Special Numbers & Counting Problems

Certain sequences of numbers occur often enough in counting problems that they are given names and are worth knowing, such as the binomial coefficients

When faced with a combinatorial problem, your first question should always be:

Can I find a recurrence?

The sequences we shall see today are all nicely defined by recurrences.

How many ways are there to partition N things into k non-empty subsets?

$N=4$
$k=2$

$\{1,2,3\} \cup \{4\}$ $\{1,2\} \cup \{3,4\}$
$\{1,2,4\} \cup \{3\}$ $\{1,3\} \cup \{2,4\}$
$\{1,3,4\} \cup \{2\}$ $\{1,4\} \cup \{2,3\}$
$\{2,3,4\} \cup \{1\}$

Thus $\left\{ {4 \atop 2} \right\} = 7.$

Consider special cases:

$k=1$. only one subset so, $\left\{ {N \atop 1} \right\} = 1$, $N>0$

when $N=0$, no non-empty subsets, so $\left\{ {0 \atop k} \right\} = 0$ $k>0$.

for convenience, $\left\{ {0 \atop 0} \right\} = 1$

$k=2$. If the first set contains the last element, plus any subset of the rest so the second is non-empty,

$$\left\{ {N \atop 2} \right\} = 2^{N-1} - 1$$

What about the general case?

Suppose we wish to partition N things into k subsets. The $N^{\underline{th}}$ thing is either in its own subset or with something else.

If it is alone, the other $N-1$ things form $k-1$ subsets, which can be done $\left\{ {N-1 \atop k-1} \right\}$ ways.

If not, it can be with any of the k subsets formed from $N-1$ things, so:

$$\left\{ {N \atop k} \right\} = k\left\{ {N-1 \atop k} \right\} + \left\{ {N-1 \atop k-1} \right\}$$

The basis cases are: $\left\{ {N \atop 0} \right\} = 0,\ N > 0.$

$\left\{ {N \atop N} \right\} = 1,\ N \geq 0$

These numbers are famous as the Stirling Numbers of the Second Kind.

Space Aliens use the (Rod) Serling numbers of the (Close Encounters of the) Third kind.

Stirling Numbers + Falling Factorials

$$x^0 = \underline{x^{\underline{0}}} = 1$$

$$x^1 = \underline{x^{\underline{1}}} = x$$

$$x^2 = \underline{x^{\underline{2}} + x^{\underline{1}}} = x(x-1) + x = x^2$$

$$x^3 = \underline{x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}} = x(x-1)(x-2) + 3(x)(x-1) + x$$

$$= x^3 - 3x^2 + 2x + 3x^2 - 3x + x = x^3$$

$$x^4 = \underline{x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{3}} + x^{\underline{1}}}$$

Earlier, we saw how the finite calculus could efficiently give us the sum of falling factorials, but we wanted powers. Stirling numbers give us an easy conversion:

$$x^n = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}}$$

This can be proven by induction given:

$$x \cdot x^{\underline{k}} = x^{\underline{k}} x - k x^{\underline{k}} + k x^{\underline{k}}$$

$$= (x-k) x^{\underline{k}} + k x^{\underline{k}}$$

$$= x^{\underline{k+1}} + k x^{\underline{k}}$$

Theorem: $x^N = \sum_k \left\{ {N \atop k} \right\} x^{\underline{k}}$

Proof: Assume $x^{N-1} = \sum_k \left\{ {N-1 \atop k} \right\} x^{\underline{k}}$.

$$x \sum_k \left\{ {N-1 \atop k} \right\} x^{\underline{k}} = \sum_k \left\{ {N-1 \atop k} \right\} x \cdot x^{\underline{k}}$$

since $x \cdot x^{\underline{k}} = x^{\underline{k+1}} + k x^{\underline{k}}$

relabel k!

$$= \sum_k \left\{ {N-1 \atop k} \right\} x^{\underline{k+1}} + \sum_k \left\{ {N-1 \atop k} \right\} k x^{\underline{k}}$$

$$= \sum_k \left\{ {N-1 \atop k-1} \right\} x^{\underline{k}} + \sum_k \left\{ {N-1 \atop k} \right\} k x^{\underline{k}}$$

$$= \sum_k \left(k \left\{ {N-1 \atop k} \right\} + \left\{ {N-1 \atop k-1} \right\} \right) x^{\underline{k}} = \sum_k \left\{ {N \atop k} \right\} x^{\underline{k}}$$

$\square$

Thus if we need a closed form for $\sum_{x=1}^{N} x^k$ for some k, we can use the Stirling numbers + finite calculus to get it without work.

Another way uses the Bernoulli numbers, discussed in Graham, Knuth, + Patashnik.

