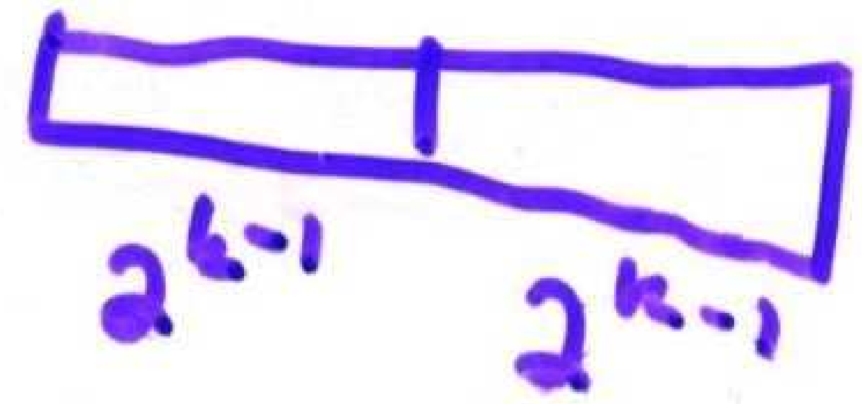
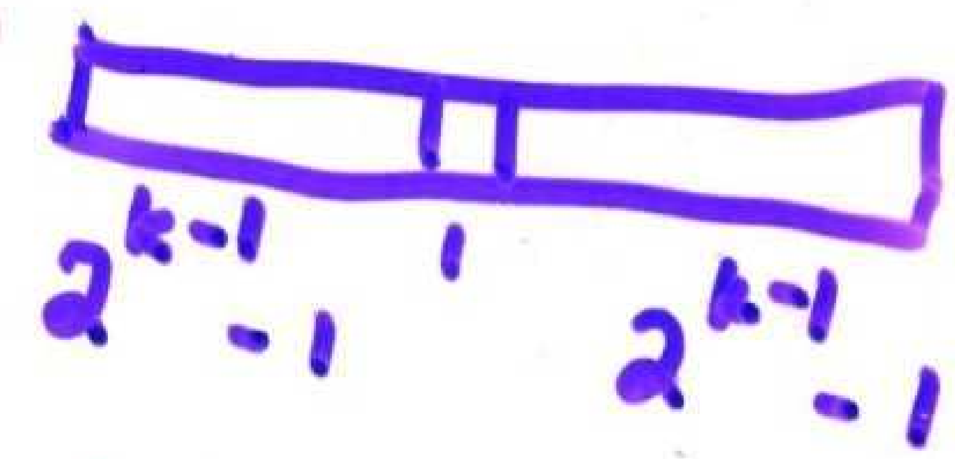
Floor / Ceiling Recurrences

Divide and conquer algorithms work by dividing the problem into as evenly sized pieces as possible.

Unless $N = 2^k$ for a mergesort



or $N = 2^k - 1$ for a binary search



our recurrences will have to deal with ceilings and floors to be correct.

Since $N = \lceil N/2 \rceil + \lfloor N/2 \rfloor$, the correct mergesort recurrence is

$$f(1) = 0$$
$$f(N) = f(\lceil N/2 \rceil) + f(\lfloor N/2 \rfloor) + N - 1 \qquad N > 1$$

Since taking the floor or ceiling changes the value of the index by at most 1, asymptotically we usually get the right answer if we ignore the ceilings / floors. However, we can get the exact answer by being careful & working harder.

The Knuth Numbers

$$K_0 = 1$$
$$K_{N+1} = 1 + \min\left(2K_{\lfloor N/2 \rfloor}, 3K_{\lfloor N/3 \rfloor}\right)$$

N	0	1	2	3	4	5	6	7	8	9	10
K_N	1	3	3	4	7	7	7	9	9	10	13

Show that $K_N \geq N+1$, proof by induction

Clearly true for $N=0$, assume true up to N.

$$K_{N+1} \geq 1 + \min\left(2\left(\lfloor N/2 \rfloor + 1\right), 3\left(\lfloor N/3 \rfloor + 1\right)\right)$$
$$= 1 + \min\left(2\lfloor N/2 \rfloor + 2, 3\lfloor N/3 \rfloor + 3\right)$$

Since $k\lfloor N/k \rfloor = k\left\lfloor \frac{mk+r}{k} \right\rfloor = km = N-r$

is minimized when $r = k-1$

$$\geq 1 + \min\left(N-2+1+2, N-3+1+3\right)$$
$$\geq 1 + \min\left(N+1, N+1\right)$$
$$\geq N+2$$

■

The Generalized Josephus Problem

When we left Josephus, all were sharpening their swords and Knuth had shown him where to stand when executing every second man. He said thanks, but they are executing every third man!

