots of the denominator of its generating function!

$$C_1 \left(1/\rho_1\right)^N + C_2 \left(1/\rho_2\right)^N \cdots\cdots$$

But what does the partial fraction expansion look like with repeated roots?

$$R(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{q_0(1-\rho_1 z)^{d_1}(1-\rho_2 z)^{d_2}}$$

$$= \frac{A_1}{1-\rho_1 z} + \frac{A_2}{(1-\rho_1 z)^2} + \cdots \frac{A_{d_1}}{(1-\rho_1 z)^{d_1}} +$$

$$\frac{B_1}{(1-\rho_2 z)} + \frac{B_2}{(1-\rho_2 z)^2} + \cdots \frac{B_{d_2}}{(1-\rho_2 z)^{d_2}}$$

where A_i + B_i are all constants if $Q(z)$ is of higher degree than $P(z)$.

Thus we can partition any rational generating function in d functions $\frac{c}{(1-\rho z)^{d_i}}$

where d is the degree of $Q(z)$,

EACH OF WHICH WE KNOW HOW TO EXPAND!

Further, what does the expansion look like?
If the i^{th} root ρ_i has multiplicity d_i,

$$\frac{A_1}{(1-\rho_i z)} + \frac{A_2}{(1-\rho_i z)^2} + \cdots\cdots \frac{A_{d_i}}{(1-\rho_i z)^{d_i}}$$

has as the k^{th} term in its expansion,

$$A_1 \rho_i^k z^k + A_2 \binom{k+1}{k} \rho_i^k z^k + \cdots \quad A_{d_i} \binom{k+d_i-1}{k} \rho_i^k z^k$$

$$\boxed{\text{which has the form} \quad \left(c_{d_i-1} k^{d_i-1} + \cdots c_1 k + c_0 \right) (\rho_i)^k z^k}$$

So we know the solution to any recurrence whose generating function is rational, except for the constants c_i, and these can be found by solving a system of linear equations from the first d values of the recurrence.

Ex: $g_0 = g_1 = 1$

$$g_N = g_{N-1} + 2g_{N-2} + (-1)^N$$

Solve using the four step method

1. Write it as a single recurrence

$$g_N = g_{N-1} + 2g_{N-2} + (-1)^N[N \geq 0] + [N = 1]$$

The assumption with generating functions is that the coefficients are 0 for z^{-N}, $N > 0$

2. Turn it into a generating function

$$G(z) = \sum_{N \geq 0} g_N z^N = \sum g_{N-1} z^N + 2\sum g_{N-2} z^N + \frac{1}{1+z} + z$$

$$= \underbrace{zG(z)} + \underbrace{2z^2 G(z)} + \frac{1 + z(1+z)}{1+z}$$

simple shifts are handled by multiplying by z.

3. Solve for $G(z)$

$$G(z) = \frac{1 + z + z^2}{(1+z)(1-z-2z^2)}$$

4. Expand into power series

$$G(z) = \frac{1+z+z^2}{(1+z)^2(1-2z)}$$

The roots to the denominator are $z = -1, -1, 1/2$

$$g_N = (-1)^N(c_1 + c_2 N) + 2^N \cdot c_3$$

$g_0 : 1 = c_1 + c_3$

$g_1 = 1 = 2c_3 - c_1 - c_2$

$g_2 = 1 + 2 + 1 = 4 = c_1 + 2c_2 + 4c_3$

$c_1 = 2/9$

$c_2 = 1/3$

$c_3 = 7/9$

we could have gotten the actual constants directly from the Expansion Theorem, but I find it easy to solve equations once I know the form.