How many ways are there to arrange N things into k cycles?

$N=4$ $\{1,2,3\},\{4\}$ $\{124\},\{3\}$ $\{134\}\{2\}$, $\{234\},\{1\}$

$k=2$ $\{1,3,2\},\{4\}$ $\{142\},\{3\}$ $\{143\}\{2\}$, $\{243\}\{1\}$

$\{1,2\},\{3,4\}$

$\{1,3\},\{2,4\}$ so $\left[{4 \atop 2}\right] = 11$

$\{1,4\},\{2,3\}$

Each cycle of N elements describes N permutations:

$\rightarrow$ ABCD, DABC, CDAB, BCDA

Consider special cases:

$k=1$ There are $N!$ permutations, and each cycle accounts for N, so

$$\left[{N \atop 1}\right] = (N-1)!, \quad N>0$$

Alternately, in any cycle the smallest element can be written first, meaning the cycles are all permutations of $N-1$ elements.

$k=N$ To partition N things into N cycles, each must go in its own cycle, so

$$\begin{bmatrix} N \\ N \end{bmatrix} = 1$$

$k=N-1$ Since one cycle contains two elements and any pair of elements is ok,

$$\begin{bmatrix} N \\ N-1 \end{bmatrix} = \binom{N}{2}$$

What about the general case?

Either the $N^{\underline{th}}$ key forms its own cycle $\begin{bmatrix} N-1 \\ k-1 \end{bmatrix}$

Or else it goes in one of then $N-1$ slots on the other k cycles; since each cycle of length

$$[x], \ [x, y, z], \ \ldots [w, x, y, z]$$

$\uparrow$ N $\quad$ $\uparrow \uparrow \uparrow$ $N?$ $\quad$ $\uparrow \uparrow \uparrow \uparrow$

k has k distinct slots:

Thus: $$\begin{bmatrix} N \\ k \end{bmatrix} = (N-1)\begin{bmatrix} N-1 \\ k \end{bmatrix} + \begin{bmatrix} N-1 \\ k-1 \end{bmatrix},$$

basis cases: $\begin{bmatrix} N \\ 0 \end{bmatrix} = 0, \ N \geq 0 \quad \begin{bmatrix} N \\ N \end{bmatrix} = 1, \ N \geq 0$

These numbers are famous as the:

Stirling Numbers of the First kind

Table ~~244~~ 258 **Stirling's triangle for subsets.**

n	$\left\{ n \atop 0 \right\}$	$\left\{ n \atop 1 \right\}$	$\left\{ n \atop 2 \right\}$	$\left\{ n \atop 3 \right\}$	$\left\{ n \atop 4 \right\}$	$\left\{ n \atop 5 \right\}$	$\left\{ n \atop 6 \right\}$	$\left\{ n \atop 7 \right\}$	$\left\{ n \atop 8 \right\}$	$\left\{ n \atop 9 \right\}$
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

Not Symmetric!

Table ~~245~~ 259 **Stirling's triangle for cycles.**

n	$\left[n \atop 0 \right]$	$\left[n \atop 1 \right]$	$\left[n \atop 2 \right]$	$\left[n \atop 3 \right]$	$\left[n \atop 4 \right]$	$\left[n \atop 5 \right]$	$\left[n \atop 6 \right]$	$\left[n \atop 7 \right]$	$\left[n \atop 8 \right]$	$\left[n \atop 9 \right]$
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

Not symmetric!

Note: knuth changed the definition!

Table ~~253~~ 287 **Stirling's triangles in tandem.**

n	$\left\{ n \atop -5 \right\}$	$\left\{ n \atop -4 \right\}$	$\left\{ n \atop -3 \right\}$	$\left\{ n \atop -2 \right\}$	$\left\{ n \atop -1 \right\}$	$\left\{ n \atop 0 \right\}$	$\left\{ n \atop 1 \right\}$	$\left\{ n \atop 2 \right\}$	$\left\{ n \atop 3 \right\}$	$\left\{ n \atop 4 \right\}$	$\left\{ n \atop 5 \right\}$
−5	1										
−4	10	1									
−3	35	6	1								
−2	50	11	3	1							
−1	24	6	2	1	1						
0	0	0	0	0	0	1					
1	0	0	0	0	0	0	1				
2	0	0	0	0	0	0	1	1			
3	0	0	0	0	0	0	1	3	1		
4	0	0	0	0	0	0	1	7	6	1	
5	0	0	0	0	0	0	1	15	25	10	1

A duality relationship:

$$\left[n \atop k \right] = \left\{ -k \atop -n \right\}$$

Cycles and Permutations

A permutation may be viewed either as an object or an arrangement operation:

1	2	3	4	5	6	7	8	9	10
3	8	4	7	2	9	1	5	6	10

$1 \to 3 \to 4 \to 7$ $\quad 2 \to 8 \to 5 \to 2$ $\quad 6 \to 9$ $\quad 10$

Every permutation can be written as a disjoint set of cycles, and visa versa.