The recurrence we got,

$$J(1) = 1$$
$$J(n) = 2J(\lfloor n/2 \rfloor) - (-1)^n$$

was messy to solve using standard functions. It is even worse to solve the case where $k=3$.

A different approach is to count off the people as we encounter them:

1	2	~~3~~	4	5	~~6~~	7	8	~~9~~	10
11	~~12~~		13	14		~~15~~	16		17
~~18~~			19	20			~~21~~		22
			23	~~24~~					25
			26						~~27~~
			28						
			29						
			~~30~~						

Executions occur when the number is $3k$.
The last survivor is the one numbered $3n$.

The trick is to figure out what $3n$'s original number was, which can be found by finding its immediate ancestor and trace up the list.

Consider person # $N > n$. He was assigned this number before slaying the $(k+1)^{st}$ person, thus.

$$N = n + 2k + 1 \quad \underline{\text{or}} \quad \Rightarrow \quad k = (N-n-1)/2$$
$$N = n + 2k + 2 \qquad\qquad\qquad k = (N-n-2)/2$$

Thus the iteration he was labelled was:

$$k = \lfloor (N-n-1)/2 \rfloor$$

Thus his previous number was

$3k+1$ if $N = n + 2k + 1$
$3k+2$ if $N = n + 2k + 2$
$3k$ - can't be, would have been killed off

So his previous number is $3k + (N-n-2k)$

$$= \underline{k + N - n}$$

Ex: $n = 10$

$N = 30$

$k = \lfloor (30-10-1)/2 \rfloor = 9 \quad = n-1$

previous $= 9 + 30 - 10 = 29 \quad = k + N - n$

$N = 29$

$k = \lfloor (29-10-1)/2 \rfloor = 9$

previous $= 9 + 29 - 10 = 28$

$N = 28$

$k = \lfloor (28-10-1)/2 \rfloor = 8$

previous $= 8 + 28 - 10 = 26$

This gives an iterative algorithm to find the survivors original number:

$N = 3n$

while $N > n$ do $N = \left\lfloor \frac{N-n-1}{2} \right\rfloor + N - n;$

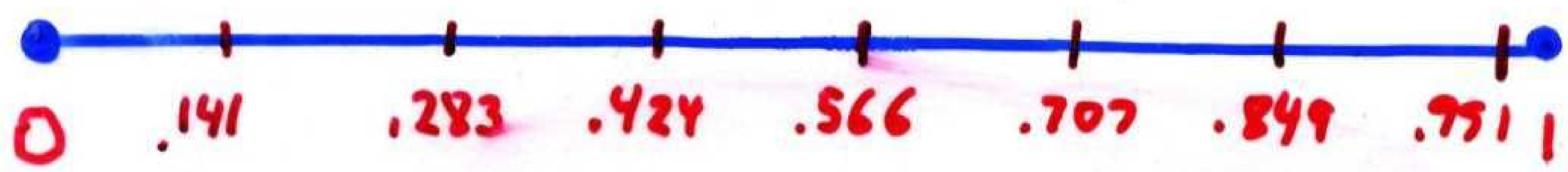
$J_3(n) = N$

The Three Distance Theorem

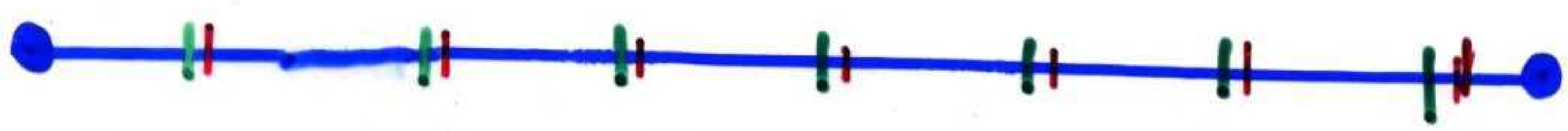
Consider the sequence $\{\alpha i\}$, where α is an irrational number and $i = 1, 2, 3, \ldots, N$

The result is a sequence of points between 0 + 1.

Ex. $\alpha = \pi$
$N = 7$



Before the sequence "wraps around", note that only two distict distances have been defined between successive points – .141592... and 1 – .991149...