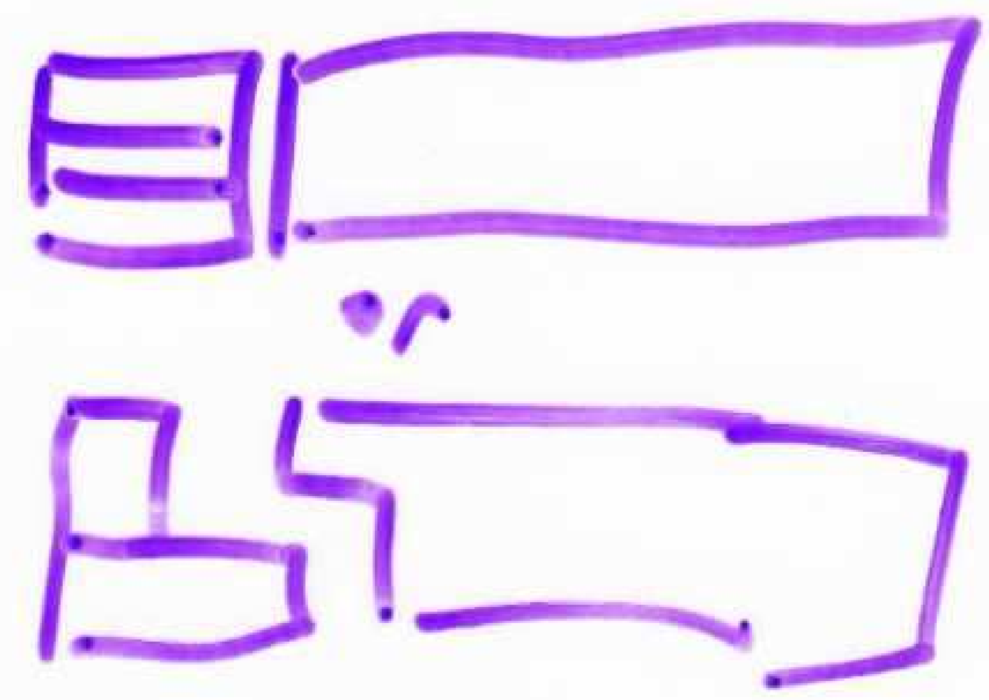
One caution. This method only works when the degree of the numerator is less than the denominator. If not, the problem often can be resolved easily:

$$\frac{z^{100}}{z^5 + z^3 + z + 1} \Rightarrow$$ right shift 100 places the solution to $$\frac{1}{z^5 + z^3 + z + 1}$$

Ex: $3 \times N$ Domino Tilings

The total number of ways to tile a $3 \times N$ region is governed by the recurrence

$$U_0 = 1$$
$$U_1 = 0$$
$$U_N = U_{N-2} + 2V_{N-1}$$



where

$$V_0 = 0$$
$$V_1 = 1$$
$$V_N = U_{N-1} + V_{N-2}$$

Thus: $$U_N = U_{N-2} + 2V_{N-1} + [N=0]$$
$$V_N = U_{N-1} + V_{N-\cancel{1}2}$$

So, $$U(z) = z^2 U(z) + \overset{2\times}{z} V(z) + 1$$
$$V(N) = z\,U(z) + z^2 V(z)$$

These are two intertwined generating functions. With two equations and two unknowns, we can solve for $U(z)$ + $V(z)$:

$$u(z) = \frac{1-z^2}{1-4z^2+z^4} \qquad v(z) = \frac{z}{1-4z^2+z^4}$$

Since $u(z)$ is what we are interested in, and it is the proper ratio of polynomials, we can solve it as before, but it involves finding four roots.

The slick solution observes all exponents in the denominator are even, so if:

$$w(z) = \frac{1}{1-4z+z^2}, \text{ then}$$

$$u(z) = (1-z^2)\,w(z^2) \qquad v(z) = z\,w(z^2)$$

Since $W_N = \frac{3+2\sqrt{3}}{6}(2+\sqrt{3})^N + \frac{3-2\sqrt{3}}{6}(2-\sqrt{3})^N$

$$U_{2N} = W_N - \underbrace{W_{N-1}}_{\text{shift right one place}} = \frac{(2+\sqrt{3})^N}{3-\sqrt{3}} + \frac{(2-\sqrt{3})^N}{3+\sqrt{3}}$$

The moral,
understanding the combinatorial significance of GF can save much calculation!