Cycles for a permutation must be disjoint, since each element has a unique predecessor.

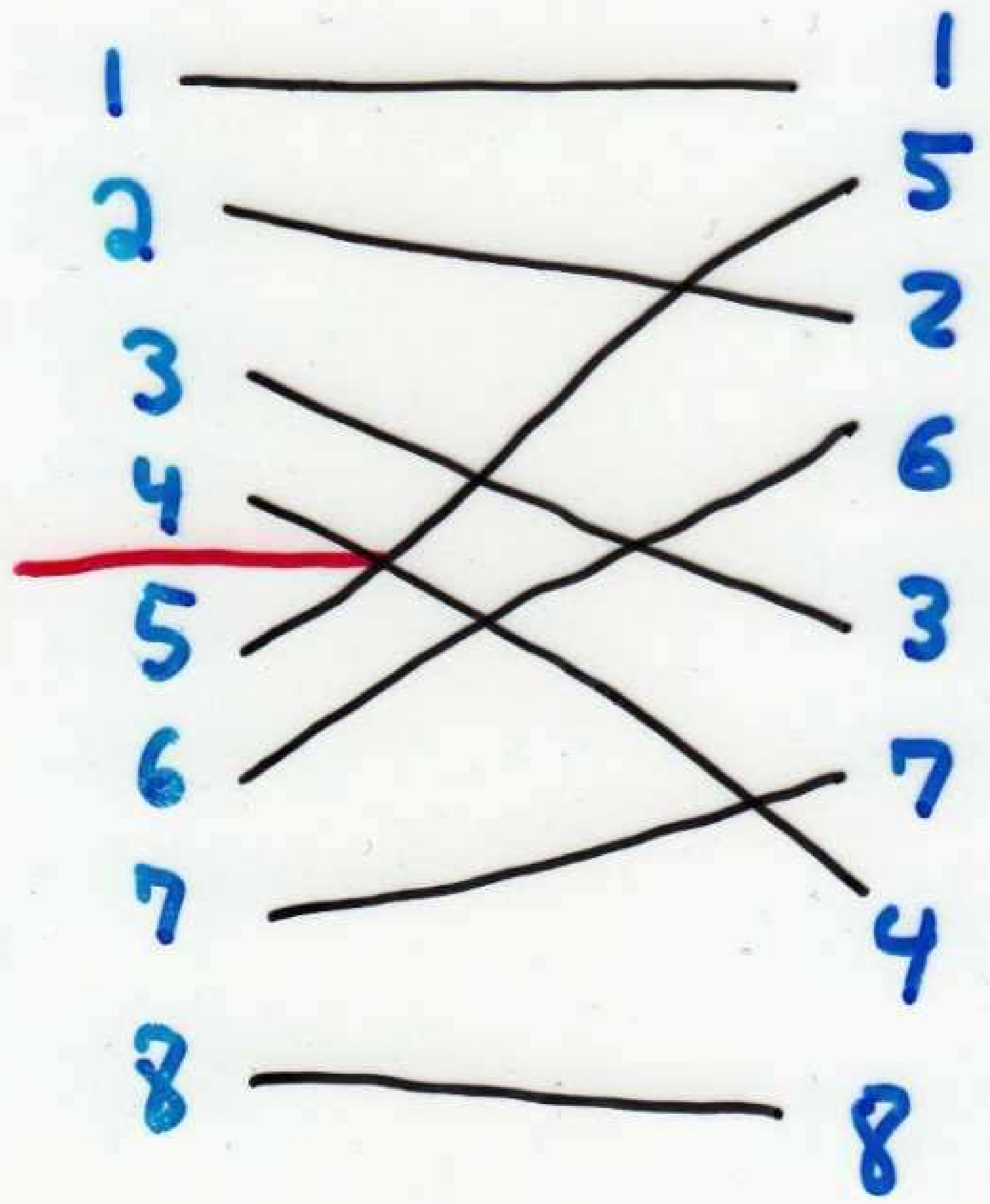
This 1-1 correspondence is very important for studying permutations.

Since we know how count permutations + permutations with k cycles,

$$\sum_{k=0}^{N} \left[{N \atop k} \right] = N!$$

Perfect Shuffles

The poker players among you are familiar with perfect shuffles, an operation which takes one permutation to another.