After the second trip through $N = 14$, there are still only two distinct distances .132741... and .00885142...

For any N + α, at most three distinct distances are ever defined!

This result is, to me, very surprising and counter-intuitive. It helps justify the use of multiplicative hash functions for table lookup, since clusters of consecutive keys get spread uniformly through the search space.

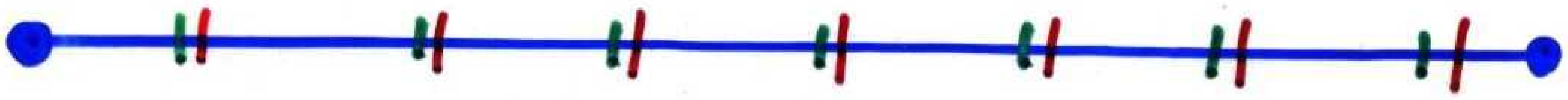
The next point always splits a region of largest length - further regions are split in a first-in, first out order.

What happens if α is rational?

$\left\{ \frac{a}{b} i \right\}$ will repeat when $i \backslash b$, and so all future distances will be zero, for a total of 3 distinct distances.

The proof is by induction and is fairly technical. It appears in Knuth, volume 3.

We can readily verify that the last two regions in our example are exactly the same lengths:



Last region $= 1 - \{7\pi\}$

Next to last $= \{7\pi\} - \{14\pi\}$

How can we prove equality?

$$\{7\pi\} - \{14\pi\} = 7\pi - \lfloor 7\pi \rfloor - (14\pi - \lfloor 14\pi \rfloor)$$

$$= \lfloor 14\pi \rfloor - 7\pi - \lfloor 7\pi \rfloor$$

$$= \lfloor 14\pi \rfloor - (\lfloor 7\pi \rfloor + \{7\pi\}) - \lfloor 7\pi \rfloor$$

$$= \lfloor 14\pi \rfloor - 2\lfloor 7\pi \rfloor - \{7\pi\}$$

$$= 43 - 2(21) - \{7\pi\}$$

$$= 1 - \{7\pi\} \quad \blacksquare$$

The Mod operator

Before people got calculators with decimal displays, the answer to a division problem was a quotient and a remainder.

$$N = qm + r, \text{ where}$$

$$q = \lfloor N/m \rfloor \text{ and } r = N - m\lfloor N/m \rfloor$$

The remainder is useful enough to define a binary operation to yield it:

$$x \bmod y = x - y\lfloor x/y \rfloor$$

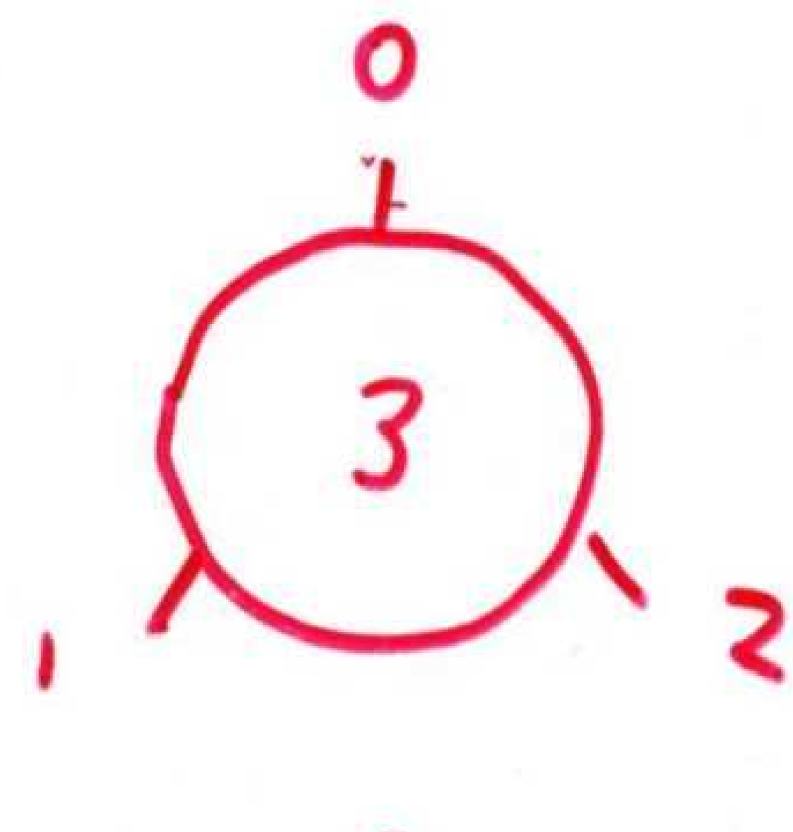
This is defined on negative values as well as positive:

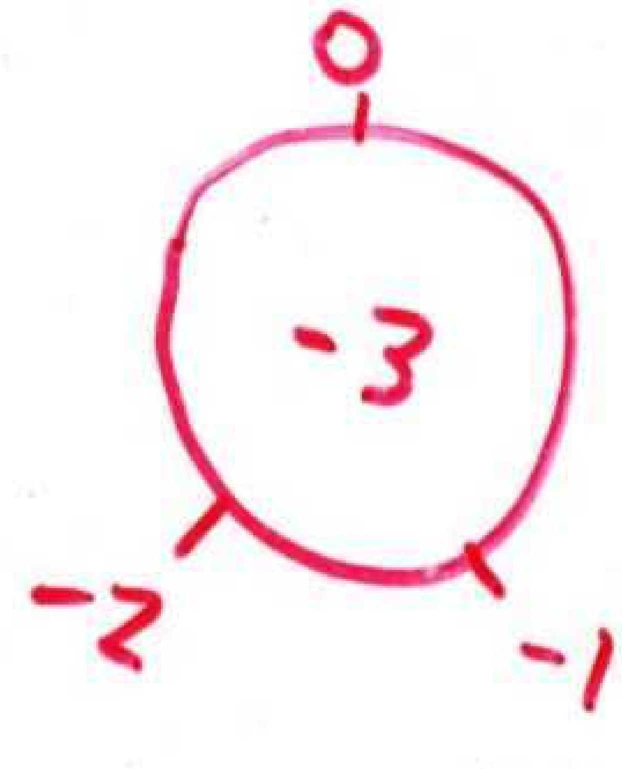
$$5 \bmod 3 = 5 - 3\lfloor 5/3 \rfloor = 2$$

$$5 \bmod -3 = 5 + 3\lfloor 5/-3 \rfloor = -1$$

$$-5 \bmod 3 = -5 - 3\lfloor -5/3 \rfloor = 1$$

$$-5 \bmod -3 = -5 + 3\lfloor 5/3 \rfloor = -2$$





Note $x \bmod 1 = \{x\}$

The mod function obeys the distributive law:

$$c(x \bmod y) = c\left(x - y\lfloor x/y \rfloor\right)$$
$$= cx - cy\lfloor x/y \rfloor$$
$$= cx - cy\left\lfloor cx/cy \right\rfloor \leftarrow \text{top \& bottom by } c.$$
$$= (cx \bmod cy)$$

thus

$$x \bmod y = -(-x \bmod -y)$$

as a special case.

Modular arithmetic is one of the most important aspects of number theory. We will see it in more detail when we study congruences in the next chapter.

Ex: The LS command in UNIX

The prettiest way to arrange N items in M columns makes the columns as even as possible:

1	6	11	15
2	7	12	16
3	8	13	17
4	9	14	18
5	10		

~~1~~	~~6~~	~~11~~	~~16~~
~~2~~	~~7~~	~~12~~	~~17~~
~~3~~	~~8~~	~~13~~	~~18~~
~~4~~	~~9~~	~~14~~	
~~5~~	~~10~~	~~15~~	

Using a row-by-row greedy algorithm gives the right shape, but the items are not laid out correctly.

Observe $N = M\lfloor N/M \rfloor + (N \bmod M)$

This means $(N \bmod M)$ columns of $\lfloor N/M \rfloor + 1$ and $M-(N \bmod M)$ columns of $\lfloor N/M \rfloor$

If $(N \bmod M) > 0$, $\lfloor N/M \rfloor + 1 = \lceil N/M \rceil$, so we can change the top condition

Note that the first column contains $\lceil N/M \rceil$ elements. This suggests a recursive algorithm, outputting the first column + then solving it with $N - \lceil N/M \rceil$ elements and $M-1$ columns.

To determine the number of elements in the k^{th} column, we need a formula which is $\lceil N/M \rceil$ when $1 \le k \le N \bmod M$ and $\lfloor N/M \rfloor$ when $k > N \bmod M$

Try $\left\lceil \frac{N-k+1}{M} \right\rceil$, which implies

$$N = \left\lceil \frac{N}{M} \right\rceil + \left\lceil \frac{N-1}{M} \right\rceil + \cdots + \left\lceil \frac{N-M+1}{M} \right\rceil$$