Split the 2N cards evenly, and interleave every other card.

Perfect shuffles have application to routing networks in parallel computer architectures.

A perfect shuffle of permutation p can be performed by multiplying p by

$(1, N/2+1, 2, N/2+2, \ldots, N/2, N)$

1	1	1	1	1	1	1	1	1
2	27	14	33	17	9	5	3	2
3	2	27	14	33	17	9	5	3
4	28	40	46	49	25	13	7	4
5	3	2	27	14	33	17	9	5
6	29	15	8	30	41	21	11	6
7	4	28	40	46	49	25	13	7
8	30	41	21	11	6	29	15	8
9	5	3	2	27	14	33	17	9
10	31	16	34	43	22	37	19	10
11	6	29	15	8	30	41	21	11
12	32	42	47	24	38	45	23	12
13	7	4	28	40	46	49	25	13
14	33	17	9	5	3	2	27	14
15	8	30	41	21	11	6	29	15
16	34	43	22	37	19	10	31	16
17	9	5	3	2	27	14	33	17
18	35	18	35	18	35	18	35	18
19	10	31	16	34	43	22	37	19
20	36	44	48	50	51	26	39	20
21	11	6	29	15	8	30	41	21
22	37	19	10	31	16	34	43	22
23	12	32	42	47	24	38	45	23
24	38	45	23	12	32	42	47	24
25	13	7	4	28	40	46	49	25
26	39	20	36	44	48	50	51	26
27	14	33	17	9	5	3	2	27
28	40	46	49	25	13	7	4	28
29	15	8	30	41	21	11	6	29
30	41	21	11	6	29	15	8	30
31	16	34	43	22	37	19	10	31
32	42	47	24	38	45	23	12	32
33	17	9	5	3	2	27	14	33
34	43	22	37	19	10	31	16	34
35	18	35	18	35	18	35	18	35
36	44	48	50	51	26	39	20	36
37	19	10	31	16	34	43	22	37
38	45	23	12	32	42	47	24	38
39	20	36	44	48	50	51	26	39
40	46	49	25	13	7	4	28	40
41	21	11	6	29	15	8	30	41
42	47	24	38	45	23	12	32	42
43	22	37	19	10	31	16	34	43
44	48	50	51	26	39	20	36	44
45	23	12	32	42	47	24	38	45
46	49	25	13	7	4	28	40	46
47	24	38	45	23	12	32	42	47
48	50	51	26	39	20	36	44	48
49	25	13	7	4	28	40	46	49
50	51	26	39	20	36	44	48	50
51	26	39	20	36	44	48	50	51
52	52	52	52	52	52	52	52	52

N	iterations
50	21
52	8
54	52
56	20
58	18
60	58

How many perfect shuffles until we return to the original condition of the deck?

But why does it only take eight perfect shuffles to get back to where we started from? Consider the cycle structure of a perfect shuffle:

$$\{\{1\}, \{27, 14, 33, 17, 9, 5, 3, 2\}, \{28, 40, 46, 49, 25, 13, 7, 4\}, \{29, 15, 8, 30, 41, 21, 11, 6\}, \{31, 16, 34, 43, 22, 37, 19, 10\}, \{32, 42, 47, 24, 38, 45, 23, 12\}, \{35, 18\}, \{36, 44, 48, 50, 51, 26, 39, 20\}, \{52\}\}$$

There are cycles of lengths 1, 2, 8.

In a cycle of length k, every k multiplications each element in the cycle returns home.

> Thus the deck repeats when the iteration number is the **least common multiple** of the cycle lengths

To maximize the repetition length, have cycle lengths of different small primes!

$$\{\underbrace{2, 1}_{2},\ \underbrace{4, 5, 3}_{3},\ \underbrace{7, 8, 9, 10, 6}_{5}\}$$

Perfect Shuffles + Modular Arithmetic

Another way to view a perfect shuffle of $2N$ cards is as a multiplication by 2 (mod $2N-1$)

card position	$\times 2 \pmod{2N-1}$
0	0
1	2
⋮	4
⋮	⋮
$N-1$	$2N-2$
N	$2N \equiv 1$
⋮	$2N+2 \equiv 3$
	⋮
$N+k$	$2(N+k) \equiv 2k+1$
⋮	⋮
$2N-1$	$2(2N-1) \equiv 0$

First and last cards always stay in place in a shuffle

Thus the number of perfect shuffles to restore the deck is the number of times we multiply by 2 until it is an identity operation, ie.

$$2^x \equiv 1 \mod 2N-1$$

ex: $2N = 52$

$$2^8 = 256 \equiv 1 \mod 51$$